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Competition for Territories under the Switching Cost

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Abstract

This article theoretically and empirically considers an effective policy to mitigate market powers held by retail liquefied petroleum gas distributors in Japan. A bilateral spatial competition model is constructed to analyze the effect of the switching cost of consumers on the retail spatial competition. The strategic space in the competition includes the declared free delivery area and the uniform price. Model analysis shows that, if stores commit to the regional designation for free delivery the equilibrium price increases up to the reservation price, and if stores do not commit, there exists an interior equilibrium. At both equilibria stores have their own exclusive delivery areas where rival stores do not intrude. The exclusive areas are natural territories of the stores given endogenously. If the competition is better described by a game where stores commit to their territories, promoting competition with the adjacent market would be effective to mitigate the market power. Whereas, if the competition is described by a game where stores do not commit, policies to lower the switching cost are effective. This paper shows an empirical method to identify which theory best explains the market. The method is applied to the liquefied petroleum gas retail delivery market in Japan.

keywords: Switching Cost, Undercut Proof Equilibrium, LP Gas, Retail territory, Auto–Regressive Analysis, Estimation of Coefficient of the pass–through

1 Introduction: Competition with Free Delivery Service Area

Free delivery service is common in some service industries, such as in ready-to-eat food delivery. Such services are especially important for competing with rivals in the retail industry, where the commodity sold is heavy, bulky, or cumbersome to carry. As delivery costs increase with the distance from the store, retailers must decide on the area within which they guarantee to provide free delivery services. When the delivery area is limited to a neighborhood, charging delivery fees by distance is difficult. For example, a pizza delivery service does not offer discounts for neighbors in an adjacent building. Retailers compete not only on uniform pricing, but also on the area of free delivery. In this article, the outcome of this kind of extended competition is analyzed theoretically and empirically. We show that various kinds of equilibria are possible theoretically and can be identified empirically by estimating the coefficient of the pass-through in the markets.

Consider a simple model of competition between two food delivery stores located at both ends of a standard Hotelling-type linear market. Both stores choose a strategy that is a combination of a free delivery area and a uniform price within the declared delivery area. Shipping costs are not charged to the consumers other than the uniform price. Hereafter, such delivery areas are denoted as DA. DAs are subsets of the linear market. The stores neither accept orders from consumers outside of their DA, nor refuse orders if from customers are within the DA. Consumers have a common reservation price for a unit of the service.

One possible construction of the game is a two-stage game, where the stores choose their DAs in the first stage and uniform prices in the second stage. In the second stage, however, there is no equilibrium, as long as either of the stores has its exclusive supply area that is a part of the DA of the store and not included in the DA of the rival. This is because the store that holds its exclusive supply area can set the reservation price as its uniform price. Choosing a price as high as the reservation price may risk the store losing demand from the customers in the area overlapped by DA of the other store, although the store secures a positive profit from the customers in the exclusive supply area. With this option, the stores need not lower the price to their marginal cost to compete with their respective rival. The consumers in the exclusive supply area are a kind of captive consumer.[1] Thus, in the

[1] Varian (1980) shows this non-existence of pure-strategy equilibria with captive consumers. Oertel and Schmutzler (2021) also show non-existence under consumers' hetero-

second stage of a two-stage game, there is no Nash equilibrium, as long as the stores have their exclusive supply areas. The only exception is the case wherein they have no exclusive supply area, that is, both stores choose DAs to cover the entire area, and they earn zero profits, the typical outcome of Bertrand price competition.

An alternative construction is to consider the undercut proof equilibrium (UPE), assuming switching costs for consumers (Shy (1996), Shy (2002), Shy and Oz (2001), Morgan and Shy (2015)) in the second stage in the two-stage game.[2] Although the option to choose the reservation price reduces the undercut proof sets in the strategy space, the equilibrium exists unless the set is empty. Given the rival's DA, the store faces severer competition by extending its own DA. Undercutting the rival becomes more effective strategy when the store extends the DA because the share of the profit earned from the customers competing with the rival increases, whereas the demand from the customers in the exclusive supply area cannot be undercut from the rival. Therefore, at the first stage, the stores prefer the smaller DA to avoid competition. At the equilibrium, the stores choose the size of the DA such that the price at the UPE increases to the reservation price. There is no need to retreat further, because the price reaches its upper limit. Thus, the price is determined at the level of the reservation price and the stores have their exclusive supply areas and the area where they compete with their rivals at the sub-game perfect equilibrium. The exclusive supply areas are a kind of territories for the stores, although they are not assigned by upstream firms as vertical constraints.[3] The exclusive territory endogenously determined by competition among firms might convey the meaning of "territory" in the biological world more faithfully. Construction of such exclusive territories is closely related to switching costs.

In the other construction of the game, the UPE is extended to allow the stores to choose both the DA and the price simultaneously. In this configuration, undercutting the rival is a severer strategy because the stores reach even the customers who are in the exclusive supply areas of their rivals

geneity.

[2]Byford (2015) showed that the UPE is the core of a cooperative game, although the game analyzed here does not satisfy the condition required.

[3]Since Smithies (1944), "territories" has been sometimes used to describe the set of locations of consumers who prefer to buy from a store in spatial competition models. Here, the exclusive supply areas are those the rival stores would not intrude over even when the consumers want to buy from the rival stores. In more recent literature, "territory" refers to a type of vertical constraint that is used by an upstream firm to mitigate competition among downstream firms. See Mathewson and Winter (1984), Rey and Tirole (1986), and Rey and Stiglitz (1995) among other notable papers on this subject.

when the stores decide to undercut. Compared with this configuration, in the second stage under the two-stage game, the stores are assumed to commit not to reach the customers who are in the exclusive supply areas of their rivals. At the equilibrium, the price is determined to be the sum of the cost and a margin that is determined by the size of the switching cost, say the switching cost premium, and there exist exclusive supply areas and an intersection of the DAs of the two stores.

Another possible equilibrium is the outcome of the collusion between the stores. The stores would not leave any intersection of the DAs to avoid competition and enjoy the highest price of the reservation price in their exclusive territories as monopolists.

Thus, there are four possible outcomes of the competition for the DA and the price. There may exist exclusive supply areas for stores where consumers have no other option but to order from the designated stores. There may also exist regions where consumers can order services from either of the nearby stores. All the areas are divided as the former exclusive supply areas, or all the areas are the latter and consumers can choose stores, or stores have the former exclusive areas that are surrounded by the latter areas. The price may be set as high as customers' reservation price for the service, or stores sell the services with profit margins that are determined by the switching costs of customers, or competition brings prices down to a marginal cost level.

It is important to determine which model is the most appropriate to describe an actual competitive situation in a market because each model has different implications for pro-competition policy to the markets. Consider a market where the exercise of market power based on consumers' switching cost is suspected. If the competition in the markets is better described by the two-stage game, it does not matter how much it costs consumers to switch suppliers, because the price eventually goes up to the reservation price at the equilibrium. Even a small amount of switching cost can cause this outcome. Promoting competition with the adjacent market to lower the reservation price, or invalidating the declaration of the DA, can effectively suppress the market price in this case.

However, if the competition is better described by the simultaneous determination of the DAs and prices, policies to lower the switching cost become effective. Then, removing the obstacles for consumers to switch the suppliers is recommended. If no territory is observed, as in food-delivery services, then no government intervention in the market is necessary, because they would have to compete with severe price competition.

The identification of the type of equilibrium is possible by estimating the coefficient of the pass-through empirically. Under the two-stage game, zero coefficients of the pass-through are expected because the retail price is determined solely by the reservation price: fluctuations of wholesale price are not reflected in retail prices in this case. Under the one-stage game, the coefficient is expected to take the value of 1 because the price is determined as the sum of the wholesale price and the switching cost premium. Fluctuations of wholesale price are fully passed through to retail prices in the long run.

In this article, this identification proves valid for the Japanese liquefied petroleum (LP) gas retail market. About half of the households in Japan rely on nearby stations' LP gas delivery and maintenance contracts. Small-scale retailers provide the supply of LP gas to domestic customers. The distribution costs include transportation, replacement of cylinders, and metering. It is an important decision for retail suppliers to determine the region to promote demand from consumers and guarantee gas distribution service.

The problem in the retail LP gas distribution markets in Japan is the exorbitant service charge. Kojima (2012) reports that, among 52 countries, the price of retail LPG in January 2012 was the highest in Japan at USD 4.56 per kg, followed by USD 2.94 per kg in Turkey, and a mean of USD 1.24 per kg. The high price is not due to high fuel price. The cost, insurance, and freight (CIF) price in Japan was USD 0.86 per kg for propane and USD 0.92 per kg for butane in January 2012, while the free on board (FOB) price in the United States was USD 0.67 per kg and USD 0.91 per kg for butane. The fuel prices in Japan are not so expensive compared to those in the United States. The share of fuel cost on retail price was only 18.7 percent in 2010 and the share of the retail gross margin was 63 percent.[4] High variance of retail prices within the domestic markets has been remarked upon in government reports.[5] Thus, market power is suspected to exist in retail LP gas markets in Japan.

Switching costs are known to prevent competition in the market. Generally, there is no technical barrier in switching retail LPG suppliers. However, in Japan's market, institutional obstacles deter switching of suppliers. First,

[4] "Liquefied Petroleum Gas Guide 2017," LP Gas Center, `http://www.lpgc.or.jp/corporate/information/guide.pdf`

[5] See Ministry of Economy, Trade and Industry (2013). By the author's calculation on the data from "The Oil Information Center, The Institute of Energy Economics," while the standard error and the mean of the average prices in 47 prefectures is JPY 1,090 and JPY 13,099 per 20 squares meters use of LP gas, the mean of the ratios of the difference between the maximum and the minimum to the mean is 0.498 for 47 prefectures. The maximum of the ratios is 1.02.

survey reports show that a considerable share of consumers stated they incorrectly assumed LP gas prices were regulated to be uniform. LP gas supply is apt to be confused with regulated pipeline gas supply by utility companies. In a 2014 survey, 62.8 percent were shown to be misled,[6] while this proportion was 31 percent in a 2008 survey.[7] Second, 61.7 percent of retail LP gas suppliers provide free interior piping works when consumers build their new houses, and more than 80 percent of such suppliers think they have right to the ownership of the interior piping, while only 17.9 percent of such consumers agree with the right, which is a hindrance when consumers try to switch their supplier.[8]

The other institutional problem in retail LP gas distribution is the territories held by the retail distributing stores. The existence of territories in this sector has been neglected in official documents of the Japan Fair Trade Commission.[9] This is because territories there arise naturally as retailers' turf and are not imposed from upstream firms as vertical constraints. Unless imposed from the outside, natural territories would simply represent insufficient competition from an anti-competitive perspective. Except for safety regulations avoiding risks in gas use, the LP gas retail market is by its very nature a free competitive market. However, there are many documents that reveal the non-competitive nature of the industry. The documents are rife with such expressions as "We don't touch other suppliers' customers," "A written pledge not to infringe on each other's customers," "Decisions regarding restrictions on the movement of clients of (regional trade association) members."[10] The fact that many consumers misunderstand LP gas rates as regulated utility rates and that a considerable share of consumers are unaware of their right to change suppliers points to a lack of outreach

[6] "Survey Report on Consumers Behavior of Petroleum Gas," Nippon Consultant Group Inc., https://www.kanagawalpg.or.jp/images/201509_shouhishajittai.pdf

[7] "Consumer Consciousness Survey on LP Gas," National Federation of Regional Women's Organization, http : / / www . chifuren . gr . jp / kikanshi / news-bk / 380/newsback-380_5.html

[8] "Survey Report on Trade Practice of Retail LP gas in 1999," Japan Fair Trade Commission, http://www.jftc.go.jp/info/nenpou/h10/02070005.html

[9] Regarding restrictions on customer movement, the Annual report of Japan Fair Trade Commission, 1998, admits widespread practice of free piping, stating that "If customer movement is restricted due to the practice of free piping, it may cause problems under the Antimonopoly Law." https://www.jftc.go.jp/info/nenpou/h10/02070005.html

[10] See, for example, Kido (2002). In 2018, The Fair Trade Commission issued a cease and desist order against a regional LP gas trade association in Kanagawa Prefecture that attempted to prevent new entrants from entering the market in an attempt to gain customers from incumbent businesses. https://www.jftc.go.jp/houdou/pressrelease/h30/mar/180309_1.html

to customers already served by other retailors.[11]

Using gas price panel data, I try to empirically clarify the circumstances underpinning the high price problem in the Japanese retail LP gas market by estimating the coefficient of the pass-through in each regional market. The recorded prices are the means of prices of samples from each regional market. The coefficients of the pass-through are estimated as an accumulated impulse response of the retail price level to wholesale price shock.

In the first half of the data set, that is before April 2006, I found the estimated coefficients of the pass-through are distributed around 1.0, although the standard deviation is not as little. This implies the market equilibrium is described better as that in the one-stage simultaneous decision game. In the second half of the data, that is after April 2006, the estimated coefficients are distributed around 0.6, which implies each regional market is a mixture of two types of competition—the one-stage and the two-stage games. Therefore, a mixture of heterogenous results, positive but less than 1 coefficient, is observed. The recorded price is the mean of samples from each regional market, which is expected to increase with the share of the results from the two-stage game because the equilibrium price is higher in the markets under the two-stage game, such that a positive correlation is expected between the mean prices and the coefficients. This is positively examined in the second half of the data. This change in the distribution of the coefficient of the pass-through indicates a change of competition in the market and a change of effective policy to deal with the market power in the market. The policy has been in place for more than 20 years in the form of the provision of guidelines on proper trade. The fact that these policies are still needed[12] may indicate that they have not been effective in lowering switching costs or reducing reservation prices.

The remaining article is structured as follows. A bilateral spatial competition model is constructed in section 2. The model has variations: one-stage/two-stage games and with/without switching cost. After examining the absence of Nash equilibrium with territory formation, the undercutting proof property (UPP) is introduced following Shy (1996), Shy (2002), Shy and Oz (2001) and is extended to cover both the price and the size of the dis-

[11] A survey report in 2014 shows 46.4 percent of consumers did not notice that they have a choice of LP gas provider.`https://www.kanagawalpg.or.jp/images/201509_shouhishajittai.pdf`

[12] For example, see Kaisei-Ekisekihou-Syoureitou-Torihiki-Tekiseika-Guideline-Setsumeikai-Siryo (Briefing Materials: Guidelines for Proper Trading based on the Revised Liquified Petroleum Gas Act) `https://www.enecho.meti.go.jp/category/resources_and_fuel/distribution/notice/170130/handout/pdf/handout_001.pdf`

tribution area as a strategy. Then, the problems to be solved are specified. Examining the possible solutions to the problems lead us to the conclusion that each model implies different outcomes of the competition that can be identified by estimating the coefficient of the pass-through through empirical analyses. In section 3, for each regional retail LP gas market in Japan, applying auto-regressive process in first differences, the coefficients of the pass-through are estimated as accumulated impulse responses. Finally, the distribution of the coefficient of the pass-through and the association between the mean prices of regional markets and the coefficient are examined. The last section concludes the study.

2 The Model Analysis

One of the features of the model constructed here is the expansion of the strategic space of retailers in their spatial competition. The extended strategic space consists of the price and the size of the DA. Another feature of the model is its incorporation of the switching cost in competition. Basically, the analysis follows Shy's model (Shy (1996), Shy (2002), Shy and Oz (2001)), which proposes the UPP. We extend the UPP concept to accommodate the extended strategic space.

2.1 The Retail Market

The market analyzed here is a linear market following Hotelling (1929). The location on the linear market is expressed as a real number on $U \stackrel{\text{def}}{=} [0, 1]$. Consumers distribute on U uniformly with a density measuring 1. Each consumer takes one unit of service in the period if the charge for the service is not more than the uniform reservation price, v ($v \in \Re^{++}$), where $\Re^{++} \stackrel{\text{def}}{=} \{x \in \Re | x > 0\}$.

Two stores supply the service to consumers. Their location is fixed at the two ends of the market, 0 and 1. The store located at i is denoted as store i. Hereafter, the expression i means "i where $i \in \{0, 1\}$" for simplicity. They provide the service to each consumer on request. Customers are charged the uniform price irrespective of their distance from the store, and no other surcharge is collected for the delivery. Thus, price discrimination by the location is prohibited. No cost is borne by the stores other than delivery costs. The stores incur costs for each unit of delivery proportional to the distance required for the delivery. The unit delivery cost is denoted as t ($t \in \Re^{++}$). Thus, store 0 expends tx to deliver a unit of service to the customer located at x ($x \in U$), while store 1 expends $t(1-x)$ for delivery to the same customer. Among parameters, a constraint is assumed in order to restrict the solution.

Assumption 1.

$$v > t.$$

By this assumption, only that situation is considered wherein providing service for the consumer located at the rival store is profitable if the store sets the reservation price as the uniform price.

The stores declare their uniform service prices and the DA. The timing of the declaration determines two versions of the game. The timing of the game is explained in the latter part of the article. The prices declared are denoted as p_i, respectively. At these prices, the stores cannot refuse requests from customers in their DAs. Requests from customers outside their DAs will be automatically refused. In these circumstances, the DAs are naturally assumed to be one continuous zone including the location of the store, $A_0 = [0, r_0]$ and $A_1 = [1-r_1, 1]$, where A_i denotes the DA of store i.[13] The variables r_i are the sizes of the DAs.

No delivery service is available if a consumer is located outside both DAs ($A_0^c \cap A_1^c$). If a consumer is located in $A_i \cap A_j^c$, the consumer has no choice but to buy from store i; this region is called the exclusive DA of store i. Hereafter, if i and j appear in a sentence simultaneously, it is automatically implied that "$i \in \{0,1\}, j \in \{0,1\}$, and $i \neq j$" for simplicity. When a location is covered by both DAs ($A_0 \cap A_1$), the consumer has to decide which store to order from. In this decision, a transcendental preference for stores is assumed. If a consumer has the transcendental preference for store i, he(she) chooses

store i, if $p_i \leq p_j + \delta$,

store j, otherwise.

Consumers bear the switching cost δ when they choose against their own transcendental preference. They are assumed to have the uniform switching cost δ ($\delta \in \Re^+$), where $\Re^+ \stackrel{\text{def}}{=} \{x \in \Re | x \geq 0\}$, and the switching cost may possibly be zero. At location x, the ratio of the consumers with a preference for store 0 is denoted as $q(x)$, a function of location x. All the consumers are assumed to have their own transcendental preference for store 0 or for

[13] Although it is easy to start with a more basic assumption, for the purpose of saving space, we assume the convexity of the DAs.

store 1. As the density of consumers is already assumed to be 1, the density of consumers with a preference for store 0 is $q(x)$ at location x, and that for store 1 is $1 - q(x)$.

To exclude trivial equilibriums, it is assumed that at any location of the consumer, there is a positive possibility that each store is chosen, and the preference decreases as the distance of consumers from the store increases. [14]

Assumption 2.
$q : U \to (0, 1),\ q(x)$ *is differentiable and* $q'(x) \leq 0,$ *for* $x \in U$.

Hereafter, all lemmata and propositions below will be proved on the assumptions 1 and 2.

For simplicity of expression in the analysis, some denotations are provided.

$$1 > D_0(B) \stackrel{\text{def}}{=} \int_B q(x)dx > 0,\ \frac{1}{2} > D_1(B) \stackrel{\text{def}}{=} \int_B q(x)xdx > 0,\ \text{for } B \subseteq U.$$

The strategy s is defined as a vector with two elements $s \stackrel{\text{def}}{=} (p, r)$, where p and r are the price set by the store and the declared size of the DA. The strategy space is also defined as $S \stackrel{\text{def}}{=} \{(p, r)|\ p \in [0, v],\ r \in U\}$.

