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A Dynamic Optimization Model of Monetary Stabilization Policy

Ken Matsumoto
Chuo University

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INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

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Abstract

By solving a dynamic optimization structure, this paper aims to achieve an optimal monetary policy which enables the central bank to minimize a Taylor-rule embedded social loss function of inflation and employment. Three different types of inflation expectations are tested: forward-looking, backward-looking, and mixed expectation formation. As a result, under several assumptions, we found stable and optimal policy paths with all of the three scenarios. The results suggest that when the expected inflation rate is below the target, the central bankers need to initially set the interest rate at a sufficiently low level but gradually raise it, until they reach the equilibrium. Opposite solution stands if the inflation expectation is above the target. These may be counterfactual to a commonly seen monetary policy, though we believe this analysis is meaningful, in a way that our solution can be regarded as a "reference" or "yardstick", and indicate how the real-life monetary policy is deviating from the optimal path.

Keywords: Optimal monetary policy, Taylor Rule, (in)stability, dynamic optimization, inflation expectation

JEL Classification: E5, E12, E52

1. Introduction

At this moment in 2022, most of the major central banks have been struggling from the 40-year high inflation which was triggered by the combination of supply bottlenecks and expansive fiscal policy as a response to the Covid-19 pandemic, which has been spread from 2020. As a response, a rapid pace of tightening has been taken which should eventually lower the inflation at some point, but a trade-off between a loss of employment seems to be inevitable. If the economy is overkilled, a deep recession is the possible outcome, and disinflation risk could even emerge at some point. However, forecasting these scenarios are always a big challenge. For example, the Federal Reserve System (Fed) has already admitted that they overlooked the inflation outlook after the pandemic. Their outlooks on policy rate (dot plot) have been revised upward repeatedly, adding uncertainty to the financial market. Their stance of data-by-data policy decisions is widely known as a discretionary policy. The opposite method is a rule-based policy, for example, a Taylor Rule. This comparison is originally termed by Kydland and Prescott (1977).

In this paper, a welfare loss function is formulated by inducing a simple Taylor-rule function in the model. Taylor-rule, which was elaborated by Taylor (1993) is not explicitly adopted as a policy instrument to determine a policy rate, yet it is still a very influential notion widely among policy makers. For example, in the Federal Reserve Bank (Fed) semiannual Monetary Policy Report[1], they provide Taylor Rule with several different versions since 2017 as a reference to theoretically show the appropriate policy rate. The current Fed chairman Jerome Powell pointed out on Taylor Rule that it never has been used as a strict policy instrument, though acknowledged it is still a useful reference to signal an appropriate interest rate[2].

In the Monetary Policy Report which was published in June 2022, all the versions of Taylor Rule are implying that the Federal Funds rate should have been higher than what it was at that moment[3]. The personal consumption expenditure (PCE) deflator as of Oct 2022 was 6.0% year on year, which is substantially higher than the Fed's target of 2%. The unemployment rate as of Nov 2022 was 3.7%, which is well below the Fed's longer-run projection of 4%[4]. The Fed intends to raise the policy rate expeditiously until the inflation

[1] For example, the Monetary Policy Report (June 17, 2022)
https://www.federalreserve.gov/monetarypolicy/2022-06-mpr-summary.htm

[2] Chair Powell's quote on Taylor Rule mentioned in John Taylor's blog
https://economicsone.com/2022/06/25/play-by-the-rules/

[3] Monetary Policy Report (June 17, 2022)

[4] December 14, 2022 FOMC Projection materials,
https://www.federalreserve.gov/monetarypolicy/fomcprojtabl20221214.htm

rate heads back to the target, acknowledging the risk of facing higher unemployment rate. However, from its significant gap from the target, a concern of over-killing the economy has been arising and the Fed's handling of the policy rate setting seems more and more challenging. It would be beneficial if the Fed owns a certain instrument which guides an appropriate policy path, or an "optimal path" to achieve their policy goals.

There are several different approaches on the study of optimal policy. Rotemberg and Woodford (1997) analyzed monetary policy optimization through econometric approach, using vector autoregression models. As a variation of optimal monetary policy, for instance, Acharya et al. (2020) takes into account consumption inequality in their model, by assuming a heterogeneous agent, instead of a representative agent model. Also, Brayton et al. (2014) tackled this problem using the FRB/US Model to figure what they call "optimal-control policy". We mention the comparison with this paper in the end as well, to highlight this paper's policy implication more effectively.

We analyze central bank's optimal policy path that minimizes the social loss function using a dynamic optimization structure and test its stability at the equilibrium. The social loss function here is defined by the Taylor Rule-like output/inflation gap combination, and the dynamic optimization is solved subject to the inflation expectation function constraint. This paper inherits the work by Matsumoto (2022), which assumed consolidated government (central bank and government) could control its real output to minimize the social loss function of inflation and unemployment gap. Here, we change the control variable to nominal interest rate and assess its policy implications. Similar approach was taken by Asada (2010) and Semmler and Zhang (2004) to derive the appropriate monetary policy and its characteristics at the equilibrium point, but this paper takes further steps, by switching the parameter of the inflation expectation function, which represents people's stance on their inflation expectation. Three different cases of optimization are tested: forward-looking, backward-looking, and mixed expectation formation scenarios.

2. Formulation of the Model

Basic formulation of the model used in this paper is based on Matsumoto (2022). Eq.(1) is a linear "expectations-augmented Phillips curve", which variable π and r represents inflation and nominal interest rate, respectively. As such, $r-\pi^e$ is the expected real rate of interest. Parameter ε is a reaction parameter from the output gap, π^e is inflation expectation, Y is the real output, and $\bar{Y}$ represents natural output, or real output target. $Y(r-\pi^e)$ below represents a reduced form of the equilibrium condition of the goods market. Real output is a decreasing function of the expected real rate of interest, from a decrease in investment expenditure. What is different from Matsumoto (2022) is the control variable, which this paper uses nominal interest rate r to indirectly control the real output.

$$\pi = \varepsilon(Y(r-\pi^e) - \bar{Y}) + \pi^e \quad ; \; \varepsilon > 0 \; , \; Y_{(r-\pi^e)} = \frac{\partial Y}{\partial (r-\pi^e)} < 0 \tag{1}$$