Under the demand structure constructed here, given strategies (s_i, s_j) chosen by firm i and j, respectively, the profits of firm i are

$$\pi_i(s_i, s_j) = E_i(p_i) + \begin{cases} J_i(p_i) & \text{if } p_i < p_j - \delta, & (1a) \\ Q_i(p_i) & \text{if } p_j - \delta \leq p_i \leq p_j + \delta, & (1b) \\ 0 & \text{if } p_j + \delta < p_i, & (1c) \end{cases}$$

where

$$E_i(p) \stackrel{\text{def}}{=} \int_{A_i \cap A_j^c} (p - tx_i^*)\, dx, \qquad J_i(p) \stackrel{\text{def}}{=} \int_{A_i \cap A_j} (p - tx_i^*)dx,$$
$$Q_i(p) \stackrel{\text{def}}{=} \int_{A_i \cap A_j} q_i^*(x)(p - tx_i^*)dx,$$

[14]A transcendental preference can be considered a result of the consumers' last purchase record, although the model analyzes only one period here. If such a scenario is considered, the switching cost is borne when a consumer attempts to purchase from the other store, rather than the one they purchased from during the last period.

for $s_i \in S$, and

$$x_i^* \stackrel{\text{def}}{=} \begin{cases} x & \text{for } i = 0, \\ 1 - x & \text{for } i = 1, \end{cases} \quad q_i^*(x) \stackrel{\text{def}}{=} \begin{cases} q(x) & \text{for } i = 0, \\ 1 - q(x) & \text{for } i = 1. \end{cases}$$

These functions defined here are better denoted exactly as $E_0(p; \tilde{r}_0, \tilde{r}_1)$ or $E_0(p; A_0, A_1)$, for example. Simple expressions are chosen here.

Note that

$$E_i'(p_i) > 0, \text{ if } A_i \cap A_j^c \neq \emptyset$$
$$J_i'(p_i) > 0, \ Q_i'(p_i) > 0 \text{ if } A_0 \cap A_1 \neq \emptyset,$$

where dashes denote differentiation with the variable in the argument.

A lemma is provided to facilitate proofs hereafter.

Lemma 1. *For $(\tilde{r}_0, \tilde{r}_1) \in U^2$, when $r_0 + r_1 - 1 > 0$,*

(A) $Q_i\left(\frac{t}{2}\right) \begin{cases} \geq 0 & \textit{when } \ r_j = r_i, \\ > 0 & \textit{when } \ r_j > r_i, \end{cases}$

(B) $\frac{1}{2} D_0\left(A_0 \cap A_1\right) - D_1\left(A_0 \cap A_1\right) \begin{cases} \geq 0 & \textit{when } \ r_1 = r_0, \\ > 0 & \textit{when } \ r_1 > r_0, \end{cases}$

(C) $J_i(v) > Q_i(v), \quad J_i(0) < Q_i(0),$

(D) $0 < D_0(A_0 \cap A_1) < \tilde{r}_0 + \tilde{r}_1 - 1.$

2.2 Two-stage Game

In this subsection, a two-stage game is analyzed; in the first stage, both stores declare their DAs simultaneously; then, in the second stage, stores determine their uniform price simultaneously. When there is no switching cost, the subgame Nash equilibrium in the second stage exists only if in the first stage stores choose DAs that cover most of the area, although they have incentives to shrink their DAs in the first stage. There is no subgame perfect equilibrium here.

Further, when positive switching cost is assumed, the subgame perfect Nash equilibrium exists in limited cases, where the switching cost is sufficiently large. At the equilibrium, both DAs cover all the area, and the prices are increased up to the ceiling reservation price. This means no territories are established in this model, although the model aims to investigate equilibria where territories are established, as is observed in the LP gas distribution market in Japan.

2.2.1 Definitions of Critical Price Levels

Four definitions of critical price levels are given to facilitate the later proofs.

Definition 1. *Given* $(\tilde{r}_0, \tilde{r}_1) \in U^2$, p_i^J, p_i^Q, p_i^{JQ}, *and* p_i^{VQ} *are defined as*

(A) $E_i(p_i^J) + J_i(p_i^J) = E_i(v)$,

(B) $E_i(p_i^Q) + J_i(p_i^Q) = E_i(v) + Q_i(v)$,

(C) $E_i(p_i^{JQ}) + J_i(p_i^{JQ}) = E_i(p_i^{JQ}) + Q_i(p_i^{JQ})$,

(D) $E_i(p_i^{VQ}) + Q_i(p_i^{VQ}) = E_i(v)$.

For p_i^ks, $k \in \{J, Q, JQ, VQ\}$, Lemma 2 is proved.

Lemma 2. *For* $(\tilde{r}_0, \tilde{r}_1) \in U^2$ *such that* $\tilde{r}_0 + \tilde{r}_1 > 1$,

(A) $\exists\, p_i^k,\ p_i^k \in (0, v),\ k \in \{J, Q, VQ\}$,

(B) $\exists\, p_i^{JQ},\ p_i^{JQ} \in \left[\dfrac{t(1 + \tilde{r}_i - \tilde{r}_j)}{2}, t\tilde{r}_i\right)$,

(C) $E_i(v) \gtreqless E_i(p) + J_i(p)$ *when* $p_i^J \gtreqless p,\ p \in [0, v]$,

(D) $E_i(v) + Q_i(v) \gtreqless E_i(p) + J_i(p)$ *when* $p_i^Q \gtreqless p,\ p \in [0, v]$,

(E) $E_i(p) + Q_i(p) \gtreqless E_i(p) + J_i(p)$ *when* $p_i^{JQ} \gtreqless p,\ p \in [0, v]$,

(F) $E_i(v) \gtreqless E_i(p) + Q_i(p)$ *when* $p_i^{VQ} \gtreqless p,\ p \in [0, v]$,

(G) $p_i^J \geq p_j^J$, *if* $\tilde{r}_i \geq \tilde{r}_j$,

(H) $p_i^Q \geq p_i^J$,

(I) *If* $p_i^J \gtreqless p_i^{JQ}$, *then* $p_i^J \lesseqgtr p_i^{VQ}$.

2.2.2 Two-stage Game without Switching Cost

Consider a two-stage game without switching cost, Game 1.

Game 1 (two-stage game with no switching cost)**.**

- $\delta = 0$.
- *The first stage: both stores declare their DAs simultaneously.*

- *The second stage: stores choose their uniform price simultaneously.*

Except when both DAs were declared in the first stage to cover most of the market area, or when they are separated, Nash equilibrium does not exist in the second stage.

Lemma 3. *Assume $(\tilde{r}_0, \tilde{r}_1)$ such that $\tilde{r}_0 + \tilde{r}_1 > 1$ is selected in the first stage of Game 1. In the second stage of the game, denote the combination of strategies $\langle \tilde{s}_0, \tilde{s}_1 \rangle = \langle (\tilde{p}_0, \tilde{r}_0),\ (\tilde{p}_1, \tilde{r}_1) \rangle$ as that satisfying*

$$\pi_i(\tilde{s}_i, \tilde{s}_j) = \max_{p_i \in [0,v]} (\pi_i((p_i, \tilde{r}_i), \tilde{s}_j)) \geq 0. \tag{2}$$

The necessary conditions for such combinations of strategies $\langle \tilde{s}_0, \tilde{s}_1 \rangle$ exist include $\tilde{r}_i > 1 - t/(4v)$. When $\tilde{r}_0 = \tilde{r}_1 = 1$, $\langle \tilde{s}_0, \tilde{s}_1 \rangle$ exist. Further, $\tilde{p}_i < t\tilde{r}_i$.

When v is almost as low as value t, the necessary condition in Lemma 3 requires $\tilde{r}_i > 3/4$. As v increases, the sizes of the DAs are required to be near to 1: stores have to cover the whole area. Only when both stores cover most of the market area, there exist equilibria in the second stage. When $(\tilde{r}_0, \tilde{r}_1)$ is chosen in the first stage such that $r_0 + r_1 \leq 1$, both DAs are separated from each other. It is easy to check that the Nash equilibrium is achieved when (p_0, p_1) is (v, v), which is a trivial outcome wherein the optimization simply requires the maximum price in isolated markets. Obviously, there is no subgame perfect Nash equilibrium, because for every combination of $(\acute{r}_0, \acute{r}_1) \in U^2$, the set

$$R_1 \cap (\{(\acute{r}_0, r_1) | r_1 \in U\} \cup \{(r_0, \acute{r}_1) | r_0 \in U\})$$

is not empty, where,

Definition 2.

$$R_i \stackrel{\text{def}}{=} \left\{ (r_0, r_1) \middle| \begin{array}{c} (r_0, r_1) \in U^2, \textit{there is no Nash equilibrium} \\ \textit{for } (r_0, r_1) \textit{ in the second stage of Game } i \end{array} \right\}.$$

Furthermore, even when an equilibrium exists in the second stage of Game 1, stores can increase their profit by shrinking their DAs because $\tilde{p}_i < t\tilde{r}_i$, as shown in the lemma. They always have an incentive to leave from their front line because it is not profitable to supply customers in the line at the equilibrium in the second stage, which implies that the possible equilibrium in the second stage is much vulnerable.

Thus, we have the following proposition:

Proposition 1. *For every combination of the DAs in the first stage of Game 1, there always exists a deviation of the choice of the DA by either of the store under which there exists no Nash equilibrium in the second-stage subgames.*

In the proposition, there is no pure-strategy subgame perfect Nash equilibrium for Game 1.

2.2.3 Two-stage Game with Positive Switching Cost

When $\delta > 0$, in limited cases, there exists Nash equilibrium in the second-stage subgame:

Game 2 (two-stage game with positive switching cost)**.**

- $\delta > 0$.
- *The first stage: both stores declare their DAs simultaneously.*
- *The second stage: both stores choose their uniform price simultaneously.*

Specifically, Lemma 4 is proved.

Lemma 4. *Given* $(\tilde{r}_0, \tilde{r}_1) \in U^2$ *such that* $\tilde{r}_0 + \tilde{r}_1 > 1$*, in the second stage of Game* 2*, the combination of strategies* $\langle \tilde{s}_0, \tilde{s}_1 \rangle = \langle (v, \tilde{r}_0), (v, \tilde{r}_1) \rangle$ *satisfies* (2) *only when* $\min(p_0^Q, p_1^Q) \geq v - \delta$.

Because $p_i^Q > 0$ from Lemma 2(A), given $(\tilde{r}_0, \tilde{r}_1)$ when the switching cost is sufficiently large, there exists Nash equilibrium in the second stage of the game. If such equilibrium exists anyhow, in the first stage of the game, stores expect an increase in their profit by expanding their DAs, as shown in the next lemma.

Lemma 5. *Given* $(\tilde{r}_0, \tilde{r}_1) \in U^2$ *such that* $\tilde{r}_0 + \tilde{r}_1 > 1$*, in the second stage of Game* 2*, subgame Nash equilibrium exists in the neighborhood of the point only when* δ *is sufficiently large. At this point,*

$$\frac{\partial}{\partial \tilde{r}_i} \pi_i((v, \tilde{r}_0), (v, \tilde{r}_1)) > 0.$$

Note that Lemma 5 is valid even when $\tilde{r}_0 = 1$ or $\tilde{r}_1 = 1$. Then, for the subgame perfect Nash equilibrium for Game 2, territories are not constructed, and the prices increase up to the ceiling reservation price.

Proposition 2. *If there exists a pure-strategy subgame perfect Nash equilibrium for Game 2, then the price is* v *and the DAs cover the whole area under the equilibrium.*

2.3 One-stage Game

In the two-stage game analyzed in the previous subsection, the stores are assumed to keep the DA when they compete by prices in the second stage. This drives Nash equilibrium out to trivial cases. Keeping the DA is accommodating to rival stores, although the purpose of undercutting the price of the rival stores should be extending the DA. Considering this facet of undercutting strategy, we investigate a one-stage game; that is,

Game 3.

- $\delta \geq 0$.
- *The single stage: both stores declare DAs and uniform price simultaneously.*

In a one-stage game, stores can undercut their rivals with extended DAs. Nevertheless, the result of competition in Game 3 is not much different from that in Game 1 or Game 2.

Lemma 6. *In Game* 3*, there is no combination of strategies* $\langle \tilde{s}_0, \tilde{s}_1 \rangle$ *such that*

$$\pi_i(\tilde{s}_i, \tilde{s}_j) = \max_{s_i \in S} \pi_i(s_i, \tilde{s}_j) \geq 0, \tag{3}$$

except when $\tilde{s}_i = (v, 1)$ *and* δ *is sufficiently large.*

For example, when $v > 4t/3$, and $q(x)$ is specified as $q(x) = 1 - x$, the condition for existence of Nash equilibrium requires $\delta > (3v - 2t)/6$. Intuitively, if the switching cost is sufficiently large, stores do not have incentives to undercut their rivals. So, they extend their DAs to cover the whole area and set their maximum prices. Thus,

Proposition 3. *If there exists a pure-strategy equilibrium for Game* 3*, then at the equilibrium, the price is* v *and the DAs cover the whole area.*

2.4 UPP in the two-stage game

Under the only possible equilibrium implied by proposition 2 and 3 for Games 2 and 3, respectively, both stores cover the whole market as their DAs. So, the competition is the same as that under the pure price competition, which does not influence the determination of the DAs. Nash equilibrium is possible even when the switching cost is positive, but it is because the ceiling of the price is assumed as the reservation price in this model. The problem of Nash equilibrium with a positive switching cost is the upward adjustments of the price levels, which are restricted in this model by the assumption of the reservation price.

In fact, territories have long been observed in the LP gas retail market in Japan, as we explained in section 1. Therefore, the models above that seeks Nash equilibrium and the outcome that all the consumers have an alternative service provider might be inadequate to discuss the problems in Japan's LP gas retail market. We are concerned with cases wherein natural territories not a forced product of vertical constraint are constructed as a result of competition. Therefore, we replace the Nash equilibrium with the UPP in spatial competition to fix the outcomes of the competition of Games 1 and 2. (Shy, 2002, sec. 6) presents the UPP under a Hotelling-type Hotelling (1929) environment where consumers bear the switching cost with uniform transcendental preference. Undercutting was defined as a capture of the whole market through sufficient price reductions. The model developed therein was a Hotelling-type shopping model, where consumers pay their own trip costs that are proportional to the distance between their location and the store. This causes heterogeneity of consumers and a downward-sloping demand function. Then, the share of consumers expands continuously against reducing price sizes.

The model constructed herein, however, is not a shopping model but a shipping model with declared DAs. There is no heterogeneity among consumers in the DAs except their transcendental preference. The demand is discontinuous in regard to the uniform price by the effect of the switching cost: a set of consumers switch at the same price. Hence, undercutting is defined in the same manner as in Shy (2002), that is, as a strategy to capture all the demands in the DA through sufficiently low prices. The stores increase their revenue discontinuously at prices that undercut their rivals', in contrast to the case of the shopping model. This definition may approximate the nuance of the word *undercutting*.

In this subsection, we introduce the UPP in the two-stage Game 2, and in the next subsection, the UPP in the one-stage Game 3 is considered. In Game 2, the DAs have been declared in the first stage. Even if the stores undercut the rival's strategy, they do not intend to expand their DAs in the second stage beyond the declared DA in the first stage.

The maximum profit earned by store i when store i successfully undercuts its rival store j is defined as $\pi_i^{u1}(p_j)$ when a one-stage game is considered and as $\pi_i^{u2}(p_j|\tilde{r}_i, \tilde{r}_j)$ given $(\tilde{r}_i, \tilde{r}_j)$ when a two-stage game is considered. Hereafter, $\pi_i^{u2}(p_j|\tilde{r}_i, \tilde{r}_j)$ is denoted by $\pi_i^{u2}(p_j)$ for simplicity.

Definition 3.

$$\pi_i^{u1}(p_j) \stackrel{\text{def}}{=} \max_{s_i \in \{(p,r) | p < p_j - \delta,\ 0 \le r \le 1\}} \pi_i(s_i, s_j)$$

$$\pi_i^{u2}(p_j) = \pi_i^{u2}(p_j | \tilde{r}_i, \tilde{r}_j) \stackrel{\text{def}}{=} \begin{cases} \max\limits_{s_i \in \{(p,\tilde{r}_i) | p < p_j - \delta\}} \pi_i(s_i, s_j) & \textit{if } \tilde{r}_0 + \tilde{r}_1 > 1 \\ -K & \textit{otherwise} \end{cases}$$

$$K \in \Re^{++}.$$

Note that $\pi_i^{u1}(p_j) \ge 0$ for any $s_j \in S$ because, for any price p of its rival, stores can retain positive profits by setting their DA sufficiently small when $p - \delta > 0$, or they declare no DAs and get zero profits when $p - \delta \le 0$. When $\tilde{r}_0 + \tilde{r}_1 \le 1$, π_i^{u2} is defined for the sake of formality: it is not possible to undercut the rival when DAs do not overlap each other.

These profit functions when stores successfully undercut their rivals have the following property:

Lemma 7. *For all $r_i \in U$ and $s_i, s_j \in S$,*

$$\pi_i^{u1}(p_j) \ge \pi_i^{u2}(p_j) = E_i(p_j - \delta) + J_i(p_j - \delta)$$
$$\pi_i^{u1}(p_j) \ge E_i(p_j - \delta) + Q_i(p_j - \delta), \quad \pi_i^{u1}(p_j) \ge 0.$$

In the remaining part of this subsection, Game 2, where the equilibrium prices are determined as having the UPP in the second stage, is considered. Following the definition in Shy (2002), the UPP is defined for a pair of strategies such that no store can increase its profit by switching to a strategy that undercuts its rival's strategy. That is,

Definition 4. A pair of strategies $\langle s_0, s_1 \rangle$ is said to have an **undercut-proof property (UPP)** if store i chooses the strategy s_i $(s_i \in S)$ that maximizes the profit $\pi_i(s_i, s_j)$ subject to

$$\pi_j(s_j, s_i) \ge \pi_j^{u1}(p_i), \qquad \text{at one-stage game}$$
$$\pi_j(s_j, s_i) \ge \pi_j^{u2}(p_i), \qquad \text{at two-stage game}$$

given strategy $s_j = (p_j, r_j)$ of store j, $(s_j \in S)$.

Note that, under this definition, the UPP when $\tilde{r}_0 + \tilde{r}_1 \le 1$ is a mere maximization of the profit.

The optimization problems that frame the definition of the UPP (definition 4) in the second stage of Game 2 are restated to find the solution $\langle s_0^*, s_1^* \rangle \stackrel{\text{def}}{=} \langle (p_0^*, \tilde{r}_0), (p_1^*, \tilde{r}_1) \rangle$, $s_i^* \in S$ for the problem:

Problem 1.
Given $(\tilde{r}_0, \tilde{r}_1) \in U^2$,

$$p_i^* = \underset{p \in [0,v]}{\operatorname{argmax}} \ \pi_i((p, \tilde{r}_i), (p_j^*, \tilde{r}_j)) \text{ subject to } \pi_j((p_j^*, \tilde{r}_j), (p, \tilde{r}_i)) \geq \pi_j^{u2}(p).$$

Note that the participation constraint $\pi_i \geq 0$ is not considered as the constraint here. The constraint will be checked later when the outcome of the first stage of the game is examined because negative profits cause no trouble if the sizes of the DA $(\tilde{r}_0, \tilde{r}_1)$ that yield negative profits are not chosen in the first stage.

There does exist a unique solution to Problem 1, as proven by three steps. First, if the solution (p_0^*, p_1^*) exists, it is shown to fall in area Q, where $Q \overset{\text{def}}{=} \{(p_0, p_1) | p_i \in [0, v], |p_0 - p_1| \leq \delta\}$. Second, if the prices (p_0^*, p_1^*) that stores can set are limited to area Q, it is shown that there exists a unique solution to the optimization problem. Lastly, the unique solution identified at the second step is shown to be the solution to Problem 1. These steps will be verified sequentially as follows:

Lemma 8. *Given* $(\tilde{r}_0, \tilde{r}_1) \in (0, 1]^2$, *if* $\langle s_0^*, s_1^* \rangle = \langle (p_0^*, \tilde{r}_0), (p_1^*, \tilde{r}_1) \rangle$, $s_i^* \in S$ *is a solution to Problem 1 for* $\delta > 0$. *Then,*

$$|p_0^* - p_1^*| \leq \delta.$$

Confine the area where the combination of strategies $\langle s_i, s_j \rangle$ takes places to Q defined above, and consider Problem 2 within the restricted area:

Problem 2.
Given $(\tilde{r}_0, \tilde{r}_1) \in U^2$,

$$p_i^* = \underset{p \in [\max(0, p_j^* - \delta), \min(v, p_j^* + \delta)]}{\operatorname{argmax}} \ \pi_i((p, \tilde{r}_i), (p_j^*, \tilde{r}_j))$$

$$10pt] \qquad \text{subject to } \pi_j((p_j^*, \tilde{r}_j), (p, \tilde{r}_i)) \geq \pi_j^{u2}(p).$$

There always exists a unique solution to Problem 2. To show this, the function p_i^{**} is defined:

Definition 5. *Functions* $p_i^{**} : [0, v] \to \Re$ *are such that*

$$E_j(p_j) + Q_j(p_j) = E_j(p_i^{**}(p_j) - \delta) + J_j(p_i^{**}(p_j) - \delta).$$

The functions have the following characteristics:

Lemma 9. *For a given* $(\tilde{r}_i, \tilde{r}_j) \in U^2$ *such that* $\tilde{r}_0 + \tilde{r}_1 > 1$*, the functions* $p_i^{**}(p_j)$ *are linear functions of* $p_j \in [0, v]$ *and*

$$p_i^{**}(p_j^{JQ}) = p_j^{JQ} + \delta, \quad p_i^{**}(0) > \delta, \quad 0 < \frac{d}{dp_j} p_i^{**}(p_j) < 1.$$

Lemma 10 shows that the existence for the solution to Problem 2 is proved.