Eq.(2) is the dynamic inflation expectation formation that is based on Asada (2010), which combines forward-looking and backward-looking expectations. Parameter ξ represents the weight of people's stance on their inflation expectation. If ξ =1, the equation of motion for $\dot{\pi}^e$ is dependent on the inflation gap, which here is represented by the difference between actual inflation π and inflation expectation π^e. This shows that people have adaptive behavior on inflation. On the other hand, if ξ =0 , the equation expresses people's behavior as forward-looking, since $\dot{\pi}^e$ is now dependent on the gap between government's inflation target $\bar{\pi}$ and π^e. If $0 < \xi < 1$, then people's behavior on inflation is mixed between forward and backward-looking. The parameter α is the reaction parameter to the inflation expectation dynamics.

$$\dot{\pi}^e = \alpha\{\xi(\pi - \pi^e) + (1-\xi)(\bar{\pi} - \pi^e)\} \, ; \; \alpha > 0, \; 0 \leqq \xi \leqq 1 \tag{2}$$

Eq.(3) is derived by substituting Eq.(1) into Eq.(2), and now the equation of motion for $\dot{\pi}^e$ has real output as one of the variables.

$$\dot{\pi}^e = \alpha\{\xi\varepsilon(Y(r-\pi^e) - \bar{Y}) + (1-\xi)(\bar{\pi} - \pi^e)\} \tag{3}$$

For simplicity on solving the upcoming equations, linear approximation of $Y(r-\pi^e)$ below is used in the further analyses.

$$Y = -a(r-\pi^e) + b = Y(\pi^e, r) \, ; \; \frac{\partial Y}{\partial r} = -a < 0 \, , \; \frac{\partial Y}{\partial \pi^e} = a > 0, b > 0 \tag{4}$$

2.1 Case of ξ =0

Based on the model mentioned above, in this section, we work on the case of ξ =0. As previously mentioned, this transforms the equation of $\dot{\pi}^e$ into a forward-looking system. Eq.(3) now becomes Eq.(5).

$$\dot{\pi}^e = \alpha(\bar{\pi} - \pi^e) \, ; \alpha > 0 \tag{5}$$

The social loss function is defined as below. This transformation is similar to the method used in Taylor(1989), Chiang(1992), Woodford(2001) and Asada(2010)[5].

$$V = \theta(Y(r-\pi^e)-\bar{Y})^2 + (1-\theta)(\pi-\bar{\pi})^2 = V(Y(r-\pi^e),\pi) \ \ ; \ \ 0<\theta<1 \tag{6}$$

The parameter θ is the positive parameter which represents the weight of how much the policy maker prioritize between real output and inflation gap. Eq.(1) is now substituted into Eq.(6), and in order to minimize the loss function, Eq.(6) is turned into negative.

$$-V = -[\theta(Y(r-\pi^e)-\bar{Y})^2 + (1-\theta)\{\varepsilon(Y(r-\pi^e)-\bar{Y}) + \pi^e - \bar{\pi}\}^2] = W(Y(r-\pi^e),\pi^e)$$
$$0<\theta<1 \tag{7}$$

Combining Eq.(5) and Eq.(7) gives the following dynamic optimization problem subject to the constraint equation.

$$\max_{r\geq 0}\int_0^\infty W(Y(r-\pi^e),\pi^e)e^{-\rho t}\,dt \ \ ; \ \ \rho>0$$
$$s.t.\ \dot{\pi}^e = \alpha(\bar{\pi}-\pi^e) \tag{8}$$

In order to derive the optimal macroeconomic policy path, a current-value Hamiltonian is formalized as below, where λ is the co-state variable.

$$H = -[\theta((-a(r-\pi^e)+b)-\bar{Y})^2 + (1-\theta)\{\varepsilon((-a(r-\pi^e)+b)-\bar{Y}) + \pi^e - \bar{\pi}\}^2] + \lambda\alpha(\bar{\pi}-\pi^e) \tag{9}$$

As explained in Chiang (1992), Pontryagin's maximum principle conditions are given as follows[6].

$$\underset{r\geq 0}{\mathrm{Max}}\, H(Y(r-\pi^e),\pi^e,\lambda)\ for\ all\ t\in[0,\infty)$$
$$\dot{\pi}^e = \frac{\partial H}{\partial\lambda} \quad [equation\ of\ motion\ for\ \dot{\pi}^e]$$

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[5] Consideration in monetary policy tradeoffs, welfare loss function, and problem solving of optimal monetary policy are also conducted by Galí (2015) and Woodford(2002).

[6] Explanation on Pontryagin's maximum principle conditions and dynamic optimization are also available from Chiang and Wainwright (2010).

$$\dot{\lambda} = -\frac{\partial H}{\partial \pi^e} + \rho\lambda \qquad [equation\ of\ motion\ for\ \lambda]$$

$$\lim_{t \to \infty} \lambda e^{-\rho t} = 0 \qquad [transversality\ condition]$$

(10)

The Hamiltonian system here is solved with respect to the nominal interest rate, r. This implies that policy maker is controlling nominal interest rate r to minimize (optimize) the social loss function previously proposed.

A first order condition is required to show that the optimal control of r will be an interior solution.

$$\frac{\partial H}{\partial r} = 2a[\theta((-a(r-\pi^e)+b)-\bar{Y}) + \varepsilon^2(1-\theta)\{((-a(r-\pi^e)+b)-\bar{Y}) + \pi^e - \bar{\pi}\}] = 0$$

(11)

Further differentiation of Eq.(11) with the result of negative shows that the control variable r does maximize the Hamiltonian system.

$$\frac{\partial^2 H}{\partial r^2} = 2a[-\theta a - \varepsilon^2 a(1-\theta)] < 0 \qquad (12)$$

Equation of motion for $\dot{\pi}^e$ is defined as below.

$$\dot{\pi}^e = \frac{\partial H}{\partial \lambda} = \alpha(\bar{\pi} - \pi^e) = F_1^1(\pi^e) \qquad (13)$$

Equation of motion for λ, which is a costate variable, is described below.

$$\dot{\lambda} = -\frac{\partial H}{\partial \pi^e} + \rho\lambda = 2[\theta a((-a(r-\pi^e)+b)-\bar{Y}) + (1-\theta)(\varepsilon a + 1)\{\varepsilon((-a(r-\pi^e)+b)-\bar{Y}) + \pi^e - \bar{\pi}\}] + \lambda(\alpha+\rho)$$

(14)

Real output Y of Eq.(11) can be transformed as below.

$$((-a(r-\pi^e)+b)-\bar{Y}) = \frac{\varepsilon(1-\theta)(\bar{\pi}-\pi^e)}{\theta+\varepsilon^2(1-\theta)} = f(\pi^e) \qquad (15)$$