Lemma 10. *Given* $(\tilde{r}_0, \tilde{r}_1) \in (0, 1]^2$*, there exists a unique solution to Problem 2.*

The solution to Problem 2 shown is also proved to be the solution for Problem 1 using the next Lemma 11.

Lemma 11. *Given* $(\tilde{r}_0, \tilde{r}_1) \in U^2$*, the solution to Problem 2 shown in Lemma 10 is a solution to Problem 1.*

Thus, there exists a unique solution to Problem 1.

The condition for the price reaching its ceiling reservation price v is stated in the next Lemma 12:

Lemma 12. *Given* $(\tilde{r}_0, \tilde{r}_1) \in (0, 1]^2$ *such that* $\tilde{r}_0 + \tilde{r}_1 > 1$*, the condition for the solution to Problem 1 being* $(p_0^*, p_1^*) = (v, v)$ *is* $E_i(v) + Q_i(v) \geq E_i(v - \delta) + J_i(v - \delta)$ *for both* $i \in \{0, 1\}$*, and if* $E_i(v) + Q_i(v) < E_i(v - \delta) + J_i(v - \delta)$ *for both* $i \in \{0, 1\}$*, then* $(p_0^*, p_1^*) < (v, v)$*.*

When the switching cost is sufficiently small, the prices that have the UPP in the second stage of Game 2 are lower than the reservation price. However, when the DA overlaps in very small areas, the prices that have the UPP increase to the ceiling reservation price. Check next lemmata.

Lemma 13. *Given* $(\tilde{r}_0, \tilde{r}_1) \in (0, 1]^2$ *such that* $\tilde{r}_0 + \tilde{r}_1 > 1$*, there exists* $\delta^* > 0$*, such that for all* $\tilde{\delta} \in (0, \delta^*)$*,* $(p_0^*, p_1^*) < (v, v)$*, where* (p_0^*, p_1^*) *is the solution to Problem 1 when* δ *takes the value of* $\tilde{\delta}$*.*

Lemma 14. *Given the value of* δ *such that* $v - t > \delta$*, there exists* $\epsilon > 0$*, such that if* $\epsilon > \tilde{r}_0 + \tilde{r}_1 - 1 > 0$*. Then, the unique solution to Problem 1 is* $(p_0^*, p_1^*) = (v, v)$*.*

Lemma 14 implies that the level of prices generally increases as stores shrink their DAs. Specifically, as shown in Remark 1, if the transcendental preference function $q(x)$ is symmetrical, which is defined as $q(x) + q(1-x) = 1$ for $x \in [0, 1]$, and the sizes of both DAs are identical, the price levels of the solution to Problem 1 decrease as the store extends the size of the DA if every customer in the DAs contributes to the profit of the store. This means $p_i^* > t\tilde{r}_i$, that is, the demand from the consumer located at the border of the DA contributes to the profit of the store. The numerical size of the regions where the condition $p_0^* = p_1^* > t\tilde{r}_0 = t\tilde{r}_1$ is satisfied is, for example, if $q(x) = 1 - x$ is assumed, $\tilde{r}_i < 0.824$ for $\delta = 0.1$, and $\tilde{r}_i \leq 1.000$ for $\delta = 0.2$; alternatively, if $q(x) = 0.5$ is assumed, $\tilde{r}_i < 0.779$ for $\delta = 0.1$, and $\tilde{r}_i < 0.932$ for $\delta = 0.2$. $t = 1$ is assumed for all cases. Thus, when the sizes of the DAs are large enough that most of the consumers are covered by both stores, competition between the stores lowers the price as low as below the transportation cost for consumers at the boundary of the DA. In those cases, the retreat of the DA, say DA of store 0, leads to decreasing profit of store 0, as expected by undercutting rival store 1. From the standpoint of store 1, the restriction to keep store 0 not to undercut store 1 loosens, and store 1 can increase its price. This also loosens the restriction for store 0 to keep store 1 from undercutting it, and finally store 0 can increase its price too.

Remark 1. *If the function $q(x)$ is such that $q(x) + q(1-x) = 1$, at the symmetric solution to Problem 1 for $(\tilde{r}_0, \tilde{r}_1) = (\tilde{r}, \tilde{r})$ $(\tilde{r} > 1/2)$, such that $(p_0^*, p_1^*) = (p^*, p^*) \in (t\tilde{r}, v)^2$,*

$$\left.\frac{\partial p_i^*}{\partial \tilde{r}_i}\right|_{p_i^* = p^*,\ \tilde{r}_i = \tilde{r}} < 0.$$

as long as $p^* > t\tilde{r}$.

The existence of the prices with the UPP are certified by Lemma 10 and 11. In the first stage, the stores decide the scales of their DAs. The next lemma shows that, at least at the boundary, the retreat of the DA results in increased prices with the UPP that the stores attain to set at the ceiling reservation price v. Any further retreat leads to decreased profit because the prices are kept unchanged at price v. Thus, equilibrium is attained for Game 2 when the DAs overlap in very narrow areas, that is, the stores are securing their closed territories.

Lemma 15. *Given a sufficiently small value of $\delta > 0$, there exists equilibrium for Game 2 such that, in the second stage, the combination of prices (v, v) satisfies the property of the UPP and $\tilde{r}_0 + \tilde{r}_1 - 1$ is sufficiently small.*

The prices v attained in the second stage secure the stores' positive profits, $\pi_i\left((v, \tilde{r}_i), (v, \tilde{r}_j)\right) > 0$, because stores charge more than transportation cost

tx_i^* everywhere in their DAs. Thus, the solution satisfies the participation constraints.

That the equilibrium implied by the lemma is unique is not secured by Lemma 15. When both $\tilde{r}_i$ are large, and the resulting prices are low enough, there remains the possibility that another equilibrium exists. However, as long as stores can secure their own DAs, such equilibrium does not exist, as shown in Lemma 16. This is because, when the value of v is high enough compared with the value of unit transportation cost t, for any given rival DA, stores can always attain the highest price v by shrinking their DA as to be so small that the DAs are hardly overlap.

Lemma 16. *Given sufficiently small value of δ, and $(\tilde{r}_0, \tilde{r}_1) \in (0,1)^2$ such that $\tilde{r}_0 + \tilde{r}_1 > 1$, there exists $\{(v^*, r^*) | v^* \in \Re^+, r_i^* \in (1 - \tilde{r}_j, 1)\}$, such that for all $v > v^*$*

$$\pi_0((v, r_0^*), (v, \tilde{r}_1)) > \pi_0((p_0^*, \tilde{r}_0), (p_1^*, \tilde{r}_1)) \text{ or,}$$
$$\pi_1((v, r_1^*), (v, \tilde{r}_0)) > \pi_1((p_1^*, \tilde{r}_1), (p_0^*, \tilde{r}_0)),$$

where the solutions to Problem 1 under $(r_0, r_1) = (r_0^*, \tilde{r}_1)$, $(\tilde{r}_0, r_1^*)$, and $(\tilde{r}_0, \tilde{r}_1)$ are (v, v), (v, v), and (p_0^*, p_1^*), respectively.

Thus, if either of the sizes of the DA is less than 1, the solution to Problem 1 provides stores profits lower than those that can be attained by setting their DAs small enough to raise the price to the reservation price. Therefore, the situation does not have equilibrium. Nevertheless, this is not applied when $\tilde{r}_0 = \tilde{r}_1 = 1$. Indeed, in such a case, there remains the possibility that another equilibrium exists, as shown in Lemma 17.

Lemma 17. *Given sufficiently small value of δ, when the function $q(x)$ is symmetrical and $q(1)$ is sufficiently small, the combination of the sizes of the DAs $(\tilde{r}_0, \tilde{r}_1) = (1, 1)$ locally maximize stores' profits that are determined in the second stage of Game 2 with the solution to Problem 1 such that $(p_0^*, p_1^*) = (t(1 - 2D_1(U)) + 2\delta, t(1 - 2D_1(U)) + 2\delta)$.*

The analyses for Game 2, when the result of the second stage is given by the UPP, is summarized as the proposition below.

Proposition 4. *Given sufficiently small value of δ, and v is sufficiently large compared with t, the equilibrium of Game 2, when the result of the second stage is given by the UPP, is limited to the case wherein $\tilde{r}_0 + \tilde{r}_1 - 1$ is sufficiently small and the combination of prices is (v, v), or the case where $\tilde{r}_0 = \tilde{r}_1 = 1$.*

2.5 UPP in the one-stage game

2.5.1 Problem to be solved for the UPP in the one-stage game

In this subsection, Game 3, where the prices are determined as having the UPP at the single stage, is considered. The optimization problems that frame the definition of the UPP (definition 4) of Game 3 are restated as follows:

Given $\delta > 0$, if $\langle s_0^*, s_1^* \rangle \stackrel{\text{def}}{=} \langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$ such that $s_0^*, s_1^* \in S$ is a combination with the UPP of Game 3, we have the solution to

Problem 3.

$$s_i^* = \underset{s=(p,r)\in S}{\operatorname{argmax}} \ \pi_i(s, s_j^*) \qquad \textit{subject to } \pi_j(s_j^*, s) \geq \pi_j^{u1}(p).$$

Note that the participation constraints are not necessary, because π_j^{u1} has a non-negative value (Lemma 7).

First, check that conditions $\pi_j(s_j, s) \geq \pi_j^{u1}(p)$ are always binding; that is,

Lemma 18. *Assume $v > \delta$. If $\langle s_0^*, s_1^* \rangle \equiv \langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$, $s_i^* \in S$ is a solution to Problem 3, then $\pi_i(s_i^*, s_j^*) = \pi_i^{u1}(p_j^*)$ and $p_i^* \in [\delta, v]$.*

Under the construction here, if $r_0 + r_1 < 1$, that is, if there are areas that are not covered by any store's DA, a pair of such strategies $\langle s_0, s_1 \rangle$ cannot be the solution to Problem 3. This property may appear obvious because both stores can increase their profits by selecting an expanded DA with a sufficiently high price given the strategy of their rivals when $r_0 + r_1 < 1$. Nevertheless, expanding their DAs into blank areas does not always yield higher profits because the marginal profits are the difference between the price levels and the increasing cost of transportation at the boundaries of their existing DAs. Price levels face the constraints of possible undercutting strategies by their rivals.

Lemma 19. *Given $\delta > 0$, there is no combination of strategies $\langle s_0^*, s_1^* \rangle = \langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$, $s_i^* \in S$, which is the solution to Problem 3 such that $r_0^* + r_1^* < 1$.*

Next, we examine the case wherein the combination of strategies in which a store is undercutting the other store is not the solution to Problem 3. That is:

Lemma 20. *If a combination of strategies $\langle s_0^*, s_1^* \rangle = \langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$ $s_i^* \in S$ is the solution to Problem 3 with $\delta > 0$,*

$$|p_0^* - p_1^*| \leq \delta.$$

With Lemma 18 $\sim$ 20, where the combination of strategy $\langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$ is the solution to Problem 3, it can be restricted to the area $r_0^* + r_1^* \geq 1$ and the prices $(p_0^*, p_1^*) \in [\delta, v]^2$ should satisfy the condition $|p_0^* - p_1^*| \leq \delta$. Based on this property, Problem 3 can be transformed to the problem with the condition $r_0 + r_1 \geq 1$ and the specified profit functions, $\pi_i(s_i, s_j) = E_i(p_i) + Q_i(p_i)$. Given $\delta > 0$, if $\langle s_0^*, s_1^* \rangle \equiv \langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$ is a combination with the UPP in Game 3, it should satisfy

Problem 4.

$$\begin{aligned} s_i^* = &\operatorname*{argmax}_{s_i \in S} \pi_i(s_i, s_j^*), \\ &\quad \textit{subject to } \pi_j(s_i, s_j^*) = \pi_j^{u1}(p_i),\ \delta \leq p_i \leq v,\ 1 - r_j^* \leq r_i \leq 1, \end{aligned}$$

and $|p_0^* - p_1^*| \leq \delta$, *where*

$$\begin{aligned} \pi_0(s_0, s_1) &= p_0(1 - r_1) - \frac{t(1 - r_1)^2}{2} + p_0 D_0(A_0 \cap A_1) - t D_1(A_0 \cap A_1), \\ \pi_1(s_1, s_0) &= p_1 r_1 - \frac{t r_1^2}{2} - (p_1 - t) D_0(A_0 \cap A_1) - t D_1(A_0 \cap A_1), \\ \pi_i^{u1}(p) &= \psi^0(p - \delta), \\ \psi^0(p) &\stackrel{\text{def}}{=} \begin{cases} \dfrac{p^2}{2t} & \textit{when } p \leq t, \\ p - \dfrac{t}{2} & \textit{when } p \geq t. \end{cases} \end{aligned}$$

Note that

$$\begin{aligned} \pi_0^{u1}(p) &= \int_0^{\max\left(1, \frac{p-\delta}{t}\right)} (p - \delta - t x_0^*) dx = \psi^0(p - \delta), \\ \pi_1^{u1}(p) &= \int_{1-\max\left(1, \frac{p-\delta}{t}\right)}^{1} (p - \delta - t x_1^*))\, dx = \psi^0(p - \delta). \end{aligned}$$

The solution to Problem 4 has to satisfy the Kuhn–Tucker conditions stated below. The condition $|p_0^* - p_1^*| \leq \delta$ is not a part of the restrictions of the maximization problem, and thus does not appear in the conditions. The inequality should be examined afterward for a possible solution, as it specifies the profit functions. When these conditions are discussed in the latter parts of the article, the difference between the logical sum and the logical product is crucial in the proof, and the conditions are shown for each store's strategy specifically, with $i = 0$ and/or $i = 1$, not i or j.

Condition 1 (**UPE** (Undercut Proof Equilibrium) of one-stage game)**.**

$$1 - (r_1^* - D_0^*) - \lambda_0 \psi^1 (p_0^* - \delta) + \xi_0 - \zeta_0 = 0, \tag{4}$$
$$q_0^* \left((p_0^* - t r_0^*) - \lambda_0 \left(p_1^* - t(1 - r_0^*) \right) \right) + \mu_0 - \eta_0 = 0, \tag{5}$$
$$p_0^* (1 - r_1^*) - \frac{t(1 - r_1^*)^2}{2} + p_0^* D_0^* - t D_1^* - \psi^0 (p_1^* - \delta) = 0, \tag{6}$$
$$\xi_0 (p_0^* - \delta) = 0, \tag{7}$$
$$\xi_0 \geq 0, \tag{8}$$
$$p_0^* \geq \delta, \tag{9}$$
$$\zeta_0 (v - p_0^*) = 0, \tag{10}$$
$$\zeta_0 \geq 0, \tag{11}$$
$$v - p_0^* \geq 0, \tag{12}$$
$$\mu_0 (r_0^* + r_1^* - 1) = 0, \tag{13}$$
$$\mu_0 \geq 0, \tag{14}$$
$$r_0^* + r_1^* - 1 \geq 0, \tag{15}$$
$$\eta_0 (1 - r_0^*) = 0, \tag{16}$$
$$\eta_0 \geq 0, \tag{17}$$
$$1 - r_0^* \geq 0, \tag{18}$$
$$(r_1^* - D_0^*) - \lambda_1 \psi^1 (p_1^* - \delta) + \xi_1 - \zeta_1 = 0, \tag{19}$$
$$q_1^* \left((p_1^* - t r_1^*) - \lambda_1 \left(p_0^* - t(1 - r_1^*) \right) \right) + \mu_1 - \eta_1 = 0, \tag{20}$$
$$p_1^* r_1^* - \frac{t {r_1^*}^2}{2} - p_1^* D_0^* + t(D_0^* - D_1^*) - \psi^0 (p_0^* - \delta) = 0, \tag{21}$$
$$\xi_1 (p_1^* - \delta) = 0, \tag{22}$$
$$\xi_1 \geq 0, \tag{23}$$
$$p_1^* \geq \delta, \tag{24}$$
$$\zeta_1 (v - p_1^*) = 0, \tag{25}$$
$$\zeta_1 \geq 0, \tag{26}$$
$$v - p_1^* \geq 0, \tag{27}$$

$$\mu_1(r_0^* + r_1^* - 1) = 0, \tag{28}$$
$$\mu_1 \geq 0, \tag{29}$$
$$r_0^* + r_1^* - 1 \geq 0, \tag{30}$$
$$\eta_1(1 - r_1^*) = 0, \tag{31}$$
$$\eta_1 \geq 0, \tag{32}$$
$$1 - r_1^* \geq 0 \tag{33}$$

where

$$\begin{aligned} q_0^* &= q(r_0^*), & q_1^* &= 1 - q(1 - r_1^*), \\ D_0^* &= D_0([1 - r_1^*, r_0^*]), & D_1^* &= D_1([1 - r_1^*, r_0^*]), \\ \psi^1(p) &\overset{\text{def}}{=} \min\left(\frac{p}{t}, 1\right) \end{aligned}$$

2.5.2 Combination of Strategies with the UPP as the Corner Solutions of One-stage Game

First, the cases when $\mu_0 > 0$ and/or $\mu_1 > 0$ in the conditions UPE (13) and/or (28) are investigated. The consumers are divided by the DAs of the stores as their territories. The territories do not overlap. We examine this case for the same reason explaining why Lemma 19 is not self-evident. In Game 3, the size of the DA can be expanded at the same time to determine the price level. Then, the possibility of undercutting a rival should be examined even in the cases wherein the DAs do not overlap.

Lemma 21. *If $\delta \in (0, t)$ and $r_0^* + r_1^* = 1$, then $\eta_i = \xi_i = 0$, $r_i^* \in (0, 1)$, and $p_i^* > \delta$. If $v > t + 3\delta$ is assumed additionally, then $\zeta_i = 0$, $\lambda_i > 0$, $p_i^* - tr_i^* \geq 0$ and*

$$(p_i - tr_i) - \lambda_i (p_j - tr_j) \leq 0. \tag{34}$$

Furthermore $p_0^* - tr_0^* > 0$ and/or $p_1^* - tr_1^* > 0$.

Using this lemma, we can ascertain that there are no solutions such that $r_0^* + r_1^* = 1$ for Problem 4.

Proposition 5. *When $v > t + 3\delta$, there is no combination of strategies with the UPP, $\langle s_0^*, s_1^* \rangle = \langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$, $s_i^* \in S$, that is a solution to Problem 4 with $\delta \in (0, t)$, such that $r_0^* + r_1^* = 1$.*

The inequality conditions (34) require that the stores have no incentive to expand their DAs when they face each other at the boundary. Expansion of their territories has the direct effect of increasing their revenue by

$q(r_i)(p_i - tr_i)$ for store i. At the same time, rival store j is partly deprived of revenue, which may compel it to undertake an undercutting strategy because profits from such a strategy will be kept constant as long as store i does not alter the price level, whereas the current profits of store j are certainly decreased. To prevent its rival from opting for the undercutting strategy, store i has to lower its price level to suppress the undercutting strategy of store j; this revenue reduction is expressed as $q(r_i)\lambda_i(p_j - tr_j)$. If the corner solution is with the UPP, the net effect of the expansion of the DA into the rival's territory should be negative. Note that the relative net incentive to expand the DA is independent of the possible share $q(r_i)$ at the boundary, and Proposition 5 suggests that such an incentive is significant enough to consider the boundary transgression.

Next, consider the solutions where both stores cover the whole market. The cases are when $\eta_0 > 0$ and $\eta_1 > 0$ in the UPE condition. For the case, the next three lemmata show that there exists such a solution to Problem 4 that satisfies the UPE condition only when switching cost δ is sufficiently large compared with unit transportation cost t.

Lemma 22. *There exists a value of κ and a region Δ such that—if and only if $v > \kappa t$ and $\delta \in \Delta$, where $0 \notin \Delta$—a solution to Problem 4, where $r_i^* = 1$, $p_i^* \geq t + \delta$, satisfies the UPE condition.*

Lemma 23. *There exists δ^* such that, if $\delta < \delta^*$, no solution to Problem 4 exists, where $r_i^* = 1$, $p_i^* < t + \delta$, $i \in \{0, 1\}$ satisfies the UPE condition. If such a solution exists, the prices should be such that $(p_0^*, p_1^*) \in [t, t+\delta)^2$.*

Lemma 24. *There exist δ^{**} such that, if $\delta < \delta^{**}$, no solution to Problem 4 exists, where $r_i^* = 1$, $p_i^* \geq t + \delta$, $p_j^* < t + \delta$, $i, j \in \{0, 1\}$, $i \neq j$ satisfies the UPE condition. If such a solution exists, the price p_j^* should be such that $p_j^* \in (t, t+\delta)$.*

These lemmata are summarized as the following proposition:

Proposition 6. *When δ is sufficiently small, there is no combination of strategies with the UPP, $\langle s_0^*, s_1^* \rangle = \langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$, that is a solution to Problem 4 such that $r_0^* = r_1^* = 1$. If a solution exists to Problem 4 such that $r_0^* = r_1^* = 1$, the prices at the solution should be greater than unit transportation cost t.*

When the whole market is covered by both stores, the model is not much different from a simple Bertrand price competition model, because competition for expanding territories is restricted. Nevertheless, when switching cost δ is small enough, Lemmata 22 and 23 imply that there exists no solution with the UPP such that both stores cover the whole market. Thus, even the UPE may not exist in this circumstance.

At the end of this subsection, we examine the case wherein the prices stick to the ceiling price.

When the game is a two-stage game, an equilibrium is attained when $p_0 = p_1 = v$, as shown in Proposition 4. In the one-stage game, contrarily, there is no such equilibrium when v is sufficiently larger compared with t, as shown in the next proposition.

Proposition 7. *When $v > t + 2\delta$, there is no combination of strategies with the UPP, $\langle s_0^*, s_1^* \rangle = \langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$, that is a solution to Problem 4 and $p_0^* = p_1^* = v$.*

2.5.3 Combination of Strategies with the UPP as the Interior Solution of a One-stage Game

In this subsection, the interior solution to Problem 4 is considered. They are solutions such that $r_0 + r_1 > 1$, $r_i < 1$, and $p_i < v$, meaning some of the areas covered by the DAs overlap, with all stores having their exclusive distribution areas, but none covering the whole market. It is the situation of $\mu_i = \eta_i = \zeta_i = 0$ in the UPE condition.

First, check that, for sufficiently small switching cost δ, The prices that satisfy the UPE condition do not exceed $t + \delta$ as proved in the next lemma.

Lemma 25. *There exists $\delta^* > 0$ such that, if $\delta \in (0, \delta^*)$, there exists no solution to Problem 4 that satisfies the UPE condition and $p_i^* > t + \delta$ for either/both of i, $i \in \{0, 1\}$.*

By Lemma 25, we can confine possible solutions to the UPE condition, as shown in Lemma 26.

Lemma 26. *Assume $v > t + \delta$. If $(r_0^*, r_1^*, p_0^*, p_1^*)$ such that $r_0^* + r_1^* > 1$, $r_i^* < 1$, and $p_i^* \leq t + \delta \quad (i \in \{0, 1\})$, satisfies the UPE condition, then*

$$(p_0^* - \delta)^2 - 2t(r_1^* - D_0^*)p_1^* + t^2(r_1^{*2} - 2D_0^* + 2D_1^*) = 0, \tag{35}$$

$$(p_1^* - \delta)^2 - 2t(1 - r_1^* + D_0^*)p_0^* + t^2((1 - r_1^*)^2 + 2D_1^*) = 0, \tag{36}$$

$$(p_0^* - \delta)(p_0^* - tr_0^*) - t(1 - r_1^* + D_0^*)(p_1^* - t(1 - r_0^*)) = 0, \tag{37}$$

$$(p_1^* - \delta)(p_1^* - tr_1^*) - t(r_1^* - D_0^*)(p_0^* - t(1 - r_1^*)) = 0, \tag{38}$$

where

$$D_0^* = D_0([1 - r_1^*, r_0^*]), \quad D_1^* = D_1([1 - r_1^*, r_0^*]).$$

Thus, the four equations in Lemma 26 with four unknowns determine the interior solution $(p_0^*, r_0^*, p_0^*, r_1^*)$ of the UPE condition. When the switching cost δ is sufficiently small, the solution certainly exists, as shown in Lemma 27.

Lemma 27. *There exists $\delta^* \in \Re^{++}$ such that, if $\delta < \delta^*$, then solution $(p_0^*, r_0^*, p_1^*, r_1^*)$ for equations (35) $\sim$ (38) exists. The solution satisfies $|p_0^* - p_1^*| < \delta$, $t + 2\delta < p_0^* + p_1^* < t + 2\delta + \sqrt{t\delta}$, $p_i^* < t + \delta$, $r_i^* < 1$, and $r_0^* + r_1^* > 1$ for $i \in \{0, 1\}$.*

Lemma 25 and Lemma 27 are combined to be the next proposition, choosing the minimum value of δ^*s in both lemmata as the value of δ^* of this lemma.

Proposition 8. *There exists $\delta^* \in \Re^{++}$ such that, if $\delta < \delta^*$, only the solution wherein $p_i^* < t + \delta$ exists for Problem 4, that is, only strategy $\langle s_0^*, s_1^* \rangle = \langle (p_0^*, r_0^*), (p_1^*, r_1^*) \rangle$ such that $r_0^* + r_1^* > 1, r_i^* < 1, p_i^* < t + \delta$, $i \in \{0, 1\}$ has the UPP in Game 3.*

For any form of decreasing the transcendental preference of consumers, $q(x)$, if the switching cost is sufficiently small, an interior solution always exists. Because the marginal cost for providing services is assumed to be 0, the price here is the profit margin of the store, called the switching cost premium. The margin or the switching cost premium is determined by the magnitude of the switching cost and the unit transportation cost. As the sizes of the DAs of the stores are less than 1 at the equilibrium, territories are constructed naturally rather than as deliberate vertical constraints. Such territories are surrounded by areas of competition, where consumers select the store that serves them.

3 Estimation of coefficients of the pass-through in LP gas retail delivery prices in Japan

In this section, the type of competition in the regional retail LP gas delivery market in Japan is estimated based on the analysis results derived by the theoretical model.

The analysis so far has shown three possible outcomes in retail delivery markets: the two-stage game that results in equilibrium with the reservation price; the one-stage game that results in equilibrium with the switching cost premium; and the collusion that results in the reservation price.

Because the third outcome is expected to be accompanied with a strictly closed territory that is not observed in Japan's market in reality, only the first two possibilities are considered. When the price is determined by the reservation price, it is expected that the coefficient of the pass-through from the marginal cost onto the equilibrium price level takes the value 0. However, when the price is determined by the switching cost premium, the coefficient takes the value 1. Thus, it is possible to identify the type of competition by estimating the coefficient of the pass-through.

3.1 Estimating the coefficient of the pass-through by auto-regressive model

3.1.1 Data

The source of the data investigated is a survey by the Institute of Energy Economics Japan, the Oil Information Center. The survey was conducted to inquire about the LP gas retail delivery price of around 3,000 stores all over Japan in a bi-monthly duration. For the period from June 1996 to February 2006, all areas of Japan were divided to 291 regions, and price statistics were shown for each region. The means of the standing charges and of total charges when the customer used 5, 10, 20, and 50 cubic meters of LP gas were reported. For the periods after April 2006 to October 2019, the number of divided regions is 268: the division of regions was modified between the periods. Although the same price statistics are shown, the price data cannot be linked for the two periods. Therefore, two panel data sets are investigated separately. [15]

The time series data of the means of the total charges when the customer used 10 cubic meters of LP gas are investigated for each region. This is because 10 cubic meters consumption is close to the typical household consumption.[16]

Besides the retail price data, the time series wholesale price of propane amounting to 10 cubic meter of LP gas was prepared.[17] The retrieved whole-

[15]The data are retrieved from, Japan LP Gas Association, `http://www.j-lpgas.gr.jp/stat/kakaku/files/kakakusuii12.xls`, and Oil Information Center, Institute of Energy Economics, Japan, `https://oil-info.ieej.or.jp/price/data/zenkoku.xls`

[16]According to Petrochemical Press (2011) in 2006, 2.570×10^7 households consumed 5.480×10^9kg of LP gas in total and 0.098×10^7 commercial users consumed 2.489×10^9kg, meaning the monthly average usage of household was $8.92m^3$, while that of household and commercial users was $11.48m^3$.

[17]LP gas for household usage in Japan is regulated to include more than 95 percent of propane or propylene by Act on the Securing of Safety and the Optimization of Transaction of Liquefied Petroleum Gas, `https://elaws.e-gov.go.jp/search/elawsSearch/`

sale price is one series that shows that the wholesale price is uniform all over Japan.

3.1.2 Unit root and Cointegration tests

First, an augmented Dickey–Fuller unit root test is conducted to examine the stationarity of the retail and wholesale prices. The result is summarized in Table 1. For any lag length and for both periods, the null hypothesis that the wholesale price series has a unit root is not rejected with 0.05 percent p-value. For the retail price, although the null hypothesis is rejected for some regions, it holds for most of the observed regions. Specifically, the null hypothesis is rejected for at most 51 regions out of 281 in total.

elaws_search/lsg0500/detail?lawId=409M50000400011. The wholesale price data are retrieved from Japan LP Gas Association, http://www.j-lpgas.gr.jp/stat/kakaku/fails/kakakusuii19.xls

period (total number of regions)	number of lag	retail price: Dickey–Fuller statistic maximum	retail price: Dickey–Fuller statistic minimum	retail price: Dickey–Fuller statistic mean	retail price: number of regions with 0.05 or less p-value	wholesale price: Dickey-Fuller statistic (p-value)
~ Feb. 2006 (281)	0	0.653	-5.346	-2.521	35	-1.187(0.901)
	1	0.564	-4.549	-2.283	24	-1.256(0.875)
	3	1.065	-6.708	-2.451	51	-0.800(0.957)
	6	2.373	-3.940	-1.330	4	-0.826(0.954)
	9	2.671	-3.382	-1.424	0	-1.916(0.609)
	12	3.224	-4.095	-1.476	2	-0.622(0.972)
Apr. 2006 ~ (233)	0	0.636	-4.129	-2.114	3	-3.187(0.096)
	1	-0.618	-3.184	-2.061	0	-3.296(0.078)
	3	0.495	-3.250	-2.225	0	-2.535(0.357)
	6	-1.082	-4.853	-2.650	10	-2.453(0.390)
	9	0.662	-5.179	-2.662	24	-2.223(0.485)
	12	0.595	-4.168	-1.616	4	-2.307(0.450)

TABLE 1: Results of Dickey–Fuller unit root test for retail and wholesale price

Next, cointegration between the retail and the wholesale prices for each region is tested following the "Pz" test in Phillips and Ouliaris' (1990) study. The result is shown in Table 2. As the critical test statistics of 10, 5, and 1 percent for this case are 45.58, 55.22, and 71.92, respectively, no cointegration is detected for any regions.

	Pz test statistic	
periods	$\sim$ Feb. 2006	Apr. 2006 $\sim$
maximum	24.85	39.97
minimum	2.71	21.45
mean	9.26	27.78

Table 2: Results of the cointegration test between retail and wholesale prices

3.2 Estimation of the coefficient of the pass-through

Based on these checks, an auto-regressive model (AR(p)) is prepared for the difference in prices with the exogeneous variable in order to estimate the coefficient of the pass-through:

$$\Delta r_{jt} = \sum_{i=1}^{p} \theta_{ji}^{0} \Delta r_{j,t-i} + \sum_{i=1}^{12} \theta_{ji}^{1} \Delta w_{j,t-i} + \epsilon_{jt},$$

where $\Delta r_{ji} \stackrel{\text{def}}{=} r_{ji} - r_{j,i-1}$, r_{jt} and w_{jt} denote the retail price of region j and the wholesale price at time t, respectively, and ϵ_{jt} is white noise. Selection of lag length p is determined by the minimum value of the Akaike information criterion. Note that the lag length for the exogeneous wholesale price is fixed at 12. The effect of changes in wholesale price is considered retrospectively until 24 months before. Because presenting a price list at the time of contract is required by a guideline set by the Ministry of Economy, Trade and Industry. Suppliers must also notify consumers before revising the price list, even when the revision is due to a change in the CIF gas price. It may thus take time to pass the change in wholesale price onto the retail price. As the wholesale price is one series assuming a uniform price over Japan, this is not a vector autoregressive model on retail and wholesale prices for each region, although the actual estimation is not different.

The coefficient of the pass-through of region j is defined as

$$\Theta_j \stackrel{\text{def}}{=} \sum_{i=1}^{12} \theta_{ji}^1$$

The results of the estimation of the coefficient are summarized in Table 3. For the period before February 2006, the mean of the estimates of the coefficient is 1.014. In 114 regions among the 139 regions, the F-statistics of the regression to estimate the AR model show a validity of 0.05 probability; the null hypothesis that the coefficient is 1 is not rejected. For the period after April 2006, the mean is 0.602; in the 119 regions among 205 regions where the regression is valid, the null hypothesis is not rejected.

periods	~ Feb. 2006 (n=139)		Apr. 2006 ~ (n=205)	
	Θ_j	p-value	Θ_j	p-value
maximum	3.454	0.999	1.286	0.993
minimum	-0.881	0.000	0.094	0.000
mean	1.014	0.440	0.602	0.206
s.d.	0.681	0.320	0.218	0.260
number of regions with 0.05 or larger p-value		114		119

TABLE 3: Estimates of the coefficient of the pass-through (p-value in the table denotes the test statistic of the null hypothesis and the coefficient is 1.)

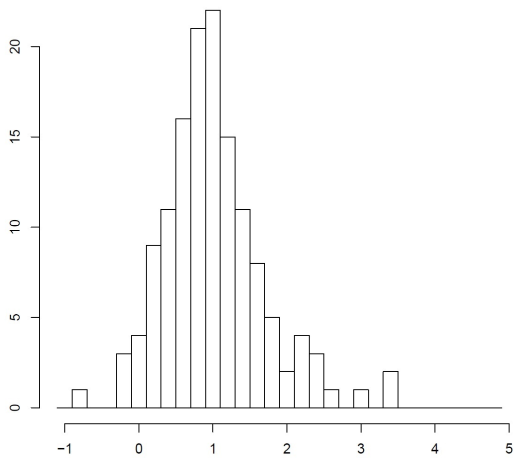

FIGURE 1: Distribution of Θ_j for the period before February 2006.

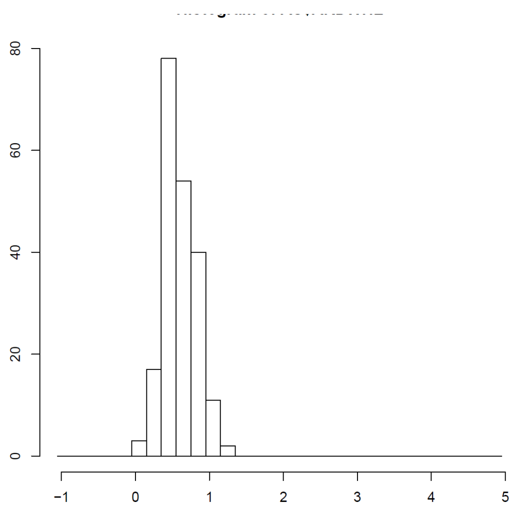

FIGURE 2: Distribution of Θ_j for the period after April 2006.

3.3 Observed coefficient of the pass-through

As shown in the table and the figures of the distribution of estimates, although the variance is high, in a typical regional market, a price shock in the wholesale price is passed on to retail prices in full within 24 months for the period before February 2006 (Figure 1). For the period after April 2006, the mean of the coefficients is less than 1 although positive, and in 86 regions among the 205, the null hypothesis that the coefficient is 1 is rejected (Figure 2). Note that a retail price series is the series of the mean of sample retail prices in a region. If in a region some stores set the retail price based on the reservation price of consumers, while other stores set the price based on the cost plus the switching cost premium, the coefficient estimated is supposed to show a value between 0 and 1. This possibility can be examined by comparing the correlation between the coefficient and the absolute price level. As the number of stores that set the price based on the reservation price increases, the coefficient is expected to be lower and the average price higher because the reservation price is higher than the sum of the gas cost and the switching cost premium.

The estimated regression equations are as follows:

$$
\begin{array}{rllll}
r_j = & -4.614 & -135.4^b\Theta_j & +1.012^a\bar{r}_j & +679.6^a DI_j \\
 & (182.9) & (64.9) & (0.0238) & (117.8) \\
 & & & \bar{R}^2 = 0.901 & F(3,201) = 621.2^a \\
r_j/\bar{r}_j = & 1.011^a & -0.0183^b\Theta_j & +0.0893^a DI_j & \\
 & (0.005517) & (0.00864) & (0.0157) & \\
 & & & \bar{R}^2 = 0.139 & F(2,202) = 17.51^a
\end{array}
$$

,

where $\bar{r}_j$ is the mean of the retail price in the prefecture where the region j belongs to, DI_j is a dummy variable that takes the value of 1 when region j is a remote island, standard deviations of the coefficient appear in parentheses, and superscripts a and b indicate that the coefficient or the test statistics are significant with 1 and 5 percent two-sided tests, respectively. The mean price of prefectures $\bar{r}_j$ is included to control the heterogeneity in demand and cost structures among regions. For example, delivery efficiency is assumed to be dependent on demand density (amount of demand per unit area). Then, if the demand density is assumed to be not different in a prefecture, the average price is supposed to control the effect of demand density. The remote island dummy is introduced because LP gas supply was charged higher in those areas as the cost structure there is different from other areas. When regions of remote islands are excluded, the estimated regression equations are as follows:

$$
\begin{array}{llll}
r_j = 5.789 & -155.8^b\Theta_j & +1.012^a\bar{r}_j & \\
\quad (182.3) & (65.5) & (0.0237) & \\
 & & \bar{R}^2 = 0.901 & F(2,199) = 916.2^a \\
r_j/\bar{r}_j = 1.013^a & -0.0210^b\Theta_j & & \\
\quad (0.005561) & (0.00872) & & \\
 & & \bar{R}^2 = 0.023 & F(1,200) = 5.788^b
\end{array}
$$

Thus, the negative correlation between the relative retail price and the coefficient of the pass-through does not contradict the hypothesis that, as the number of stores that set their retail prices based on the reservation price increases in a region, the mean of the retail price in the region increases, while the coefficient of the pass-through observed decreases.

The last check of the validity of the hypothesis is to compare the variance of the mean and that of the maximum. Currently the Oil Information Center publishes the mean, maximum, and minimum prices of the last period for each region, but only the mean price is shown as historical time-series data. Therefore, AR analysis cannot be conducted for the maximum price. The maximum price in a region is expected to reflect the reservation price, with a higher possibility than the mean price. If the result of an AR analysis on the maximum price could be compared with the result of the mean price, a more direct check on the hypothesis would have been possible. Previously, however, the time-series data of the maximum price had been available on the website of the center for a period.[18] The data available are from August 1998 till October 2002; this length is not sufficient for AR analysis, but an additional test is conducted on the limited data. In this test, the variance of the mean prices and that of the maximum price is compared. If the maximum price reflects the reservation price, while the mean price is below the reservation price, the variance of the mean should be greater than that of the maximum. This is because the reservation price is expected to fluctuate to a weaker degree compared with the cost plus switching cost premium. Electricity and kerosene are energy sources used as substitutes to LP gas. The reservation price is determined in this case by the limit price at which consumers would change their energy source, namely, the alternative fuel cost and switching cost. Electricity price is stabler than the LP gas cost. Although the kerosene price should fluctuate based on imported oil prices, switching between LP gas and kerosene is far more difficult than switching

[18]The center does not provide the historical data of the maximum price for February 2020 as of now.

between LP gas from a nearby store and LP gas from another nearby store. Therefore, the fluctuation of fuel cost in the short run is difficult to reflect in the reservation price; that is, when the prices are low for long enough, consumers might be prompted to change their source of energy. This assumption leads to the hypothesis that the variance of the mean price is greater than that of the maximum price, given that the variances are those of the sample statistics.