Eq.(15) can be substituted into Eq.(14) as below.

$$\dot{\lambda} = 2[\theta a f(\pi^e) + (1-\theta)(\varepsilon a + 1)\{\varepsilon f(\pi^e) + \pi^e - \bar{\pi}\}] + \lambda(\alpha+\rho) = F_2^1(\pi^e, \lambda) \qquad (16)$$

Where $$\frac{df(\pi^e)}{d\pi^e} = -\frac{\varepsilon(1-\theta)}{\theta+\varepsilon^2(1-\theta)} < 0$$

2.2 Phase diagram in case of $\xi = 0$

From the equilibrium point of $\dot{\pi}^e$ in Eq.(13), the solution $\bar{\pi} = \pi^e$ can be derived. $\dot{\pi}^e = 0$ line is shown in Figure 1. It is assumed here that the policy maker would indirectly maneuver the costate variable of λ by controlling the interest rate r, which can be termed as a marginal increase (or decrease) in inflation expectation.

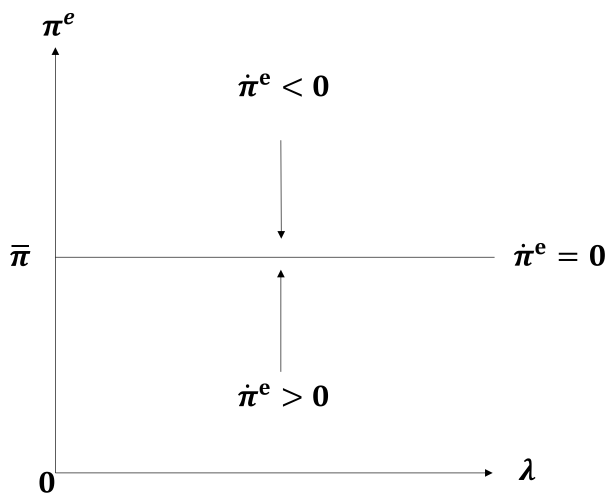

Figure1

Total differentiation of Eq.(16) is described below.

$$\left.\frac{d\pi^e}{d\lambda}\right|_{\dot{\lambda}=0} = -\frac{\frac{\partial F_2^1}{\partial \lambda}}{\frac{\partial F_2^1}{\partial \pi^e}}$$

We obtain Figure 2 by solving the equations below.

$$\frac{\partial F_2^1}{\partial \pi^e} = 2\left[-\frac{\varepsilon\theta a(1-\theta)+\theta(1-\theta)(\varepsilon a+1)}{\theta+\varepsilon^2(1-\theta)}\right] = F_{21} < 0$$

$$\frac{\partial F_2^1}{\partial \lambda} = \alpha + \rho = F_{22} > 0$$

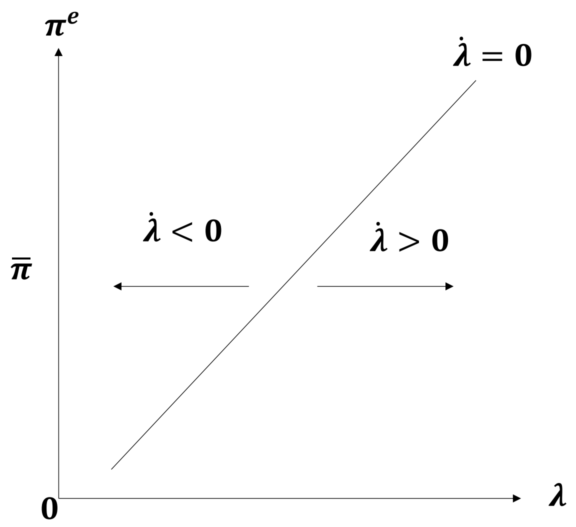

Figure 2

We now have the phase diagram of this analysis below in Figure 3, by combining Figure 1 and 2.

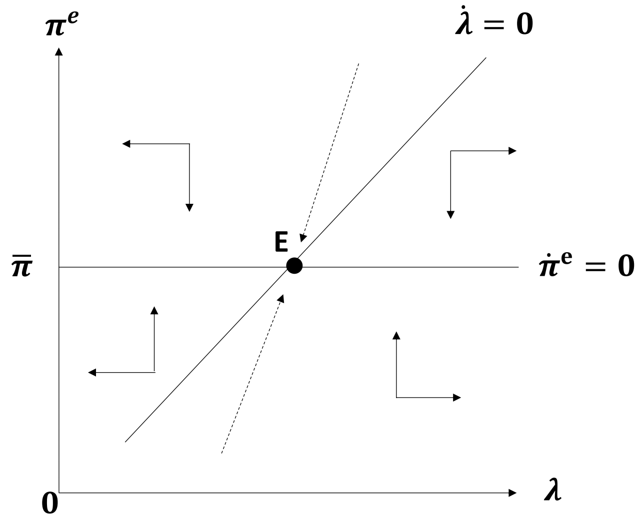

Figure 3

As depicted in Figure 3, an implication drawn under the rule $\xi = 0$ is that there is only one policy path (from both sides) which enables policy maker to achieve full employment and the inflation target goal, and this unique path toward the equilibrium is the saddle path. Only the saddle path satisfies the transversality condition in Eq.(10). From a given initial state variable $\pi^e(0)$ of inflation expectation, policy maker needs to select the initial point of λ and control it following the saddle path, by indirectly maneuvering the interest rate r. Or else, the inflation expectation and co-state variable λ would not end up at the equilibrium point of E.

If the starting point of inflation expectation is exceeding the inflation target, the policy maker would have to set the initial interest rate r at a sufficiently high point but lower it gradually afterwards in order to reach the equilibrium. On the other hand, if the initial point of inflation expectation is below the target, the starting point of r would be set sufficiently low, and the optimal path is to raise it upward until the target is achieved. This is quite a paradoxical result, since in general, output and inflation expectation have a positive correlation, which the increase in output would lead to a further rise in inflation expectation. The only way to achieve the policy target is to follow the saddle path, and if any other path is to be chosen, the output and inflation expectation would then spread out indefinitely.

By taking into account Eq.(13) and Eq.(16), the characteristic equation of this system at the equilibrium point is described below, where λ_1 and λ_2 are two characteristic roots[7].

$$\begin{aligned}|\lambda I - J_1| &= \lambda^2 - (traceJ)\lambda + (detJ) \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0\end{aligned} \tag{17}$$

Where,

$$J_1 = \begin{bmatrix} -\alpha & 0 \\ F_{21}^1 & F_{22}^1 \end{bmatrix}$$

$$trace\,J = \lambda_1 + \lambda_2 = -\alpha + F_{22}^1$$

$$detJ = \lambda_1\lambda_2 = -\alpha F_{22}^1 < 0$$

2.3 Case of ξ =1

Now, this time, a backward-looking type of inflation expectation, is considered. This is the case of ξ =1 in Eq.(3).

The equation of motion for $\dot{\pi}^e$ now becomes a function of the interest rate r.

[7] Evaluating the dynamical properties of interest rate and policy tool is also done by Drumond et al.(2022)

$$\dot{\pi}^e = \alpha\{\varepsilon(Y(r-\pi^e)-\bar{Y})\}\ ;\ \alpha>0 \tag{18}$$

Eq.(18) becomes a constraint function, and the dynamic optimization problem is formulated as below.

$$\max_{r\geq 0}\int_0^\infty W(Y(r-\pi^e),\pi^e)e^{-\rho t}\,dt$$
$$s.t.\ \dot{\pi}^e = \alpha\varepsilon(Y(r-\pi^e)-\bar{Y})$$

As similar to Eq.(9), a current value Hamiltonian is described below.

$$H = -[\theta((-a(r-\pi^e)+b)-\bar{Y})^2+(1-\theta)\{\varepsilon((-a(r-\pi^e)+b)-\bar{Y})+\pi^e-\bar{\pi}\}^2]+\lambda\alpha\varepsilon(Y(r-\pi^e)-\bar{Y}) \tag{19}$$

First order condition and second derivative are described below.

$$\frac{\partial H}{\partial r} = 2a[\theta((-a(r-\pi^e)+b)-\bar{Y})+\varepsilon^2(1-\theta)\{((-a(r-\pi^e)+b)-\bar{Y})+\pi^e-\bar{\pi}\}]-\lambda\alpha\varepsilon a = 0$$
$$\frac{\partial^2 H}{\partial r^2} = -2a^2\{\theta+\varepsilon^2(1-\theta)\}<0 \tag{20}$$

Equation of motion for the inflation expectation $\dot{\pi}^e$ is given below.

$$\dot{\pi}^e = \frac{\partial H}{\partial \lambda} = \alpha\varepsilon((-a(r-\pi^e)+b)-\bar{Y}) = F_1^2(\pi^e,r) \tag{21}$$

From the first order condition in Eq.(20), the equation of λ can be obtained.

$$\lambda = \frac{2}{\alpha}\left[\left((-a_{(r-\pi^e)}+b)-\bar{Y}\right)\left\{\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right\}+(1-\theta)(\pi^e-\bar{\pi})\right] \tag{22}$$

Differentiation of Eq.(22) with respect to time t gives Eq.(23).

$$\dot{\lambda} = \frac{2}{\alpha}\left[\{-a(\dot{r}-\dot{\pi}^e)\}\left\{\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right\}+\dot{\pi}^e(1-\theta)\right] \tag{23}$$

Additionally, from the one of the maximum principle conditions, the motion for λ is obtained below.

$$\dot{\lambda} = -\frac{\partial H}{\partial \pi^e} + \rho\lambda = 2\left[\left\{\left((-a(r-\pi^e)+b)-\overline{Y}\right)\right\}\{\theta a + \varepsilon(1-\theta)(\varepsilon a+1)\}\right.$$
$$\left. + (1-\theta)(\varepsilon a+1)(\pi^e - \overline{\pi})\right] - \lambda(\alpha\varepsilon a - \rho) \tag{24}$$

Transversality condition is termed as follows.

$$\lim_{t\to\infty} \lambda\, e^{-\rho t} = 0 \tag{25}$$

By combining Eq.(23) and (24) and bringing $\dot{r}$ to the left side, the equation of motion for the interest rate r can be described as below.

$$2\left[\left\{\left((-a(r-\pi^e)+b)-\overline{Y}\right)\right\}\{\theta a + \varepsilon(1-\theta)(\varepsilon a+1)\} + (1-\theta)(\varepsilon a+1)(\pi^e - \overline{\pi})\right] - \lambda(\alpha\varepsilon a - \rho)$$
$$= \frac{2}{\alpha}\left[\{-a(\dot{r} - \dot{\pi}^e)\}\left\{\frac{1}{\varepsilon}\theta + \varepsilon(1-\theta)\right\} + \dot{\pi}^e(1-\theta)\right]$$

$$2\left[\left\{\left((-a(r-\pi^e)+b)-\overline{Y}\right)\right\}\{\theta a + \varepsilon(1-\theta)(\varepsilon a+1)\} + (1-\theta)(\varepsilon a+1)(\pi^e - \overline{\pi})\right.$$
$$\left. - \frac{1}{\alpha}(1-\theta)F_1^2(\pi^e, r)\right] - \lambda(\alpha\varepsilon a - \rho) = -A(\dot{r} - \dot{\pi}^e)$$
$$A = \frac{2}{\alpha}\left(\frac{1}{\varepsilon}\theta + \varepsilon(1-\theta)\right)a$$

$$\dot{r} = -\frac{1}{A}\left[2\left[\left\{\left((-a(r-\pi^e)+b)-\overline{Y}\right)\right\}\{\theta a + \varepsilon(1-\theta)(\varepsilon a+1)\} + (1-\theta)(\varepsilon a+1)(\pi^e - \overline{\pi})\right.\right.$$
$$\left.\left. - \frac{1}{\alpha}(1-\theta)F_1^2(\pi^e, r)\right] - \lambda(\alpha\varepsilon a - \rho)\right] + \dot{\pi}^e = F_2^2(\pi^e, r) \tag{26}$$

2.4 Phase diagram in case of $\xi = 1$

The locus for $\dot{\pi}^e = 0$ is derived from Eq.(21) and shown in Figure 4.

$$\left.\frac{d\pi^e}{dr}\right|_{\dot{\pi}^e=0} = -\frac{\frac{\partial F_1^2}{\partial r}}{\frac{\partial F_1^2}{\partial \pi^e}} \tag{27}$$

Where numerator and denominator are solved as below.