In the period of observation, the number of regions and investigated stores are 291 and around $3,000$, which implies the number of stores in a region is 10.3 on average. Because the center does not publicize the distribution of the number of stores investigated for each region, it is assumed to be around 10 here. When the samples' x_is are extracted from a certain population, the expected values of $sd(\sum_{i=1}^{n} x_i/n)/sd(\max_{i=1 \cdot \cdot n} x_i)$ are 0.579, 0.539, and 0.508, for $n = 8$, 10, and 12, respectively, where sd is the standard deviation. The points (x_j, y_j), where $x_j = sd\left(\sum_t r_{jt}/26\right)$ and $y_j = sd\left(\max_t r_{jt}\right)$ for each region j, are plotted in Figure 3. The number of time-series price data is 26 for each region. The bold line in Figure 3 shows the result of the ordinary least squares regression with zero intercept, and the two dotted lines show the confidence intervals at 95 percent confidence. Other lines in the figure indicate $y = 0.579x$, $y = 0.539x$, and $y = 0.508x$, respectively. The variance of the maximum of the samples in each region is clearly lower than that expected from the variance of the mean and the expected number of samples. This evidence does not contradict with the theory implied by the model analyses in the previous section.

Thus, the analyses on the time-series data of the regional LP gas retail price implies that, before 2006 the fluctuations in wholesale LP gas prices in most areas had been passed through on to the retail prices in 24 months. This result best describes a one-stage game without commitment to stores' territories. After 2006, the competition in some areas changed to that described by a two-stage game with commitment to territories. The price increased up to the reservation price, and the average of the coefficient of the pass-through decreased to a value below 1.

Over the years, the government has implemented policies to lower the switching cost in order to promote competition among retailers.[19] However, as explained in section 1, these policies have not been as effective as ex-

[19]These policies of the Agency of Natural Resources and Energy in Japan include the Liquefied Petroleum Gas Act, operational and interpretive notices of the Act, and Guidelines for Proper Transactions in Retail Sales of Liquefied Petroleum Gas. These policies aim to create more transparencies in trade conditions and resolve disputes between consumers and retailers that arise when a contract is terminated.

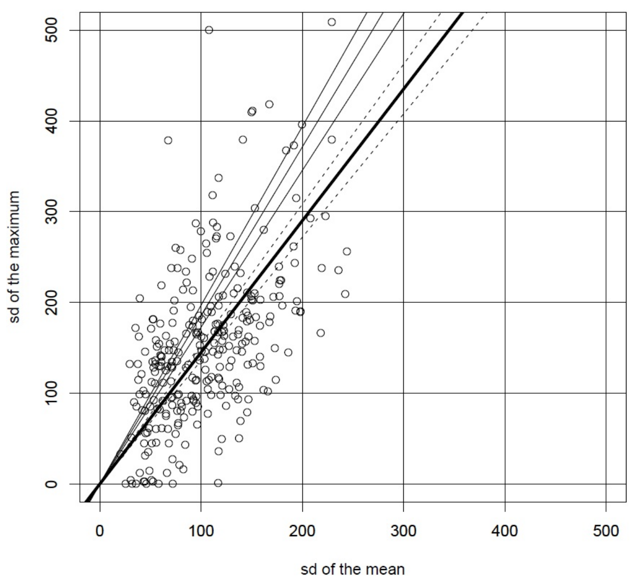

FIGURE 3: Relationship between the standard deviation of the mean and that of the maximum

pected, and there is ample evidence of unhealthy competition. We surmise that, in the areas where the prices stick to the ceiling reservation price, the policies that promote competition with the adjacent market for lowering the reservation price or invalidating the declaration of the DA become necessary for suppressing the market price.

4 Conclusion

This study constructs a bilateral spatial competition model. One of the features of the model is the expansion of the strategic space of retailers by including the size of the free-delivery area in addition to the price level. The other feature of the model is its incorporation of the switching cost in competition following Shy (1996), Shy (2002), Shy and Oz (2001). After explaining the trivial outcomes of the Nash equilibrium, where retailers cover all the markets and there is no distinguished difference from a simple price competition model, the extended UPP is applied to analyze competition wherein retailers adjust the sizes of their free delivery areas to avoid the threat of undercutting activities by rival retailers.

The model analyzed has two alternatives: a one-stage model where retailers do not commit to the sizes of their free delivery areas, and two-stage model where retailers can commit to the sizes of their free delivery areas. In the former, the interior equilibrium with the UPP is proved to exist, which implies the price is determined by the magnitude of the switching cost. For the latter mode, the price sticks to the ceiling reservation price at the equilibrium with UPP. For both equilibria, there exist exclusive delivery areas as natural territories surrounded by a competition area where both retailers provide delivery services.

These two outcomes are identified empirically by estimating how the fluctuations of wholesale price are passed through to the retail price. The estimation in the regional retail supply market of LP gas in Japan implies a gradual change from a one-stage model-type competition to a two-stage model-type competition, which may cause an increase in retail price. If so, in the market, the policies that promote competition with adjacent markets or the invalidation of the declaration of the delivery area is expected to mitigate the market power caused by the switching cost.

One possible contribution of this article is that it shows the usefulness of the UPP when combined with the assumption of ceiling reservation prices, thus mitigating the upward-stability problem of undercut proof prices. So far, there are few applications relating to the undercut proof prices other than direct estimations of the switching cost.

Further, we show that the retail territories not forced as vertical constraints are constructed endogenously in the model. Retailers, we find, may refrain from expanding their supply area for a strategic reason even when such supply contributes their profits. Standard spatial competition models seem to have territories; however, they are the mere result of consumers' selections, with the outcome of a high trip cost for some consumers.

We expect more research on the pricing schemes employed by retailers. Most retailers of LP gas in Japan charge a fixed fee as well as a gas cost, that is, a two-part tariff. The ratio of fixed costs in retail LP gas supply is not significantly higher, so retailers have no rationale for these two-part tariffs. However, as explained in section 1, household customers often misunderstand the gas service charges as a regulated "public utility rate." This is due to the similarity of the service with city gas pipeline supply, where two-part tariffs are common. This confusion may lead to a predisposition to accept two-part tariffs. We surmise that this pricing scheme affects competition among retailers. Thus, the possibility of incorporating the two-part tariff pricing scheme in the model analyzed here should be considered.

Nevertheless, our findings have scope for further exploration. For instance, we only investigate the distribution of the coefficient of the pass-through. More scrutinized analysis may shed light on the reality in each regional market. The territories should be observable geographically if they exist. The rival competitors are also different for each regional market. Further, although kerosene is assumed to be the rival household energy source, electricity may be more appropriate as a rival source in some regions. Most retailers even consider the competition with electricity as the greatest threat leading to customer attrition.[20]

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Proofs of Propositions and Lemmata

Proof of Lemma 1.

$$(\mathbf{A})\ Q_i\left(\frac{t}{2}\right) = \int_{A_0\cap A_1} q_i^*(x)\left(\frac{t}{2} - tx_i^*\right)dx \geq tq_i^*\left(\frac{1}{2}\right)\int_{1-\tilde{r}_1}^{\tilde{r}_0}\left(\frac{1}{2} - x_i^*\right)dx$$

$$= tq_i^*\left(\frac{1}{2}\right)\frac{(\tilde{r}_j - \tilde{r}_i)(\tilde{r}_0 + \tilde{r}_1 - 1)}{2}$$

$$\because \left(q_i^*(x) - q_i^*\left(\frac{1}{2}\right)\right)\left(x_i^* - \frac{1}{2}\right) \leq 0, \quad \text{for } x \in U.$$

Note that $q_i^*(x)$ do not increase as x_i^* increase.

$$(\mathbf{B})\ \frac{D_0}{2} - D_1 = \int_{1-\tilde{r}_1}^{\tilde{r}_0} q(x)\left(\frac{1}{2} - x\right)dx \geq q\left(\frac{1}{2}\right)\int_{1-\tilde{r}_1}^{\tilde{r}_0}\left(\frac{1}{2} - x\right)dx$$

$$= q\left(\frac{1}{2}\right)\frac{(\tilde{r}_1 - \tilde{r}_0)(\tilde{r}_0 + \tilde{r}_1 - 1)}{2}$$

$$(\mathbf{C})\quad J_i(v) - Q_i(v) = \int_{A_0\cap A_1}(1 - q_i^*(x))\,(v - tx_i^*)dx > 0,$$

$$J_i(0) - Q_i(0) = \int_{A_0\cap A_1}(1 - q_i^*(x))\,(-tx_i^*)dx < 0,$$

$$(\mathbf{D})\quad (\tilde{r}_0 + \tilde{r}_1 - 1) - D_0\,(A_0\cap A_1) = \int_{A_0\cap A_1}(1 - q(x))dx > 0.$$

□

Proof of Lemma 2. As $\tilde{r}_0 + \tilde{r}_1 > 1$ is assumed, $A_0 \cap A_1 \neq \emptyset$ and $\tilde{r}_i > 0$ ($\because \tilde{r}_i > 1 - \tilde{r}_j \geq 0$), as well as

$$Q_i(v) = \int_{A_0\cap A_1} q_i^*(x)(v - tx_i^*)dx > 0, \tag{A1}$$

$$Q_i(0) = \int_{A_0\cap A_1} q_i^*(x)(-tx_i^*)dx < 0, \tag{A2}$$

$$E_i(v) = v(1 - \tilde{r}_j) - t\frac{(1 - \tilde{r}_j)^2}{2} > (1 - \tilde{r}_j)\left(v - \frac{t}{2}\right) > 0, \tag{A3}$$

from assumptions 1 and 2.

Define

$$\begin{aligned}
\phi_i^1(p) &\stackrel{\text{def}}{=} E_i(p) + J_i(p) - E_i(v),\\
\phi_i^2(p) &\stackrel{\text{def}}{=} E_i(p) + J_i(p) - \left(E_i(v) + Q_i(v)\right),\\
\phi_i^3(p) &\stackrel{\text{def}}{=} E_i(p) + J_i(p) - \left(E_i(p) + Q_i(p)\right),\\
\phi_i^4(p) &\stackrel{\text{def}}{=} E_i(p) + Q_i(p) - E_i(v).
\end{aligned}$$

For $p \in [0, v]$,

$$E_i(p) + J_i(p) = p\tilde{r}_i - t\frac{\tilde{r}_i^2}{2}.$$

(A,C,D,F,H) With Assumptions 1 and 2,

$$\begin{aligned}
\phi_i^1(0) &= E_i(0) + J_i(0) - E_i(v) = -t\frac{\tilde{r}_i^2}{2} - E_i(v) < 0, \quad (\because (A3))\\
\phi_i^2(0) &= \phi_i^1(0) - Q_i(v) < 0, \quad (\because (A1))\\
\phi_i^4(0) &= E_i(0) - E_i(v) + Q_i(0) < 0, \quad (\because (A2))\\
{\phi^1}'_i(p) &= {\phi^2}'_i(p) = E_i'(p) + J_i'(p) = \tilde{r}_i > 0 \quad \text{for } p \in [0, v],\\
{\phi^4}'_i(p) &= E_i'(p) + Q_i'(p) = (1 - \tilde{r}_j) + \int_{A_0 \cap A_1} q_i^*(x)dx > 0\\
{\phi^4}'_i(p) &= (1 - \tilde{r}_j) + \int_{A_0 \cap A_1} q_i^*(x)dx < (1 - \tilde{r}_j) + \int_{A_0 \cap A_1} dx = \tilde{r}_i
\end{aligned}$$

and,

$$\begin{aligned}
\phi_i^1(v) &= \left(v\tilde{r}_i - t\frac{\tilde{r}_i^2}{2}\right) - \left(v(1 - \tilde{r}_j) - t\frac{(1 - \tilde{r}_j)^2}{2}\right)\\
&= (\tilde{r}_0 + \tilde{r}_1 - 1)\left(v - t\frac{\tilde{r}_i + 1 - \tilde{r}_j}{2}\right)\\
&> (\tilde{r}_0 + \tilde{r}_1 - 1)\left(t - t\frac{\tilde{r}_i + 1 - \tilde{r}_j}{2}\right) = t(\tilde{r}_0 + \tilde{r}_1 - 1)\frac{(1 - \tilde{r}_i) + \tilde{r}_j}{2} > 0,\\
\phi_i^2(v) &= \phi_i^1(v) + Q_i(v) > 0. \quad (\because (A1))\\
\phi_i^4(v) &= Q_i(v) > 0. \quad (\because (A1))
\end{aligned}$$

Thus, we derive a solution $p_i^k \in (0, v)$ of the equation $k \in \{J, Q, VQ\}$. **(A)**. Especially, as functions $\phi_i^k(p)$, $k \in \{1, 2, 3, 4\}$ are monotonously increasing functions, **(C,D,F)** are proved. Furthermore, because $\phi_i^2(p) = \phi_i^1(p) - Q_i(v) < \phi_i^1(p)$ for $p \in [0, v]$, the locus of the function $\phi_i^1(p)$ always locates above the locus of $\phi_i^2(p)$, so that **(H)** is proved.

(B,E) As for the case p_i^{JQ}:

$$\begin{aligned}
&{\phi^{3}}'_i(p) = J_i'(p) - Q_i'(p) = \int_{A_0\cap A_1} (1-q_i^*(x))dx > 0,\\
&\phi_i^3\left(\frac{t(\tilde{r}_i+1-\tilde{r}_j)}{2}\right) = \int_{A_0\cap A_1}(1-q_i^*(x))\left(\frac{t(\tilde{r}_i+1-\tilde{r}_j)}{2} - tx_i^*\right)dx\\
&\qquad \leq t\left(1-q_i^*\left(\frac{t(\tilde{r}_i+1-\tilde{r}_j)}{2}\right)\right)\int_{A_0\cap A_1}\left(\frac{\tilde{r}_i+1-\tilde{r}_j}{2} - x_i^*\right)dx = 0,\\
&\phi_i^3(t\tilde{r}_i) = \int_{A_0\cap A_1}(1-q_i^*(x))(t\tilde{r}_i - tx_i^*)dx > 0.
\end{aligned}$$

Note that

$$\left(q_i^*(x) - q_i^*\left(\frac{\tilde{r}_i+1-\tilde{r}_j}{2}\right)\right)\left(x_i^* - \frac{\tilde{r}_i+1-\tilde{r}_j}{2}\right) \leq 0$$

because $q_i^*(x)$ do not increase as x_i^* increase. Also, as function $\phi_i^3(p)$ is a monotonously increasing function, **(E)** is proved.

(G) As

$$\phi_i^1(p) = \tilde{r}_i p + \frac{t}{2}\left(\tilde{r}_j^2 - 2\tilde{r}_j - \tilde{r}_i^2 + 1\right) - v\left(1-\tilde{r}_j\right),$$

$$\begin{aligned}
&\tilde{r}_j\phi_i^1(p_i^J) - \tilde{r}_i\phi_j^1(p_j^J) =\\
&\quad \tilde{r}_i\tilde{r}_j(p_i^J - p_j^J) - \frac{1}{2}(\tilde{r}_i-\tilde{r}_j)(\tilde{r}_i+\tilde{r}_j-1)(2v+t(\tilde{r}_i+\tilde{r}_j-1)) = 0.
\end{aligned}$$

because $\phi_i^1(p_i^J) = 0$. Then, $\operatorname{sign}\left(p_i^J - p_j^J\right) = \operatorname{sign}\left(\tilde{r}_i - \tilde{r}_j\right)$.

(I) Note that $\phi_i^1(p_i^J) = 0$, $\phi_i^3(p_i^{JQ}) = 0$, and $\phi_i^4(p_i^{VQ}) = 0$. If $p_i^{JQ} \lesseqgtr p_i^J$, then $\phi_i^3(p_i^J) \gtreqless 0$. So, $\phi_i^4(p_i^J) = \phi_i^1(p_i^J) - \phi^3(p_i^J) \lesseqgtr 0 = \phi_i^4(p_i^{VQ})$, which means $p_i^J \lesseqgtr p_i^{VQ}$ as $\phi_i^4(p)$ is an increasing function. □

proof of Lemma 3. Because

$$\pi_0((p,\tilde{r}_0),(\tilde{p}_1,\tilde{r}_1)) = \begin{cases} E_0(p) + J_0(p) & \text{for } p\in[0,\tilde{p}_1)\\ E_0(p)+Q_0(p) & \text{for } p = \tilde{p}_1\\ E_0(p) & \text{for } p\in(\tilde{p}_1, v]. \end{cases}$$

and $E_0(p)$ and $J_0(p)$ are monotonously increasing functions of $p \in [0,v]$, maximization of $\pi_0\left((p,\tilde{r}_0),(\tilde{p}_1,\tilde{r}_1)\right)$ is attained only when $p = \tilde{p}_1 - \epsilon_0$, $p =$

$\tilde{p}_1$, or $p = v$, where ϵ_0 is an arbitrary small positive number. By the same manner, maximization of $\pi_1\left(\left(\tilde{p}_0, \tilde{r}_0\right), \left(p, \tilde{r}_1\right)\right)$ is attained only when $p = \tilde{p}_0 - \epsilon_1$, $p = \tilde{p}_0$, or $p = v$, where ϵ_1 is also an arbitrary small positive number.

For each case, the values of the profit function $\pi_i\left(\tilde{s}_0, \tilde{s}_1\right)$ are $E_i(\tilde{p}_j - \epsilon_i) + J_i(\tilde{p}_j - \epsilon_i)$, $E_i(\tilde{p}_j) + Q_i(\tilde{p}_j)$, and $E_i(v)$, respectively. In this proof, each is denoted as EJ_i, EQ_i, and EV_i, respectively. Between EJ_i, EQ_i, and EV_i, which one is chosen depends on the relationship between the prices p_i^J and p_i^{JQ}, besides the price $\tilde{p}_j$ their rival sets.

Because p_i^Js are unique solutions of equation $E_i(p) + J_i(p) - E_i(v) = 0$, they are specified as

$$p_i^J = \frac{2v(1 - \tilde{r}_j) + t\left(\tilde{r}_i^2 - (1 - \tilde{r}_j)^2\right)}{2\tilde{r}_i} \tag{A4}$$

Then,

$$\begin{aligned} p_i^J - t\tilde{r}_i &= \frac{2v(1 - \tilde{r}_j) + t\left(\tilde{r}_i^2 - (1 - \tilde{r}_j)^2\right)}{2\tilde{r}_i} - t\tilde{r}_i \\ &= \frac{1 - \tilde{r}_0^2 - \tilde{r}_1^2 + (v - t)\left(1 - \tilde{r}_j\right)}{2\tilde{r}_i}, \end{aligned}$$

Therefore, if $\tilde{r}_0^2 + \tilde{r}_1^2 < 1$, then $p_i^J > t\tilde{r}_i$, while if $\tilde{r}_0^2 + \tilde{r}_1^2 > 1$ and v is sufficiently near to t, then $p_i^J \leq t\tilde{r}_i$. When $p_i^J > t\tilde{r}_i$, Lemma 2(B) leads $p_i^J > t\tilde{r}_i > p_i^{JQ}$, while when $p_i^J \leq t\tilde{r}_i$ it is possible that $p_i^J \leq p_i^{JQ}$.

1. **The case $p_i^{JQ} < p_i^J$**
   By Lemma 2(I), $p_i^{JQ} < p_i^J < p_i^{VQ}$. There are two cases for the selection of the price by store i.

   1-1. **The case $\tilde{p}_j \in [0, p_i^J]$**
   Because $\tilde{p}_j \leq p_i^J$ and Lemma 2(C)

   $$\begin{aligned} EV_i = E_i(p_i^J) + J_i(p_i^J) &\geq E_i(\tilde{p}_j) + J_i(\tilde{p}_j) \\ &> E_i(\tilde{p}_j - \epsilon_j) + J_i(\tilde{p}_j - \epsilon_j) = EJ_i. \end{aligned}$$

   Further, because $p_i^{VQ} > p_i^J$ and Lemma 2(F)

   $$\begin{aligned} EV_i = E_i(p_i^{VQ}) + Q_i(p_i^{VQ}) &> E_i(p_i^J) + Q_i(p_i^J) \\ &\geq E_i(\tilde{p}_j) + Q_i(\tilde{p}_j) = EQ_i \end{aligned}$$

   Thus, $\tilde{p}_i = v$.