$$\frac{\partial F_1^2}{\partial r} = -\alpha \varepsilon a < 0$$

$$\frac{\partial F_1^2}{\partial \pi^e} = \alpha \varepsilon a > 0$$

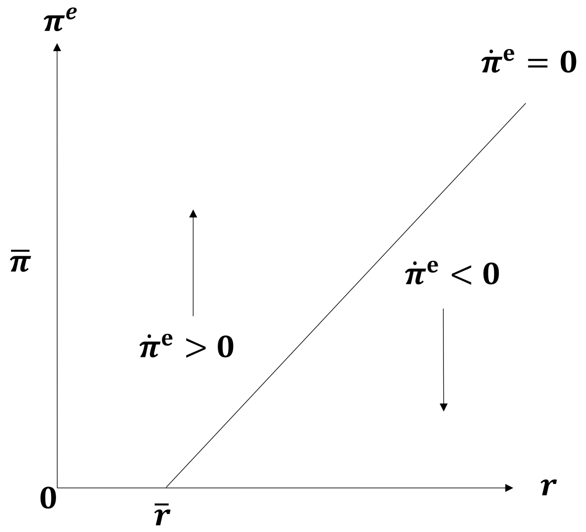

Figure 4

$$\dot{\pi}^e = \alpha \varepsilon((-a(r - \pi^e) + b) - \bar{Y}) = F_1^2(\pi^e, r)$$

$$\bar{r} = \frac{b - \bar{Y}}{a} > 0$$

Similarly, the $\dot{r} = 0$ line is depicted by solving the total derivation of Eq.(26), as shown below in Figure 5.

$$\left.\frac{d\pi^e}{dr}\right|_{\dot{r}=0} = -\frac{\frac{\partial F_2^2}{\partial r}}{\frac{\partial F_2^2}{\partial \pi^e}} \tag{28}$$

Each of the partial derivations are solved as below.

$$\frac{\partial F_2^2}{\partial r} = 0$$

$$\frac{\partial F_2^2}{\partial \pi^e} = F_{11}^2 = \frac{-2[a\{\theta a + \varepsilon^2 a(1-\theta)\} + (1-\theta)(\varepsilon a + 1)] - \varepsilon a^2 \left\{\frac{1}{\varepsilon}\theta + \varepsilon(1-\theta)\right\}}{A} < 0$$

$$A = \frac{2}{\alpha}\left\{\frac{1}{\varepsilon}\theta + (1-\theta)\right\}a$$

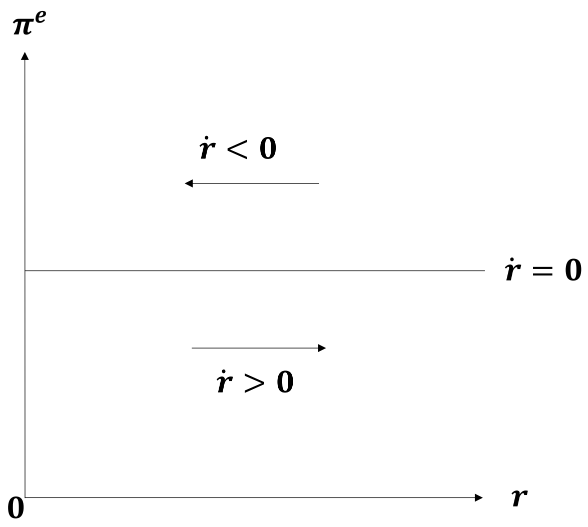

Figure 5

By combining both lines, the phase diagram is depicted below in Figure 6. It is now clear that the system with ξ =1 also has an unique path which converges into the equilibrium point E. Again, as like the case with ξ =0, the conclusion is quite paradoxical. If the starting point of inflation expectation is below the inflation target $\bar{\pi}$, the policy maker needs to set the initial interest rate r at a low point and raise it later on. On the other hand, if the inflation expectation is exceeding the target, the policy maker now needs to start from a high interest rate and lower it to suppress the inflation expectation.

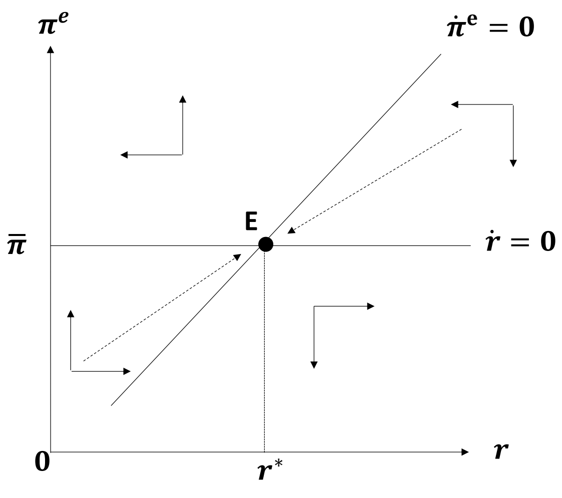

Figure 6

The evaluation of the dynamical properties of the model using the Jacobian matrix is described below, which shows that the equilibrium point becomes saddle point.

$$J_2 = \begin{bmatrix} F_{11}^2 & F_{12}^2 \\ F_{21}^2 & 0 \end{bmatrix}$$

$$trace\, J = \lambda_1 + \lambda_2 = F_{11}^2 > 0$$

$$detJ = \lambda_1 \lambda_2 < 0$$

2.5 Case of $\mathbf{0 < \xi < 1}$

Thirdly, the system with $0 < \xi < 1$ is tested to figure out its dynamic behavior. The loss function of policy maker is same as the previous cases. The dynamic optimization problem in this model is described as below.

$$\max_{r \geq 0} \int_0^\infty W(Y(r - \pi^e), \pi^e) e^{-\rho t}\, dt$$

$$s.t.\ \dot{\pi}^e = \alpha\{\xi\varepsilon(Y(r - \pi^e) - \bar{Y}) + (1 - \xi)(\bar{\pi} - \pi^e)\} \tag{29}$$

Thus, current-value Hamiltonian is formatted using the new constraint function.

$$H = -\left[\theta\big((-a(r - \pi^e) + b) - \bar{Y}\big)^2 + (1 - \theta)\big\{\varepsilon\big((-a(r - \pi^e) + b) - \bar{Y}\big) + \pi^e - \bar{\pi}\big\}^2\right]$$
$$+ \lambda\alpha\big\{\xi\varepsilon\big((-a(r - \pi^e) + b) - \bar{Y}\big) + (1 - \xi)(\bar{\pi} - \pi^e)\big\} \tag{30}$$

First order condition and second order derivative become as follows.