1-2. **The case** $\tilde{p}_j \in (p_i^J, v]$

Because $\tilde{p}_j > p_i^J > p_i^{JQ}$ and Lemma 2(E),

$$E_i(\tilde{p}_j) + J_i(\tilde{p}_j) > E_i(\tilde{p}_j) + Q_i(\tilde{p}_j).$$

As ϵ_i is an arbitrary small number,

$$E_i(\tilde{p}_j - \epsilon_i) + J_i(\tilde{p}_j - \epsilon_i) > E_i(\tilde{p}_j) + Q_i(\tilde{p}_j).$$

This implies $EJ_i > EQ_i$. Because $\tilde{p}_j > p_i^J$ and ϵ_i is an arbitrary small number,

$$EJ_i = E_i(\tilde{p}_j - \epsilon_i) + J_i(\tilde{p}_j - \epsilon_i) > E_i(p_i^J) + J_i(p_i^J) = EV_i.$$

Thus, $\tilde{p}_i = \tilde{p}_j - \epsilon_i$.

2. **The case** $p_i^J \leq p_i^{JQ}$

By Lemma 2(I), $p_i^{VQ} \leq p_i^J \leq p_i^{JQ}$. There are three cases for the selection of the price by store i.

2-1. **The case** $\tilde{p}_j \in [0, p_i^{VQ})$

Because of $\tilde{p}_j < p_i^{VQ} \leq p_i^J$ and Lemma 2(C),

$$\begin{aligned} EV_i = E_i(p_i^J) + J_i(p_i^J) &> E_i(\tilde{p}_j) + J_i(\tilde{p}_j) \\ &> E_i(\tilde{p}_j - \epsilon_i) + J_i(\tilde{p}_j - \epsilon_i) = EJ_i. \end{aligned}$$

Also, because of $\tilde{p}_j < p_i^{VQ}$ and Lemma 2(F),

$$EV_i = E_i(p_i^{VQ}) + Q_i(p_i^{VQ}) > E_i(\tilde{p}_j) + Q_i(\tilde{p}_j) = EQ_i.$$

Thus, $\tilde{p}_i = v$.

2-2. **The case** $\tilde{p}_j \in [p_i^{VQ}, p_i^{JQ}]$

Because of $\tilde{p}_j \leq p_i^{JQ}$ and Lemma 2(E),

$$\begin{aligned} EQ_i = E_i(\tilde{p}_j) + Q_i(\tilde{p}_j) &\geq E_i(\tilde{p}_j) + J_i(\tilde{p}_j) \\ &> E_i(\tilde{p}_j - \epsilon_i) + J_i(\tilde{p}_j - \epsilon_i) = EJ_i. \end{aligned}$$

Because $\tilde{p}_j \geq p_i^{VQ}$ and Lemma 2(F),

$$EQ_i = E_i(\tilde{p}_j) + Q_i(\tilde{p}_j) \geq E_i(p_i^{VQ}) + Q_i(p_i^{VQ}) = EV_i.$$

The equality holds only when $\tilde{p}_j = p_i^{VQ}$. Therefore, if $\tilde{p}_j = p_i^{VQ}$ then $\tilde{p}_i = \tilde{p}_j$ or $\tilde{p}_i = v$, while if $\tilde{p}_j \in (p_i^{VQ}, p_i^{JQ}]$, then $\tilde{p}_i = \tilde{p}_j$

2-3. **The case** $\tilde{p}_j \in (p_i^{JQ}, v]$

Because $\tilde{p}_j > p_i^{JQ}$ and Lemma 2(E),

$$E_i(\tilde{p}_j) + J_i(\tilde{p}_j) > E_i(\tilde{p}_j) + Q_i(\tilde{p}_j).$$

As ϵ_i is an arbitrary small number,

$$E_i(\tilde{p}_j - \epsilon_i) + J_i(\tilde{p}_j - \epsilon_i) > E_i(\tilde{p}_j) + Q_i(\tilde{p}_j),$$

which means $EJ_i > EQ_i$. Because $\tilde{p}_j > p_i^{JQ} \geq p_i^J$ and ϵ_i is an arbitrary small number,

$$EJ_i = E_i(\tilde{p}_j - \epsilon_i) + J_i(\tilde{p}_j - \epsilon_i) > E_i(p_i^J) + J_i(p_i^J) = EV_i$$

Thus, $\tilde{p}_i = \tilde{p}_j - \epsilon_i$.

Thus, each price strategy, $\tilde{p}_i = \tilde{p}_j - \epsilon_i,\ \tilde{p}_j,\ v$, has situations wherein it is valid. However, any combinations with strategy $\tilde{p}_i = \tilde{p}_j - \epsilon_i$ or $\tilde{p}_i = v$ are not consistent. First, if $\tilde{p}_i = \tilde{p}_j - \epsilon_i$, it contradicts both $\tilde{p}_j = \tilde{p}_i - \epsilon_j$ and $\tilde{p}_j = \tilde{p}_i$ because ϵ_is are positive. $\tilde{p}_i = \tilde{p}_j - \epsilon_i$ also contradicts $\tilde{p}_j = v$ because $\tilde{p}_j = v$ requires $\tilde{p}_i \leq p_j^J < v$ (case 1-1) or $\tilde{p}_i \leq p_j^{VQ} < v$ (cases 2-1, 2-2), where $\tilde{p}_i = v - \epsilon_i$ cannot be satisfied, as ϵ_i is an arbitrary positive small number. Second, if $\tilde{p}_i = v$, then the cases are limited to $\tilde{p}_j \leq p_i^J < v$ (case 1-1) or $\tilde{p}_j \leq p_i^{VQ} < v$ (cases 2-1, 2-2), which are not consistent with both $\tilde{p}_j = v$ and $\tilde{p}_j = \tilde{p}_i = v$. They are also not consistent with $\tilde{p}_j = \tilde{p}_i - \epsilon_j = v - \epsilon_j$, as ϵ_j is an arbitrary positive small number. Thus, there is no combination of consistent strategies remaining other than the cases of $\tilde{p}_i = \tilde{p}_j$, which are valid in the case of $\tilde{p}_i \in [p_j^{VQ}, p_j^{JQ}]$ (case 2-2). Note that the case is valid only when $p_i^{JQ} \geq p_i^J$ for both of $i \in \{0, 1\}$ (case 2), which is possible when $t\tilde{r}_i \geq p_i^J$. Lastly, as $EV_i = v(1 - \tilde{r}_j) - t\,(1 - \tilde{r}_j)^2/2 \geq 0$, and EV_i is always in the selection of the strategy of store i, the realized profits in (2) are not negative.

Therefore, the condition for the existence of the combination of strategies that satisfies (2) is that the region $\bigcap_{i=0,1} [p_i^{VQ}, p_i^{JQ}]$ is not empty. The necessary conditions for the region not being empty include

(A) $t\tilde{r}_i \geq p_i^J$, which is required for $p_i^{JQ} \geq p_i^J$; then, $p_i^{VQ} \leq p_i^{JQ}$.

(B) $p_i^{VQ} \leq p_j^{JQ}$.

As is shown earlier in this proof, $\tilde{r}_0^2+\tilde{r}_1^2-1>0$ and a sufficiently large value of v compared with the value of t are required for (A). Because p_i^{JQ} and p_i^{VQ} are the solutions of the equations $\phi_i^3(p)=0$ and $\phi_i^4(p)=0$, respectively, where $\phi_i^3(p)$ and $\phi_i^4(p)$ are functions defined in the proof of Lemma 2, and because $\phi_i^{3\prime}>0$ and $\phi_i^{4\prime}>0$, as shown in the proof, a necessary condition for (B) is $\phi_i^4(p_j^{JQ})\geq 0$. Because p_i^{JQ} is the solution to $\phi_i^3(p)=0$,

$$\begin{aligned}\phi_i^3(p_i^{JQ}) &= \int_{A_0\cap A_1}(1-q_i^*(x))\left(p_i^{JQ}-tx_i^*\right)dx\\ &= \int_{A_0\cap A_1} q_j^*(x)\left(p_i^{JQ}-tx_i^*\right)dx=0.\end{aligned} \tag{A5}$$

Then,

$$\begin{aligned}\phi_j^4(p_i^{JQ}) =& \int_{A_0\cap A_1} q_j^*(x)\left(p_i^{JQ}-tx_j^*\right)dx+\int_{A_i^c}\left(p_i^{JQ}-tx_j^*\right)dx\\ &-\int_{A_i^c}\left(v-tx_j^*\right)dx\\ =&t\int_{A_0\cap A_1} q_j^*(x)\left(2x_i^*-1\right)dx-(1-\tilde{r}_i)\left(v-p_i^{JQ}\right)\quad (\because (A5))\\ \leq& t\int_{\frac{1}{2}}^{\max\left(\tilde{r}_i,\frac{1}{2}\right)}\left(2x_i^*-1\right)dx-(1-\tilde{r}_i)\left(v-t\tilde{r}_i\right)\quad (\because \text{Lemma2}(B))\\ \leq& \frac{t}{4}-v(1-\tilde{r}_i)\end{aligned}$$

Therefore, if $\phi_j^4(p_i^{JQ})\geq 0$, then $t/4-v(1-\tilde{r}_i)\geq 0$. This is a necessary condition of (B).

When $\tilde{r}_i=1$ for both $i\in\{0,1\}$, then $p_i^J=t/2$ from (A4). Evaluating $\phi_i^3\left(t/2\right)$ with Lemma 1(B),

$$\phi_i^3\left(\frac{t}{2}\right)=\int_U(1-q_i^*(x))\left(\frac{t}{2}-tx_i^*\right)dx=-t\frac{D_0(U)-2D_1(U)}{2}\leq 0.$$

This results in $p_i^{JQ}\geq t/2=p_i^J$. Therefore, there exists $\tilde{p}$ such that $p_i^J\leq\tilde{p}\leq p_i^{JQ}$. For the existence, there is no restriction in the function $q(x)$ and the value of v.

Lastly, note that from Lemma 2(B) $p_i^{JQ}<t\tilde{r}_i$; then, $\tilde{p}_i\leq p_i^{JQ}<t\tilde{r}_i$. □

Proof of Lemma 4. Given $(\tilde{r}_0,\tilde{r}_1)$, $(\tilde{p}_0,\tilde{p}_1)$ denotes the prices of the stores such that

$$\tilde{p}_i=\operatorname*{argmax}_{p\in[0,v]}\pi_i((p,\tilde{r}_i),\tilde{s}_j). \tag{A6}$$

First, it is shown that the combination $\langle \tilde{s}_0, \tilde{s}_1 \rangle$ such that $\tilde{p}_i \in [0, v)$ and $\tilde{p}_j \in [0, v)$ does not satisfy (2). Here, without restricting the generality, $\tilde{r}_i \geq \tilde{r}_j$ is assumed. After the case $1 > \tilde{r}_j$ is considered, the other case $1 = \tilde{r}_j$ is considered.

1. **The Case** $1 > \tilde{r}_j,\ 1 \geq \tilde{r}_i \geq \tilde{r}_j,\ \tilde{p}_i < v$**, and** $\tilde{p}_j < v$
   This case is considered in three subcases: $\tilde{p}_i - \delta > \tilde{p}_j$, $\tilde{p}_i + \delta < \tilde{p}_j$, and $\tilde{p}_j \in [\tilde{p}_i - \delta, \tilde{p}_i + \delta]$.

   1-1. **Subcase** $\tilde{p}_i - \delta > \tilde{p}_j$**.**
   As $\tilde{r}_j < 1$, $\pi_i(\tilde{s}_i, \tilde{s}_j) = E_i(\tilde{p}_i) < E_i(v) = \pi_i((v, \tilde{r}_i), \tilde{s}_j)$, which contradicts (A6).

   1-2. **Subcase** $\tilde{p}_i < \tilde{p}_j - \delta$**.**
   This subcase is considered further, and divided into the following cases $\tilde{r}_i < 1$ and $\tilde{r}_i = 1$.

   (1). **The case** $\tilde{r}_i < 1$**.** Using the same logic as the case $\tilde{p}_i - \delta > \tilde{p}_j$ contradicts the definition (A6). Store j can increase its profit by choosing $\tilde{p}_j = v$.

   (2). **The case** $\tilde{r}_i = 1$**.** In this case, $\pi_j((\tilde{p}_j, \tilde{r}_j), (\tilde{p}_i, 1)) = 0$ and $\pi_j((\tilde{p}_i, \tilde{r}_j), (\tilde{p}_i, 1)) = Q_j(\tilde{p}_i)$.
   - $Q_j(\tilde{p}_i) > 0$ contradicts (A6).
   - When $Q_j(\tilde{p}_i) = 0$, setting $p \in (\tilde{p}_i, \tilde{p}_i + \delta)$ store j earns $\pi_j((p, \tilde{r}_j), (\tilde{p}_i, 1)) = Q_j(p) > Q_j(\tilde{p}_i) = 0$, another contradiction.
   - When $Q_j(\tilde{p}_i) < 0$, $\tilde{p}_i < t/2$ because $Q'_j(p) > 0$ and $Q_j(t/2) \geq 0$ as $\tilde{r}_i = 1$ by Lemma 1(A). However, because $E'_i(p) + J'_i(p) > 0$ and

     $$\begin{aligned} \pi_i\left(\left(\frac{t}{2}, 1\right), (\tilde{p}_j, \tilde{r}_j)\right) &= E_i\left(\frac{t}{2}\right) + J_i\left(\frac{t}{2}\right) \\ &= \int_U t\left(\frac{1}{2} - x_i^*\right) dx = 0, \end{aligned}$$

     then $\pi_i\left((\tilde{p}_i, 1), (\tilde{p}_j, \tilde{r}_j)\right) = E_i(\tilde{p}_i) + J_i(\tilde{p}_i) < 0$, while

     $$\begin{aligned} \pi_i\left((v, 1), (\tilde{p}_j, \tilde{r}_j)\right) &= E_i(v) + Q_i(v) \geq E_i(v) \\ &= \int_{A_j^c} (v - t x_i^*)\, dx > 0 \end{aligned}$$

     still contradicts (A6).

1-3. **Subcase $\tilde{p}_j \in [\tilde{p}_i - \delta, \tilde{p}_i + \delta]$.** Either store 0 or store 1 can increase its price without being undercut by its rival, and thus increase its profit. This contradicts (A6).

Thus, when $\tilde{r}_j < 1$, there is no combination of strategies such that $\tilde{p}_i \in [0, v)$ that satisfies (2).

2. **The case $\tilde{r}_i = 1$ and $\tilde{p}_i < v$ for $i \in \{0, 1\}$.** Next, consider the case $\tilde{r}_j = 1$. As $\tilde{r}_i \geq \tilde{r}_j$, it is the case $\tilde{r}_i = 1$. Then

$$\pi_i((\tilde{p}_i, 1), (\tilde{p}_j, 1)) = \begin{cases} J_i(\tilde{p}_i) & \text{when } \tilde{p}_i \in [0, \tilde{p}_j - \delta) \\ Q_i(\tilde{p}_i) & \text{when } \tilde{p}_i \in [\tilde{p}_j - \delta, \tilde{p}_j + \delta] \\ 0 & \text{when } \tilde{p}_i \in (\tilde{p}_j + \delta, v] \end{cases}$$

Because both $J_i(p)$ and $Q_i(p)$ are increasing functions, given $\tilde{p}_j$, the optimization of store i requires $\tilde{p}_i = \tilde{p}_j + \delta$ or $\tilde{p}_i = \tilde{p}_j - \delta - \epsilon_i$, where ϵ_i is an arbitrary positive small number. In the same manner, the optimization of store j requires $\tilde{p}_j = \tilde{p}_i + \delta$ or $\tilde{p}_j = \tilde{p}_i - \delta - \epsilon_j$, where ϵ_j is another arbitrary positive small number. Any combination of these conditions contradicts the other.

Thus, there is no combination of strategies such that $\tilde{p}_i \in [0, v)$ and $\tilde{p}_j \in [0, v)$ that satisfies (2) irrespective $\tilde{r}_j < 1$ or $\tilde{r}_j = 1$. Therefore, if there exists a combination of strategies that satisfies (2), either $\tilde{s}_i = (v, \tilde{r}_i)$, $\tilde{s}_j = (v, \tilde{r}_j)$, or both are satisfied. First, consider the case $\min(p_0^Q, p_1^Q) \geq v - \delta$, and then consider the case $\min(p_0^Q, p_1^Q) < v - \delta$.

3. **The case $\min(p_0^Q, p_1^Q) \geq v - \delta$.** Assume $\tilde{s}_j = (v, \tilde{r}_j)$. Then, for $p \in [0, v - \delta)$, as $v - \delta \leq p_i^Q$,

$$\begin{aligned} \pi_i((p, \tilde{r}_i), (v, \tilde{r}_j)) &= E_i(p) + J_i(p) < E_i(v - \delta) + J_i(v - \delta) \\ &\leq E_i(v) + Q_i(v) \quad (\because \text{Lemma } 2(D)) \\ &= \pi_i((v, \tilde{r}_i), (v, \tilde{r}_j)), \end{aligned}$$

and for $p \in [v - \delta, v)$,

$$\begin{aligned} \pi_i((p, \tilde{r}_i), (v, \tilde{r}_j)) &= E_i(p) + Q_i(p) \\ &< E_i(v) + Q_i(v) = \pi_i((v, \tilde{r}_i), (v, \tilde{r}_j)). \end{aligned}$$

Thus, $v = \underset{p \in [0,v]}{\operatorname{argmax}}\ \pi_i((p, \tilde{r}_i), (v, \tilde{r}_j))$. Then, $\langle \tilde{s}_0, \tilde{s}_1 \rangle = \langle (v, \tilde{r}_0), (v, \tilde{r}_1) \rangle$. The profits of the stores are positive because they price the reservation price.

4. **The case** $\min(p_0^Q, p_1^Q) < v - \delta$**.** Assume $v - \delta > p_i^Q$ without restricting generality. First, check the case $\tilde{p}_j = v$, and then the case $\tilde{p}_j < v$.

   4-1. **The case** $\tilde{p}_j = v$**.** $\forall p \in [v-\delta, v]$, $\exists p^* \in (p_i^Q, v-\delta)$ such that

   $$\begin{aligned} \pi_i\left((p^*, \tilde{r}_i), (v, \tilde{r}_j)\right) &= E_i(p^*) + J_i(p^*) \\ &> E_i(v) + Q_i(v) \quad (\because \text{Lemma } 2(D)) \\ &\geq E_i(p) + Q_i(p) = \pi_i((p, \tilde{r}_i), (v, \tilde{r}_j)) \end{aligned}$$

   Then, $\tilde{p}_i \notin [v-\delta, v]$. For $p \in [0, v-\delta)$, $\pi_i\left((p, \tilde{r}_i), (v, \tilde{r}_j)\right) = E_i(p) + J_i(p)$ that is increasing with p. Therefore, $\tilde{p}_i = v - \delta - \epsilon_i$, where ϵ_i is an arbitrary small positive number. However,

   $$\begin{aligned} \pi_j((v, \tilde{r}_j), (v-\delta-\epsilon_i, \tilde{r}_i)) &= E_j(v) < E_j(v-\epsilon_i) + Q_j(v-\epsilon_i) \\ &= \pi_j((v-\epsilon_i, \tilde{r}_j), (v-\delta-\epsilon_i, \tilde{r}_i)) \end{aligned}$$

   for a sufficiently small value of ϵ_i, as $Q_j(v) > 0$. This contradicts $\tilde{p}_j = v$.

   4-2. **The case** $\tilde{p}_i = v$ **and** $\tilde{p}_j < v$.
   - If $\tilde{p}_j \geq v - \delta$, $\pi_j((\tilde{p}_j, \tilde{r}_j), (v, \tilde{r}_i)) = E_j(\tilde{p}_j) + Q_j(\tilde{p}_j) < E_j(v) + Q_j(v) = \pi_j((v, \tilde{r}_j), (v, \tilde{r}_i))$, which contradicts $\tilde{p}_j < v$.
   - If $\tilde{p}_j < v - \delta$, then $\pi_j((\tilde{p}_j, \tilde{r}_j), (v, \tilde{r}_i)) = E_j(\tilde{p}_j) + J_j(\tilde{p}_j)$, so that $\tilde{p}_j = v - \delta - \epsilon_j$, where ϵ_j is an arbitrary small positive number. In this case, $\pi_i((v, \tilde{r}_i), (v-\delta-\epsilon_j, \tilde{r}_j)) = E_i(v) < E_i(v-\epsilon_j) + Q_i(v-\epsilon_j) = \pi_i((v-\epsilon_j, \tilde{r}_i), (v-\delta-\epsilon_j, \tilde{r}_j))$, when ϵ_j is sufficiently small as $Q_i(v) > 0$. This is another contradiction.

   Thus, there is no combination of strategies that satisfies (2) when $v - \delta > \min(p_0^Q, p_1^Q)$.

□

Proof of Lemma 5. In the second stage, only when $\delta > v - \min(p_0^Q, p_1^Q)$, subgame Nash equilibrium exists for $(\tilde{r}_0, \tilde{r}_1) \in U^2$ such that $\tilde{r}_0 + \tilde{r}_1 > 1$ by Lemma 4. Because $p_i^Q > 0$ from Lemma 2(A), there exists δ that yields the nontrivial equilibrium $\delta < v$. As p_i^Q is continuously determined by $(\tilde{r}_0, \tilde{r}_1)$ from the equation of the Definition 1(B), the neighborhood has also subgame Nash equilibrium given δ. Further,

$$\frac{\partial}{\partial \tilde{r}_i} \pi_i((v, \tilde{r}_i), (v, \tilde{r}_j)) = \frac{\partial Q_i(v)}{\partial \tilde{r}_i} = q_i^*(\tilde{r}_i)(v - t\tilde{r}_i) > 0, \quad \text{for } \tilde{r}_i \in U.$$

□

Proof of Lemma 6. Assume that $\langle \tilde{s}_0, \tilde{s}_1 \rangle = \langle (\tilde{p}_0, \tilde{r}_0), (\tilde{p}_1, \tilde{r}_1) \rangle$ satisfies the conditions (3). Without loss of generality, assume $\tilde{p}_i \leq \tilde{p}_j$. First, consider the case $\tilde{p}_i < \tilde{p}_j - \delta$, and then the case $\tilde{p}_i \geq \tilde{p}_j - \delta$.

1. **The case $\tilde{p}_i < \tilde{p}_j - \delta$.** If $\tilde{r}_i < 1$ and $\tilde{p}_j < v$, then π_j can be increased by choosing v as p_j, a contradiction. Then, $\tilde{r}_i = 1$ or $\tilde{p}_j = v$. In the case $\tilde{r}_i = 1$, $\pi_j((\tilde{p}_j, \tilde{r}_j), (\tilde{p}_i, 1),) = 0$ and $\pi_i((\tilde{p}_i, 1), (\tilde{p}_j, \tilde{r}_j)) = E_i(\tilde{p}_i) + J_i(\tilde{p}_i) = \tilde{p}_i - t/2$. Then, $Q_j(\tilde{p}_i) = \pi_j((\tilde{p}_i, \tilde{r}_j), (\tilde{p}_i, 1)) \leq \pi_j((\tilde{p}_j, \tilde{r}_j), (\tilde{p}_i, 1)) = 0$. Because $Q_j(t/2) \geq 0$ by Lemma 1(A) with $\tilde{r}_i = 1$ and $Q'_j(p) > 0$, $\tilde{p}_i < t/2$. This concludes $\pi_i((\tilde{p}_i, 1), (\tilde{p}_j, \tilde{r}_j)) = \tilde{p}_i - t/2 < 0$, which is a contradiction because store i can earn zero profit choosing $\tilde{r}_i = 0$. Therefore, remaining possible equilibrium is only the case $\tilde{r}_i < 1$ and $\tilde{p}_j = v$. Because $\pi_i\left((\tilde{p}_i, \tilde{r}_i), (v, \tilde{r}_j)\right) = E_i(\tilde{p}_i) + J_i(\tilde{p}_i)$ as the $\tilde{p}_i < \tilde{p}_j - \delta$ is considered here and $E'_i(p) + J'_i(p) > 0$, $\tilde{p}_i = \tilde{p}_j - \delta - \epsilon_i = v - \delta - \epsilon_i$, where ϵ_i is an arbitrary positive small number. If $v - \delta > t$, then $\exists \epsilon_i > 0$ such that $\tilde{p}_i = v - \delta - \epsilon_i > t$, which contradicts $\tilde{r}_i < 1$ because

   $$\frac{\partial}{\partial \tilde{r}_i}\left(E_i(p) + J_i(p)\right) = \frac{\partial}{\partial \tilde{r}_i} \int_{A_i} (\tilde{p}_i - t x_i^*) dx = \tilde{p}_i - t\tilde{r}_i \geq \tilde{p}_i - t > 0,$$

   leading to $\tilde{r}_i = 1$. Then, $v - \delta \leq t$ and

   $$\tilde{r}_i = \underset{r \in U}{\operatorname{argmax}} \int_{A_i} (\tilde{p}_i - t x_i^*) dx = \frac{\tilde{p}_i}{t} < \frac{v - \delta}{t} \leq 1.$$

   Therefore,

   $$\pi_i\left((\tilde{p}_i, \tilde{r}_i), (\tilde{p}_j, \tilde{r}_j)\right) = \pi_i\left(\left(\tilde{p}_i, \frac{\tilde{p}_i}{t}\right), (v, \tilde{r}_j)\right) = \frac{\tilde{p}_i^2}{2t} = \frac{(v - \delta - \epsilon_i)^2}{2t}.$$

   Meanwhile,

   $$\begin{aligned} \pi_i((\tilde{p}_i, \tilde{r}_i), (v, \tilde{r}_j)) &\geq \pi_i((v, 1), (v, \tilde{r}_j)) \\ &\geq \pi_i((v, 1), (v, 1)) = vD_0 - tD_1. \end{aligned}$$

   Hereafter, in the remaining part of this proof, D_0 and D_1 denote $D_0(U)$ and $D_1(U)$, respectively. Then,

   $$\frac{(v - \delta - \epsilon_i)^2}{2t} \geq vD_0 - tD_1. \tag{A7}$$

However,

$$\pi_j((v,\tilde{r}_j),(\tilde{p}_i,\tilde{r}_i)) = \int_{A_i^c}(v-tx_j^*)dx$$
$$=\frac{(3v-t-\delta-\epsilon_i)(t+\delta+\epsilon_i-v)}{2t},$$

where $A_0^c=[(v-\delta-\epsilon_0)/t,1]$ and $A_1^c=[0,1-(v-\delta-\epsilon_1)/t]$, respectively, while

$$\pi_j((v,\tilde{r}_j),(\tilde{p}_i,\tilde{r}_i)) \geq \pi_j((\tilde{p}_i+\delta,1),(\tilde{p}_i,\tilde{r}_i))$$
$$=\int_{A_i} q_j^*(x)\left(\tilde{p}_i+\delta-tx_j^*\right)dx+\int_{A_i^c}\left(\tilde{p}_i+\delta-tx_j^*\right)dx$$
$$>\int_U q_j^*(x)\left(v-\epsilon_i-tx_j^*\right)dx \ \left(\because \tilde{r}_i=\frac{\tilde{p}_i}{t}>0\right)$$
$$=v-\epsilon_i-\frac{t}{2}-(v-t-\epsilon_i)D_0-tD_1,$$

so that

$$\frac{(3v-t-\delta-\epsilon_i)(t+\delta+\epsilon_i-v)}{2t}\geq v-\epsilon_i-\frac{t}{2}-(v-t-\epsilon_i)D_0-tD_1. \quad \text{(A8)}$$

Adding both side of the inequations (A7) and (A8),

$$-\frac{(v-t)(v-\delta)-v\epsilon_i}{t}\geq(t+\epsilon_i)D_0-2tD_1>0$$

. Check that $v-\delta>\tilde{p}_i\geq 0$, $D_0-2D_1\geq 0$ by Lemma 1(B) for the case $\tilde{r}_0=\tilde{r}_1=1$ in the last inequality, and $D_0(U)>0$. Obviously, when ϵ_i is sufficiently small, this inequality incurs a contradiction. Thus, no combination of strategies satisfies the equations (3) for the case $\tilde{p}_i<\tilde{p}_j-\delta$.

2. **The case $\tilde{p}_i\geq\tilde{p}_j-\delta$.** This case is examined through two subcases: the case $\tilde{p}_i\leq\tilde{p}_j<v$ and the case $\tilde{p}_i\leq\tilde{p}_j=v$.

   2-1 **The case $\tilde{p}_i\leq\tilde{p}_j<v$.** If $\tilde{r}_i>0$, then π_i can be increased by choosing a higher p_i except for the cases $\tilde{p}_i=\tilde{p}_j+\delta$ with $\delta=0$. Thus, $\tilde{r}_i=0$ or $\tilde{p}_i=\tilde{p}_j+\delta$ with $\delta=0$. In the case $\tilde{r}_i=0$, π_j can be increased by choosing a higher p_j if $\tilde{r}_j>0$. Then, $\tilde{r}_j=0$ again, and $\pi_i(\tilde{s}_i,\tilde{s}_j)=\pi_j(\tilde{s}_i,\tilde{s}_j)=0$. Nevertheless, the stores can obviously choose to earn positive profits by selecting a sufficiently small but non-zero DA $\tilde{r}_i > 0$ and the price v. Thus, the case $\tilde{r}_i = 0$ can be excluded. Then, $\tilde{p}_i = \tilde{p}_j + \delta$ with $\delta = 0$. However, when $\tilde{p}_i = \tilde{p}_j$ and $\delta = 0$, the stores can increase profits by adjusting their DAs, except when $\tilde{r}_i = \tilde{r}_j = \min(\tilde{p}_i/t, 1)$. Then, $\tilde{r}_i = \tilde{r}_j = \min(\tilde{p}_i/t, 1)$. If $\min(\tilde{p}_i/t, 1) \leq 1/2$, then $\tilde{r}_i + \tilde{r}_j \leq 1$, so at least π_i increases by choosing v as its price, a contradiction. If $\min(\tilde{p}_i/t, 1) > 1/2$, then choosing $\tilde{p}_j - \epsilon_i$ as its price increases π_i by

$$\int_{A_0 \cap A_1} (1 - q_i^*(x)) (\tilde{p}_i - tx_i^*) \, dx - \epsilon_i \tilde{r}_i.$$

As ϵ_i is an arbitrary positive small number, this value is positive, which is still a contradiction.

2-2 **The case $\tilde{p}_i \leq \tilde{p}_j = v$.** In this case, $\tilde{p}_i \in [v - \delta, v]$. If $\tilde{r}_i > 0$, π_i can be increased choosing a larger $\tilde{p}_i$, except when $\tilde{p}_i = v$; therefore, $\tilde{r}_i = 0$ or $\tilde{p}_i = v$. If $\tilde{r}_i = 0$, then $\pi_i = 0$. However, by choosing small positive $\tilde{r}_i$ and $\tilde{p}_i = v$, the profit of store i yields $\pi_i ((v, \tilde{r}_i), (v, \tilde{r}_j))$, that is, $E_i(v)$ or $Q_i(v)$, positive profits, a contradiction. Then, $\tilde{p}_0 = \tilde{p}_1 = v$. When $\tilde{p}_0 = \tilde{p}_1 = v$, π_0 and π_1 are increased by choosing greater DAs, except when $\tilde{r}_0 = \tilde{r}_1 = 1$. When $\tilde{s}_j = (v, 1)$, the profit of store i is

$$\pi_i((p, \tilde{r}_i), (v, 1)) = \begin{cases} \displaystyle\int_{A_i} (p - tx_i^*) dx & \text{for } p \in [0, v - \delta) \\ \displaystyle\int_{A_i} q_i^*(x)(p - tx_i^*) dx & \text{for } p \in [v - \delta, v]. \end{cases}$$

$\tilde{r}_i$ should be chosen as $\tilde{r}_i = \min(p/t, 1)$. Then, the profit is maximized only when $\tilde{s}_i = (v - \delta - \epsilon_i, \min((v - \delta - \epsilon_i)/t, 1))$ or $\tilde{s}_i = (v, 1)$. Therefore, the condition $\tilde{s}_i = (v, 1)$ is

$$\begin{aligned} &\pi_i \left(\left(v - \delta - \epsilon_i, \min \left(\frac{v - \delta - \epsilon_i}{t}, 1 \right) \right), (v, 1) \right) \\ &\leq \pi_i((v, 1), (v, 1)) = \int_U q_i^*(x)(v - tx_i^*) dx. \end{aligned}$$

This condition is satisfied if

$$\int_U q_i^*(x)(v - tx_i^*) dx \geq \begin{cases} \dfrac{(v - \delta)^2}{2t} & \text{when } v - \delta < t \\ v - \delta - \dfrac{t}{2} & \text{when } v - \delta \geq t. \end{cases} \tag{A9}$$

Obviously, the condition (A9) for both $i \in \{0, 1\}$ is satisfied only when δ is sufficiently large. The condition is not satisfied when δ is near zero, because

$$\int_U q_i^*(x)(v - tx_i^*)dx < \int_U (v - tx_i^*)dx = v - \frac{t}{2}$$

and right-hand side of (A9) is $v - t/2 - \delta$ when δ takes an arbitrary small number.

$\square$

Proof of Lemma 7.

$$\begin{aligned}
\pi_i^{u1}(p_j) &= \max_{s_i \in \{(p,r)|p<p_j-\delta\}} \pi_i(s_i, s_j) \geq \max_{s_i \in \{(p,r_i)|p<p_j-\delta\}} \pi_i(s_i, s_j) \\
&= \pi_i^{u2}(p_j) = \lim_{\epsilon \to 0} E_i(p_j - \delta - \epsilon) + J_i(p_j - \delta - \epsilon) \\
&= E_i(p_j - \delta) + J_i(p_j - \delta) \\
\pi_i^{u1}(p) &= \max\left(\frac{(p-\delta)|p-\delta|}{2t}, \pi_i\left((p-\delta, 0), s_j\right)\right) \geq 0
\end{aligned}$$

The equality holds when $p \leq \delta$.

$$\begin{aligned}
&E_0(p_1 - \delta) + Q_0(p_1 - \delta) \\
&= \int_0^{1-\tilde{r}_1} (p_1 - \delta - tx_0^*)dx + \int_{1-\tilde{r}_1}^{\tilde{r}_0} (p_1 - \delta - tx_0^*)q_0^*(x)dx \\
&\leq \int_0^{\min\left(\frac{p_1-\delta}{t}, \tilde{r}_0\right)} (p_1 - \delta - tx_0^*)dx + \int_{\min\left(\frac{p_1-\delta}{t}, \tilde{r}_0\right)}^{\tilde{r}_0} (p_1 - \delta - tx_0^*)q_0^*(x)dx \\
&\leq \int_0^{\min\left(\frac{p_1-\delta}{t}, \tilde{r}_0\right)} (p_1 - \delta - tx_0^*)dx \\
&= \min\left(\frac{(p_1-\delta)^2}{2t}, (p_1-\delta)\tilde{r}_0 - \frac{t\tilde{r}_0^2}{2}\right) \leq \frac{(p_1-\delta)^2}{2t} = \pi_0^{u1}(p_1)
\end{aligned}$$

$$
\begin{aligned}
&E_1(p_0-\delta)+Q_1(p_0-\delta)\\
&\quad=\int_{\tilde r_0}^{1}(p_0-\delta-tx_1^*)dx+\int_{1-\tilde r_1}^{\tilde r_0}(p_0-\delta-tx_1^*)q_1^*(x)dx\\
&\quad\le\int_0^{\max\left(1-\frac{p_0-t}{t},1-\tilde r_1\right)}(p_0-\delta-tx_1^*)dx\\
&\qquad+\int_{\max\left(1-\frac{p_0-\delta}{t},1-\tilde r_1\right)}^{\tilde r_0}(p_0-\delta-tx_1^*)q_1^*(x)dx\\
&\quad\le\int_0^{\max\left(1-\frac{p_0-t}{t},1-\tilde r_1\right)}(p_0-\delta-tx_1^*)dx\\
&\quad=\min\left(\frac{(p_0-\delta)^2}{2t},(p_0-\delta)\tilde r_1-\frac{t\tilde r_1^2}{2}\right)\le\frac{(p_0-\delta)^2}{2t}=\pi_1^{u1}(p_0)
\end{aligned}
$$

□

Proof of Lemma 8. First, consider the case $\tilde r_0+\tilde r_1>1$. Assume $p_i^*-p_j^*>\delta>0$. Then, $\pi_j(s_j^*,s_i^*)=E_j(p_j^*)+J_j(p_j^*)$, and $\pi_j^{u2}(p_i^*)=E_j(p_i^*-\delta)+J_j(p_i^*-\delta)$. Because $E_j(\cdot)$ and $J_j(\cdot)$ are monotonously increasing functions, $\pi_j^{u2}(p_i^*)>\pi_j(s_j^*,s_i^*)$, which contradicts $\langle s_0^*,s_1^*\rangle$ being a solution to Problem 1. Therefore, $p_i^*-p_j^*\le\delta$.

Second, consider the case $\tilde r_0+\tilde r_1\le 1$. Obviously, this case concludes with the trivial result, $p_i^*=v$, and $|p_0^*-p_1^*|=0<\delta$. □

Proof of Lemma 9.

$$
\begin{aligned}
E_j(p_j^{JQ})+Q_j(p_j^{JQ})&=E_j(p_j^{JQ})+J_j(p_j^{JQ})\\
&=E_j(p_i^{**}(p_j^{JQ})-\delta)+J_j(p_i^{**}(p^{JQ_j})-\delta),\\
E_j(p_i^{**}(0)-\delta)+J_j(p_i^{**}(0)-\delta)&=E_j(0)+Q_j(0)>E_j(0)+J_j(0),
\end{aligned}
$$

by Lemma 1(C), and

$$
0<\frac{dp_i^{**}(p_j)}{dp_j}=\frac{D_0\left(A_0\cap A_1\right)}{\tilde r_0+\tilde r_1-1}<1,
$$

by Lemma 1(D). Given $(\tilde r_0,\tilde r_1)$, the value of the differentiation is a constant. □

Proof of Lemma 10. First, consider the case wherein $\tilde r_0+\tilde r_1>1$. Because strategy (p_i,p_j) available is restricted to $|p_0-p_1|\le\delta$, the optimization problem to be solved is specified as

$$\max_{p \in [\max(0, p_j^* - \delta), \min(v, p_j^* + \delta)]} E_i(p) + Q_i(p),$$
$$\text{s.t. } E_j(p_j) + Q_j(p_j) \geq E_j(p - \delta) + J_j(p - \delta).$$

From the definition of p_i^{**}, the constraints are equivalent to $p_i^{**}(p_j) \geq p$. As $E_i(p) + Q_i(p)$ is a strict increasing function, the optimization is achieved when

$$p_i^* = \phi_i(p_j) \stackrel{\text{def}}{=} \min\left(p_i^{**}(p_j), p_j + \delta, v\right).$$

In the square $R = [0, v] \times [0, v]$, define $I_0^- \stackrel{\text{def}}{=} \{(0, p_1) | p_1 \in [0, v]\}$, $I_0^+ \stackrel{\text{def}}{=} \{(v, p_1) | p_1 \in [0, v]\}$, $I_1^- \stackrel{\text{def}}{=} \{(p_0, 0) | p_0 \in [0, v]\}$, and $I_1^+ \stackrel{\text{def}}{=} \{(p_0, v) | p_0 \in [0, v]\}$. Consider a continuous map $(f_0, f_1) = (p_0 - \phi_0(p_1), p_1 - \phi_1(p_0)) : R \to \Re^2$. On I_0^-: $f_0(0, p_1) = 0 - \phi_0(p_1) < 0$, on I_0^+: $f_0(v, p_1) = v - \phi_0(p_1) \geq 0$, on I_1^-: $f_1(p_0, 0) = 0 - \phi_1(p_0) < 0$, and on I_1^+: $f_1(p_0, v) = v - \phi_1(p_0) \geq 0$. Thus, the conditions required for Poincaré-Miranda theorem are satisfied. Then, there exists a point $(p_0^*, p_1^*) \in R$ such that $(f_0, f_1) = (0, 0)$, which means $p_0^* = \phi_0(p_1^*)$ and $p_1^* = \phi_1(p_0^*)$. This indicates (p_0^*, p_1^*) is a solution to the problem.