$$\frac{\partial H}{\partial r} = 2a[\theta((-a(r - \pi^e) + b) - \bar{Y}) + \varepsilon(1 - \theta)\{((-a(r - \pi^e) + b) - \bar{Y}) + \pi^e - \bar{\pi}\}] - \lambda\alpha\xi\varepsilon a = 0$$

$$\frac{\partial^2 H}{\partial r^2} = -2a\{-\theta a + \varepsilon^2 a(1 - \theta)\} < 0 \tag{31}$$

We can derive the following expression of λ by transforming the result from the first derivative equation above.

$$\lambda = \frac{2}{\alpha\xi}\left[\big((-a(r - \pi^e) + b) - \bar{Y}\big)\left\{\frac{1}{\varepsilon}\theta + \varepsilon(1 - \theta)\right\} + (1 - \theta)(\pi^e - \bar{\pi})\right] = \lambda(\pi^e, r) \tag{32}$$

Equation of motion for $\dot{\pi}^e$ is obtained below.

$$\dot{\pi}^e = \frac{\partial H}{\partial \lambda} = \alpha\left[\xi\varepsilon\left((-a(r-\pi^e)+b)-\overline{Y}\right)+(1-\xi)(\overline{\pi}-\pi^e)\right] = F_1^3(\pi^e, r) \tag{33}$$

Differentiation of Eq.(32) with respect to time gives an equations of $\dot{\lambda}$.

$$\dot{\lambda} = \frac{2}{\alpha\xi}\left[\left\{-a(\dot{a}-\dot{\pi}^e)\left(\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right)\right\}+\dot{\pi}^e(1-\theta)\right] \tag{34}$$

Another equation of $\dot{\lambda}$, which is part of the maximum principle, is described below.

$$\begin{aligned}\dot{\lambda} = -\frac{\partial H}{\partial \pi^e}+\rho\lambda = 2\big[&\theta a\left((-a(r-\pi^e)+b)-\overline{Y}\right)\\ &+(1-\theta)(\varepsilon a+1)\left\{\varepsilon\left((-a(r-\pi^e)+b)-\overline{Y}\right)+\pi^e-\overline{\pi}\right\}\big]\\ &-\lambda[\alpha\{\xi\varepsilon a-(1-\xi)\}-\rho]\end{aligned} \tag{35}$$

$$\begin{aligned}&2\left[\theta a\left((-a(r-\pi^e)+b)-\overline{Y}\right)+(1-\theta)(\varepsilon a+1)\left\{\varepsilon\left((-a(r-\pi^e)+b)-\overline{Y}\right)+\pi^e-\overline{\pi}\right\}\right]\\ &\quad-\lambda[\alpha\{\xi\varepsilon a-(1-\xi)\}-\rho]\\ &\quad=\frac{2}{\alpha\xi}\left\{-a(\dot{a}-\dot{\pi}^e)\left(\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right)\right\}+\frac{2}{\alpha\xi}(1-\theta)F_1^3(\pi^e,r) = -A(\dot{r}-\dot{\pi}^e)\end{aligned}$$

$$A = \frac{2a}{\alpha\xi}\left\{\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right\} \tag{36}$$

By combining Eq.(34) and (35), and bringing $\dot{r}$ to the left side, the equation of motion for interest rate r is available.

$$\begin{aligned}\dot{r} = -\frac{1}{A}\Big[2\Big[&\theta a\left((-a(r-\pi^e)+b)-\overline{Y}\right)+(1-\theta)(\varepsilon a+1)\left\{\varepsilon\left((-a(r-\pi^e)+b)-\overline{Y}\right)+\pi^e-\overline{\pi}\right\}\\ &-\frac{1}{\alpha\xi}(1-\theta)F_1^3(\pi^e,r)\Big]-\lambda\{\alpha\{\xi\varepsilon a-(1-\xi)\}-\rho\}\Big]+F_1^3(\pi^e,r) = F_2^3(\pi^e,r)\end{aligned} \tag{37}$$

2.6 Phase Diagram in case of $0 < \xi < 1$

Phase diagram of this system can be depicted from Eq.(33) and Eq.(37).
It is clear that the slope of $\dot{\pi}^e = 0$ line is positive, as shown in Figure 7.

$$\left.\frac{d\pi^e}{dr}\right|_{\dot{\pi}=0} = -\frac{\frac{\partial F_1^3}{\partial r}}{\frac{\partial F_1^3}{\partial \pi^e}} > 0 \tag{38}$$

Where,

$$\frac{dF_1^3}{d\pi^e} = F_{11} = \alpha\{\xi\varepsilon a - (1-\xi)\} > 0$$

$$\frac{\partial F_1^3}{\partial r} = F_{12} = -\alpha\xi\varepsilon a < 0$$

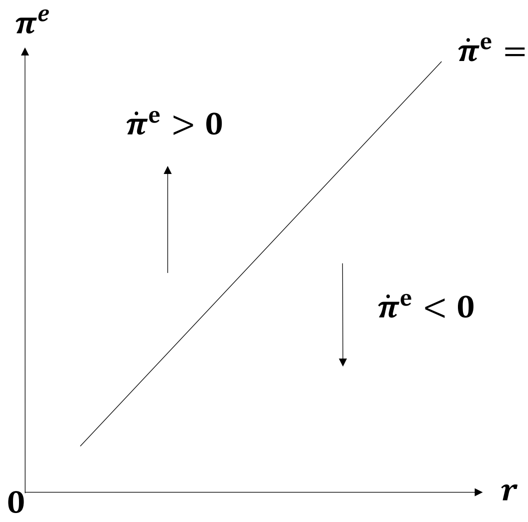

Figure 7

The slope of $\dot{r} = 0$ line is horizontal, as described below in Figure 8.

$$\frac{\partial F_2^3}{\partial \pi^e} = F_{21} = -2[a\{(\varepsilon a+1)(1-\theta)(\varepsilon(-1+a)+1)\}] - (1-\xi)\left\{2(1-\theta)\frac{1}{\xi} + \alpha A\right\}$$

$$a > 1$$

$$\frac{\partial F_2^3}{\partial r} = F_{22} = -\frac{2a[-\varepsilon(1-\theta)\{\varepsilon a - \varepsilon a\}]}{A} = 0$$

$$\left.\frac{d\pi^e}{dr}\right|_{\dot{r}=0} = -\frac{\frac{\partial F_2^3}{\partial r}}{\frac{\partial F_2^3}{\partial \pi^e}} = 0 \tag{39}$$

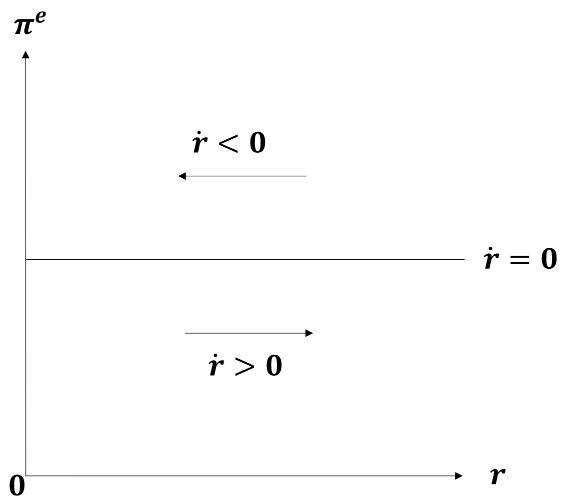

Figure 8

The combination of Figure 7 and 8 gives the phase diagram, as depicted in Figure 9.

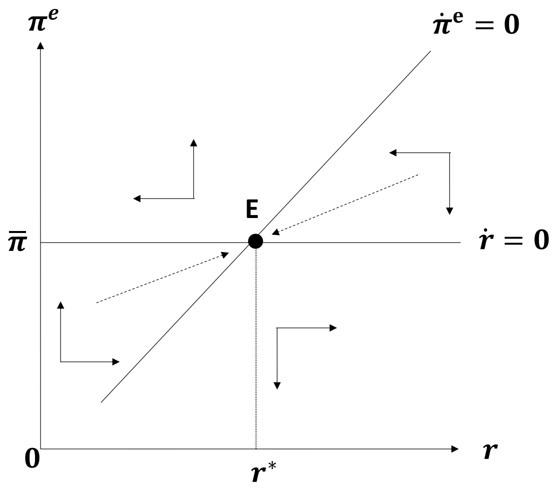

Figure 9

The system is stable as it converges to the equilibrium point, as proven in the Jacobian matrix below.

$$J_3 = \begin{bmatrix} F_{11}^3 & F_{12}^3 \\ F_{21}^3 & 0 \end{bmatrix}$$

$$trace\, J = \lambda_1 + \lambda_2 = F_{11}^3 > 0$$

$$detJ = \lambda_1 \lambda_2 < 0$$

As we have seen in the previous cases, the system with 0< ξ <1 also have the same policy implications. If the inflation expectation is above the target, the policy maker initially sets the interest rate high, but lower it gradually. If the inflation expectation is below the target, the opposite policy is required to reach the equilibrium.

3. Policy Implications

From the results of each analysis, we consider here the policy implications. First, our results tell us that if inflation expectation is above the target, the starting point of policy rate needs to be set sufficiently above the natural rate of interest, r^*. Afterwards, the policy rate should be lowered gradually until it reaches the inflation target. This is quite counterintuitive in which central banks usually deal with inflation by raising its interest rate. One way to interpret this paradox is the implication that it's more appropriate to set the initial interest rate at drastically high/low level, instead of a gradual rate hike/cut. Historically, major central banks including the Fed have responded to inflation (or potential inflation) by raising its policy rate consecutively in an orderly way, for instance, the increase of 25 basis point on each occasion. In some cases, the periods of policy tightening have lasted for few years. These are probably because the policy makers want to avoid volatile financial markets reacting to an abrupt policy change. For instance, the ongoing policy from early 2022 tightening have started from a mere 25bp, which then ratcheted up to 50bp and 75bp consecutively. The solution from our model is to set the initial rate to where the expected terminal rate is, and to lower it gradually, instead of hiking it.

If the inflation expectation is below the target, the solution from our model is to set the initial rate at a sufficiently low level, and then raise it over the optimal path until the inflation target is achieved, which also is against a standard economic theory. Unlike the former case, somewhat similar action had been taken by the Fed right after the Covid-19 in 2020. The Fed immediately lowered its policy rate from 1.75% (upper range of corridor) to 0.25% in March 2020. Disregarding the option to implement a negative interest rate policy, the Fed lowered the rate down to the minimal level instantly[8].

Regardless of people's forward/backward-looking stance on their inflation expectation, our findings imply that the optimal policy solutions for the policy makers are the same. Although Williams (2003) uses FRB/US macroeconomic model in its analysis as contrary to our analysis, the author concluded that the performance of efficient monetary policy differs under rational and backward-looking models. Carlstrom and Fuerst (2000) argues that aggressive and backward-looking stance are necessary for the monetary authority to ensure

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[8] The Fed also implemented a quantitative easing policy but we do not take into account such policy in this paper.

determinacy. Benhabib et al. (2003) tested the stabilization behavior of the backward-looking interest-rate rules, which they figured that parameters distinguish the stabilization, and not all backward-looking feed-back rules guarantee the uniqueness of equilibrium.

Our findings show policy paths which converge to the stable equilibrium point, but we need to consider here the cases when the policy actions were unsuccessful. Those cases are likely to occur when policy makers misplace the initial interest rate or mis-follow the saddle path. The output/inflation expectation paths would then diverge, letting the two variables increase/decrease indefinitely. One consequence is the upward divergence of real output and inflation expectations, what is called as an inflation spiral, and the other is the downward divergence, which is a deflation spiral. The former case is frequently observable especially among the emerging economies. Developed economies also faced somewhat similar phenomenon in the 1970s that are commonly attributed to oil embargo factor, but at the same time to a wage-price spiral. The Fed's response under then-chairman Paul Volcker was a severe tightening, which consequently led to a deep recession afterwards[9]. The opposite case, –deflationary spiral – is a relatively rare scenario. One of the few examples is Japan, particularly since the beginning of the 1990s. The sluggish economy of Japan has faced near zero or even negative year-on-year core inflation rate for nearly 20 years, which should be long enough to categorize Japan as the country which faced deflationary spiral. This phenomenon can also be described as a liquidity trap, or an absolute preference of liquidity, which was originally termed by Keynes (1936). The necessary action the Bank of Japan had to enforce was to drastically ease its policy at the onset of the crisis, but they have repeatedly failed to do so, resulting the interest rate to face lower bound while the inflation remains muted. At this juncture, the central bank's remaining solution is to stimulate the public's inflation expectation to lower the expected real rate of interest. Japan succeeded this task for a short period of time by implementing a massive quantitative easing policy from 2013, until the austere fiscal policy offset the rise of inflation expectation.

We explained that our analyses earn an optimal, unique path which converges to the equilibrium. This shows that those are the "optimal monetary policy paths", although the policy makers need to perfectly understand the exact starting point and interest rate path, otherwise the inflation expectation diverges indefinitely. A study by Drumond et al. (2022) provide us a meaningful contrast. Their model depicts a phase diagram which converges into an equilibrium, without a saddle path. The differential equations they have are described below.

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[9] For more specific history of 1970's inflation and policy actions of the Fed, see Bryan (2013).

① $\dot{r} = \gamma(\pi - \pi^T) + \beta(u - u^T)$

② $\dot{\pi}^e = k(\pi - \pi^e)$

Here, $\dot{r}$ is the time differentiated variable of real interest rate, π is actual inflation rate, $\dot{\pi}^e$ is the time differentiated variable of expected inflation rate, π^T is the inflation target, u is the rate of capacity utilization, and u^T is its target. k is a inflation memory parameter, and γ, β are positive parameters, representing the policy maker's focus on each gap from the target. After some algebraic manipulation and evaluating the dynamical properties, the phase diagram they obtained is shown below in Figure 10.

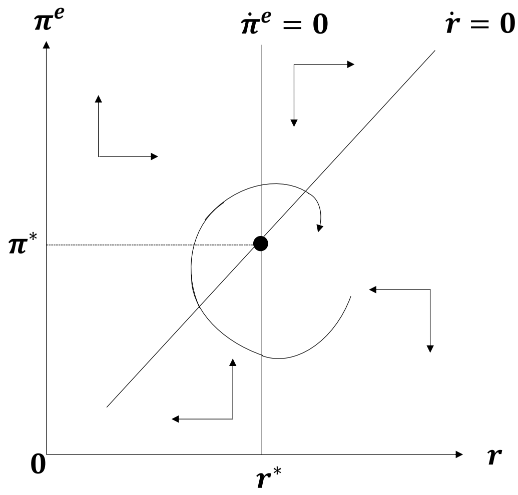

Figure 10

The analysis earns an equilibrium point with vertical $\dot{\pi}^e = 0$ line and positive slope $\dot{r} = 0$ line, which shows that the system is stable. Their result tells that there is no necessity for the policy makers to know the exact starting point and path of the interest rate. Instead of aiming for a unique, optimal policy path, this may be described as a more practical, "appropriate" policy rule.

Similar conclusion has been drawn by Asada (2010). The slope of the $\dot{\pi}^e = 0$ line differs from the previous paper, but its phase diagram shows that they have a globally stable equilibrium point, if the central bank is sufficiently credible.

Our findings provide a strict, optimal policy path, but it is rather a more theoretical solution, and may not be suitable in a real-life policy tool. However, we argue that our analysis is still meaningful since we have shown the difficulty of implementing an optimal monetary policy, and paradoxically proven that policies taken by the real-life central bankers are not ideal.

4. Conclusion

In this paper, by applying Hamiltonian equations, several optimal policy paths are derived under different scenarios of inflation expectation behavior. First was the forward-looking inflation expectation scenario, second was the backward-looking scenario, and the third was the mixed behavior of the previous two scenarios. The phase diagrams showed that each system obtains a unique converging path, namely a saddle path, which ultimately reaches a stable and unique equilibrium point.

The dynamic relationship between interest rate and inflation expectation derived here implies that if inflation expectation is below the inflation target, the optimal option for the policy maker is to select an initial point of nominal interest rate that is below the target to position itself at the appropriate trajectory (saddle path). Then the optimal policy is to raise the nominal interest rate and follow the saddle path until the inflation expectation converges to the equilibrium, or the inflation target. On the other hand, if the inflation expectation is above the target, the optimal policy is to place the initial nominal policy rate at a sufficiently high level, and afterwards lower it following the saddle path. In either case, if the policy outcome fails to follow the optimal path, inflation expectation and real output would dissipate and not end up reaching the equilibrium point. These implications are somewhat paradoxical, since an orthodox policy to stabilize the economy is to undertake counter-cyclical measures.

As a side note, another antitheoretical observations are pointed by Asada(2010) and Mankiw(2001), against the New Keynsian dynamic model which can be found in the literature by Galí (2015). They referred to their findings as a "sign-reversal" problem. "Sign-reversal" problem is a paradoxical properties of the "New Keynesian" dynamic model that shows a counterfactual relationship between inflation and real output. According to Asada(2010), the New Keynsian dynamic model implies that "the rate of inflation accelerates whenever the actual output level is below the natural output level, and it decelerates whenever the actual output level is above the natural output level". The author pointed out the same problem for the New Keynsian IS model as well.

There are some caveats on this paper's analysis. The model assumes that policy maker is omniscient and knows the best starting point and path to lead them to the equilibrium point. This assumption is referred as a "jump variable" by Asada (2010) and Asada(2013). We assume that they can control the nominal interest rate at any time, which allow them to follow the optimal path without deviation. Also, as a simplification, we defined nominal interest rate as if it's the only one that exists, and we didn't specify if it's a short term or long-term rate. These may be quite an unrealistic assumption, which similar argument has been made by Asada(2010). As mentioned previously, our findings provide an "optimal" policy path, which does not necessarily mean a realistic policy. There are some literatures which imply a more “appropriate”, or realistic policy approach. Our findings of optimal solution should be considered as a “reference” or “yardstick” to observe how the real-life monetary policy is deviating from the optimal path. As such, we may paradoxically say that the real-life policy settings by the central bankers are not optimal.

Finally, in the real-life policy setting, policy makers also have the capacity to control fiscal measures. As such, for the next step of the future study, although it may complicate the equations, we may form a more developed model with interest rate and government expenditure as the control variables.

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中央大学経済研究所
(INSTITUTE OF ECONOMIC RESEARCH, CHUO UNIVERSITY)
代表者　林　光洋　 (Director: Mitsuhiro Hayashi)
〒192-0393　東京都八王子市東中野　742-1
(742-1 Higashi-nakano, Hachioji, Tokyo 192-0393　JAPAN)
TEL: 042-674-3271　　+81 42 674 3271
FAX: 042-674-3278　　+81 42 674 3278
E-mail: keizaiken-grp@g.chuo-u.ac.jp
URL: https://www.chuo-u.ac.jp/research/institutes/economic/