For the uniqueness, consider functions

$$\phi_1(p_0)/p_0 : [0, v] \to \Re \quad \text{and} \quad p_1/\phi_0(p_1) : [0, v] \to \Re$$

Function $\phi_1(p_0)$ consists of connected line segments, each of which has positive y-intercept $(\delta, p_0^{**}(0), v)$ and non-negative slope $(1, dp_0^{**}(p_1)/dp_1, 0)$ (Lemma 9). Then, for each line segment, $\phi_1(p_0)/p_0$ monotonously decreases with respect to p_0. Furthermore, the function is continuous. Then, $\phi_1(p_0)/p_0$ is a monotonously decreasing function for $p_0 \in [0, v]$. In the same manner, $\phi_0(p_1)/p_1$ is a monotonously decreasing function, which means $p_1/\phi_0(p_1)$ is a monotonously increasing function with respect to p_1. At $(p_0, p_1) = (p_0^*, p_1^*)$, $\phi_1(p_0^*)/p_0^* = p_1^*/\phi_0(p_1^*)$. Assume there exists another $(p_0, p_1) = (p_0^+, p_1^+)$ that satisfies $p_0^+ = \phi_0(p_1^+)$ and $p_1^+ = \phi_1(p_0^+)$. If $\phi_1(p_0^+)/p_0^+ = p_1^+/\phi_0(p_1^+) > \phi_1(p_0^*)/p_0^* = p_1^*/\phi_0(p_1^*)$, then $p_0^* < p_0^+$ and $p_1^* > p_1^+$ because $\phi_1(p_0)/p_0$ and $p_1/\phi_0(p_1)$ are monotonously decreasing and increasing functions, respectively. This means $p_1^+ = \phi_1(p_0^+) > \phi_1(p_0^*) = p_1^*$, which is a contradiction. In the same manner, the assumption $\phi_1(p_0^+)/p_0^+ < \phi_1(p_0^*)/p_0^*$ leads to another contradiction. Then, $\phi_1(p_0^+)/p_0^+ = \phi_1(p_0^*)/p_0^*$ and $p_1^+/\phi_0(p_1^+) = p_1^*/\phi_0(p_0^*)$, which leads to $p_0^* = p_0^+$ and $p_1^* = p_0^+$ because $\phi_1(p_0)/p_0$ and $p_1/\phi_0(p_1)$ are monotonously decreasing and increasing functions, respectively.

Next, consider the case $\tilde{r}_0 + \tilde{r}_1 \leq 1$. In this case, the UPP constraints do not restrict the optimizations. Then, prices $p_0^* = p_1^* = v$ only maximize their profits. $\square$

Proof of Lemma 11. The difference between Problem 1 and Problem 2 is in the price strategy spaces $[0, v]$ and $[0, v] \cap [p_j - \delta, p_j + \delta]$ available for store i, given p_j, respectively. Therefore, to prove the solutions are the same for two problems, it is sufficient to prove that profits of store i, $\pi_j((p_j^*, \tilde{r}_j), (p_i, \tilde{r}_i))$ are not maximized in the extended area, $[0, v] \cap [p_j - \delta, p_j + \delta]^c$, given p_j^* and constraints.

Consider the solution to Problem 2, $p_j^* \in [0, v]$ and $\left| p_i^* - p_j^* \right| \leq \delta$. First, if store i chooses the price p_i such that $p_i > p_j^* + \delta$,

$$\pi_j((p_j^*, \tilde{r}_j), (p_i, \tilde{r}_i)) = E_j(p_j^*) + J_j(p_j^*), \text{ and } \pi_j^{2u}(p_i) = E_j(p_i - \delta) + J_j(p_i - \delta),$$

so that $\pi_j((p_j^*, \tilde{r}_j), (p_i, \tilde{r}_i)) < \pi_j^{2u}(p_i)$, then the constraint of the problem is not satisfied. Thus, store i does not choose a price p_i such that $p_i > p_j^* + \delta$.

Second, check the case when store i chooses the price such that $p^i < p_j^* - \delta$. As $p_j^* = \min(p_i^* + \delta, p_j^{**}(p_i^*), v)$, there are three subcases: $p_j^* = p_i^* + \delta$, $p_j^* = p_j^{**}(p_i^*)$, and $p_j^* = v$.

When $p_j^* = p_i^* + \delta$, $p_i^* + \delta \leq p_j^{**}(p_i^*)$, which means $p_i^* \leq p_i^{JQ}$, then $E_i(p_i^*) + Q_i(p_i^*) \geq E_i(p_i^*) + J_i(p_i^*)$ (Lemma 2(E)). Store i does not choose p_i such that $p_i < p_j^* - \delta$, because

$$\begin{aligned}\pi_i((p_i, \tilde{r}_i),(p_j^*, \tilde{r}_j)) &= E_i(p_i) + J_i(p_i) < E_i(p_j^* - \delta) + J_i(p_j^* - \delta)\\ &= E_i(p_i^*) + J_i(p_i^*) \leq E_i(p_i^*) + Q_i(p_i^*) = \pi_i((p_i^*, \tilde{r}_i), (p_j^*, \tilde{r}_j)).\end{aligned}$$

Next, when $p_j^* = p_j^{**}(p_i^*)$, again store i does not choose p_i such that $p_i < p_j^* - \delta$, because

$$\begin{aligned}\pi_i((p_i, \tilde{r}_i),(p_j^*, \tilde{r}_j)) &= E_i(p_i) + J_i(p_i) < E_i(p_j^* - \delta) + J_i(p_j^* - \delta)\\ &= E_i(p_j^{**}(p_i^*) - \delta) + J_i(p_j^{**}(p_i^*) - \delta) = E_i(p_i^*) + Q_i(p_i^*)\\ &= \pi_i((p_i^*, \tilde{r}_i), (p_j^*, \tilde{r}_j)).\end{aligned}$$

Lastly, when $p_j^* = v$, $v \leq p_j^{**}(p_i^*)$. Still, store i does not choose p_i such that $p_i < p_j^* - \delta = v - \delta$ because

$$\begin{aligned}\pi_i((p_i, \tilde{r}_i),(p_j^*, \tilde{r}_j)) &= E_i(p_i) + J_i(p_i) < E_i(v - \delta) + J_i(v - \delta)\\ &\leq E_i(p_j^{**}(p_i^*) - \delta) + J_i(p_j^{**}(p_i^*) - \delta) = E_i(p_i^*) + Q_i(p_i^*)\\ &= \pi_i((p_i^*, \tilde{r}_i), (p_j^*, \tilde{r}_j)).\end{aligned}$$

$\square$

Proof of Lemma 12. The condition for $(p_0^*, p_1^*) = (v, v)$ is $\phi_i(v) = v$. Then, $p_i^{**}(v) \geq v$, as $p_j^* + \delta = v + \delta > v$, where functions $\phi_i(p_j)$ are defined in the proof of Lemma 10. The inequality $p_i^{**}(v) \geq v$ is satisfied if $E_i(v) + Q_i(v) \geq E_i(v-\delta) + J_i(v-\delta)$ because $E_i(v) + Q_i(v) = E_i(p_j^{**}(v) - \delta) + J_i(p_j^{**}(v) - \delta)$ by definition.

If $E_i(v) + Q_i(v) < E_i(v-\delta) + J_i(v-\delta)$, then $E_i(p_j^{**}(v) - \delta) + J_i(p_j^{**}(v) - \delta) < E_i(v-\delta) + J_i(v-\delta)$, which leads $p_j^{**}(v) < v$, and so $\phi_j(v) < v$, for $i, j \in \{0, 1\}, i \neq j$. Therefore, (v, v) cannot be the solution to Problem 1. Here, assume $p_i^* = v$. Then, $p_j^* = \phi_j(p_i^*) = \phi_j(v) < v$. However, $v = \phi_i(p_j^*) < \phi_i(v)$, a contradiction. □

Proof of Lemma 13. By Lemma 1(C) $J_j(v) > Q_j(v)$. Then, $\exists \delta^{*0} > 0$, $\forall \{(\tilde{\delta}, \epsilon) | \tilde{\delta} \in (0, \delta^{*0}), \epsilon \in (0, J_j(v) - Q_j(v))\}$, $J_j(v - \tilde{\delta}) - Q_j(v) > \epsilon > 0$. Besides, for the value of such ϵ, $\exists \delta^{*1} > 0$, $\forall \{\tilde{\delta} | \tilde{\delta} \in (0, \delta^{*1})\}$, $E_j(v) - E_j(v - \tilde{\delta}) < \epsilon$. Then, $\forall \left\{ \tilde{\delta} | \tilde{\delta} \in (0, \min(\delta^{*0}, \delta^{*1})) \right\}$,

$$E_j(v) + Q_j(v) < E_j(v - \tilde{\delta}) + \epsilon + J_j(v - \tilde{\delta}) - \epsilon = E_j(v - \tilde{\delta}) + J_j(v - \tilde{\delta}).$$

Thus, from Lemma 12 $\forall \delta < \min(\delta^{*0}, \delta^{*1})$, $(p_0^*, p_1^*) < (v, v)$. □

Proof of Lemma 14. $\forall \gamma > 0, \exists \epsilon > 0$, if $\epsilon > \tilde{r}_0 + \tilde{r}_1 - 1 > 0$, then $\gamma > J_i(v)$ because $J_i(v) = \int_{A_0 \cap A_1} (v - tx_i^*) q_i^*(x) dx < \int_{A_0 \cap A_1} v dx = (\tilde{r}_0 + \tilde{r}_1 - 1)v$, and select ϵ as $\epsilon = \gamma / v$. $\gamma > J_i(v)$ leads to $\gamma > |J_i(v-\delta) - Q_i(v)|$ because $\gamma > J_i(v) > Q_i(v) > 0$ (Lemma 1(C)) and $\gamma > J_i(v) > J_i(v-\delta) > J_i(t) > 0$. Thus, selecting γ as $\gamma < \min_{i=\{0,1\}} E_i(v) - E_i(v-\delta)$,

$$\begin{aligned} &E_i(v) + Q_i(v) > E_i(v-\delta) + \gamma \\ &\qquad \begin{cases} +J_i(v-\delta) - \gamma = E_i(v-\delta) + J_i(v-\delta). & \text{when } J_i(v-\delta) \geq Q_i(v) \\ +J_i(v-\delta) > E_i(v-\delta) + J_i(v-\delta). & \text{when } J_i(v-\delta) < Q_i(v) \end{cases} \end{aligned}$$

From Lemma 12 $(p_0^*, p_1^*) = (v, v)$ for $\epsilon > \tilde{r}_0 + \tilde{r}_1 - 1 > 0$. □

Proof of Remark 1. For the solution to Problem 1 such that $(p_0^*, p_1^*) \in (t\tilde{r}, v)^2$, $p_j^* = p^* > t\tilde{r} > p_j^{JQ}$ (Lemma 2(B)). Then, as $p_i^{**}(p_j^{JQ}) = p_j^{JQ} + \delta$, $p_i^{**}(p_j^*) < p_j^* + \delta$ (Lemma 9). From the proof of Lemma 10, the solution to Problem 2 is $p_i^* = \min\left(p_j^* + \delta, p_i^{**}(p_j^*), v\right)$; here, $p_i^{**}(p_j^*) < p_j^* + \delta$, and $p^* < v$ from the assumption. Then, $p_i^* = p_i^{**}(p_j^*)$. Therefore,

$$E_j(p_j^*) + Q_j(p_j^*) = E_j(p_i^* - \delta) + J_j(p_i^* - \delta).$$

The equations ($i, j \in \{0,1\}, i \neq j$) are equivalent to the equations:

$$\tilde{r}_1(p_1^* - p_0^* + \delta) = (p_1^* - t)D_0 + tD_1, \tag{A10}$$

$$\begin{aligned}\tilde{r}_0(p_0^* - p_1^* + \delta) = (\tilde{r}_0 + \tilde{r}_1 - 1)(p_0^* - \frac{t}{2}(1 + \tilde{r}_0 - \tilde{r}_1)) \\ - p_0^* D_0 + tD_1,\end{aligned} \tag{A11}$$

where D_0 and D_1 denote $D_0([1 - \tilde{r}_1, \tilde{r}_0])$ and $D_1([1 - \tilde{r}_1, \tilde{r}_0])$ in this proof for simplicity of expression. Differentiation of these equations by $\tilde{r}_i$ leads to

$$\begin{aligned}G\frac{\partial p_0^*}{\partial \tilde{r}_0} &= (\tilde{r}_1 - D_0)(\delta - (p_1^* - t\tilde{r}_0)) \\ &+ q_0^* \left\{(\tilde{r}_1 - D_0)(p_0^* - t\tilde{r}_0) - \tilde{r}_0(p_1^* - t(1 - \tilde{r}_0))\right\}\end{aligned} \tag{A12}$$

$$\begin{aligned}G\frac{\partial p_1^*}{\partial \tilde{r}_1} &= (1 - \tilde{r}_1 + D_0)(\delta - (p_0^* - t\tilde{r}_1)) \\ &+ q_1^* \left\{(1 - \tilde{r}_1 + D_0)(p_1^* - t\tilde{r}_1) - \tilde{r}_1(p_0^* - t(1 - \tilde{r}_1))\right\},\end{aligned} \tag{A13}$$

where $G \stackrel{\text{def}}{=} \tilde{r}_0\tilde{r}_1 - (\tilde{r}_1 - D_0)(1 - \tilde{r}_1 + D_0) > 0$, as $\tilde{r}_1 > \tilde{r}_1 - D_0 > 0$, $\tilde{r}_0 > 1 - \tilde{r}_1 + D_0 > 0$ (Lemma 1(D)). q_0^* and q_1^* denote $q(\tilde{r}_0)$ and $1 - q(1 - \tilde{r}_1)$, respectively. As $q(x) + q(1 - x) = 1$ and $\tilde{r}_i = \tilde{r} > 1/2$,

$$\begin{aligned}D_0 &= \int_{1-\tilde{r}}^{\tilde{r}} q(x)dx = \int_{1-\tilde{r}}^{\frac{1}{2}} q(x)dx + \int_{\frac{1}{2}}^{\tilde{r}} q(x)dx \\ &= \int_{1-\tilde{r}}^{\frac{1}{2}} q(x)dx + \int_{1-\tilde{r}}^{\frac{1}{2}} q(1 - x)dx = \int_{1-\tilde{r}}^{\frac{1}{2}} 1 \cdot dx = \tilde{r} - \frac{1}{2}\end{aligned}$$

Hence, at the symmetric solution $(p_0^*, p_1^*) = (p^*, p^*)$, the terms in the braces in the equations (A12) and (A13) are

$$\begin{aligned}(\tilde{r}_1 - D_0)(p_0^* - t\tilde{r}_0) - \tilde{r}_0(p_1^* - t(1 - \tilde{r}_0)) &= \frac{p^* - t\tilde{r}}{2} - \tilde{r}(p^* - t(1 - \tilde{r})) \\ &= (p^* + t\tilde{r})(\frac{1}{2} - \tilde{r}) < 0,\end{aligned}$$

$$\begin{aligned}(1 - \tilde{r}_1 + D_0)(p_1^* - t\tilde{r}_1) - \tilde{r}_1(p_0^* - t(1 - \tilde{r}_1)) \\ = \frac{p^* - t\tilde{r}}{2} - \tilde{r}(p^* - t(1 - \tilde{r})) < 0.\end{aligned}$$

Therefore, if δ is sufficiently small, and $p^* > t\tilde{r}$, the differentials (A12) and (A13) take negative values. $\square$

Proof of Lemma 15. Because the given value of δ is sufficiently small, δ can be assumed to satisfy $v - t > \delta$. Then, by Lemma 14, when $\tilde{r}_0 + \tilde{r}_1 - 1$ is sufficiently small, the combination of prices (v, v) is the unique solution with UPP in the second stage. By Lemma 12, $p_i^* = v$ is attained whether

$$E_j(v) + Q_j(v) > E_j(v - \delta) + J_j(v - \delta), \tag{A14}$$

or

$$E_j(v) + Q_j(v) = E_j(v - \delta) + J_j(v - \delta). \tag{A15}$$

Consider the case (A14) where the sizes of the DAs are (r_0^*, r_1^*). If store i expands its DA, $E_j(v) + Q_j(v)$ decreases as

$$\frac{\partial}{\partial \tilde{r}_i} E_j(v) + Q_j(v) \bigg|_{\tilde{r}_i = r_i^* + 0} = -q_i^* (1 - r_i^*) \left(v - t(1 - r_i^*)\right) < 0.$$

Store i can expand its DA by a small amount without violating (A14), which increases the profit of store i as the price is kept constant at v. Therefore, as far as (A14) is valid, the equilibrium is not attained and store i expands its DA. This process continues until (A15) is attained. Note that $E_j(v - \delta) + J_j(v - \delta)$ is not affected by expansion of the DA of store i. Thus, at the equilibrium, (A15) should be satisfied for both is.

If store i expands its DAs further when (A15) is satisfied, the prices with the UPP at the second stage falls below v. The changes of the profits for store i are

$$\begin{aligned}
\frac{\partial}{\partial \tilde{r}_0} \left\{E_0(p_0^*) + Q_0(p_0^*)\right\} &= \frac{\partial p_0^*}{\partial \tilde{r}_0} (1 - \tilde{r}_1 + D_0) + q_0^* \cdot (p_0^* - t\tilde{r}_0) \\
\frac{\partial}{\partial \tilde{r}_1} \left\{E_1(p_1^*) + Q_1(p_1^*)\right\} &= \frac{\partial p_1^*}{\partial \tilde{r}_1} (\tilde{r}_1 - D_0) + q_1^* \cdot (p_1^* - t\tilde{r}_1)
\end{aligned}$$

where q_0^*, q_1^*, and D_0 denote $q(\tilde{r}_0)$, $1 - q(1 - \tilde{r}_1)$, and $D_0([1 - \tilde{r}_1, \tilde{r}_0])$, respectively, in this proof. Using (A12) and (A13) in the proof of Remark 1, at $p_i^* = v$ the differentiations are transformed to

$$\begin{aligned}
&(\,(1 - \tilde{r}_1 + D_0)\,(\tilde{r}_1 - D_0)\,(\delta - (v - t\tilde{r}_0)) \\
&\quad + \tilde{r}_0 q_0 \,(\tilde{r}_1(v - t\tilde{r}_0) - (1 - \tilde{r}_1 + D_0)\,(v - t(1 - \tilde{r}_0))))/G, \\
&(\,(1 - \tilde{r}_1 + D_0)\,(\tilde{r}_1 - D_0)\,(\delta - (v - t\tilde{r}_1)) \\
&\quad + \tilde{r}_1(1 - q_1)\,(\tilde{r}_0(v - t\tilde{r}_1) - (\tilde{r}_1 - D_0)\,(v - t(1 - \tilde{r}_1))))/G,
\end{aligned}$$

respectively, where $G \overset{\text{def}}{=} \tilde{r}_0\tilde{r}_1 - (\tilde{r}_1 - D_0)(1 - \tilde{r}_1 + D_0) > 0$. Numerators are further transformed to

$$\begin{aligned}&(G - (1 - q_0^*)\tilde{r}_0\tilde{r}_1)(v - t\tilde{r}_0) + \delta\,(\tilde{r}_0\tilde{r}_1 - G)\\ &\qquad\qquad\qquad - q_0^*\tilde{r}_0(1 - \tilde{r}_1 + D_0)(v - t(1 - \tilde{r}_0)), \qquad \text{(A16)}\\ &(G - (1 - q_1^*)\tilde{r}_0\tilde{r}_1)(v - t\tilde{r}_1) + \delta\,(\tilde{r}_0\tilde{r}_1 - G)\\ &\qquad\qquad\qquad - q_1^*\tilde{r}_1(\tilde{r}_1 - D_0)(v - t(1 - \tilde{r}_1)). \qquad \text{(A17)}\end{aligned}$$

Note that because $\tilde{r}_0 + \tilde{r}_1 - 1$ is sufficiently small, for any arbitrary small number ϵ, $\tilde{r}_0 + \tilde{r}_1 - 1 < \epsilon$. Then, $D_0 < \epsilon$ as $D_0 < \tilde{r}_0 + \tilde{r}_1 - 1 < \epsilon$ (Lemma 1(D)). Therefore, as $\epsilon > \tilde{r}_0 + \tilde{r}_1 - 1 > D_0 > 0$, $\epsilon > \tilde{r}_0 + \tilde{r}_1 - 1 - D_0$, $1 - \tilde{r}_1 + D_0 > \tilde{r}_0 - \epsilon$. Further, $\tilde{r}_1 - D_0 > \tilde{r}_1 - \epsilon$. Then,

$$G < \tilde{r}_0\tilde{r}_1 - (\tilde{r}_0 - \epsilon)\,(\tilde{r}_1 - \epsilon) = \epsilon(\tilde{r}_0 + \tilde{r}_1 - \epsilon) < \epsilon(\tilde{r}_0 + \tilde{r}_1).$$

Now, the first terms of (A16) and (A17) are sufficiently small:

$$\begin{aligned}(G - (1 - q_0^*)\tilde{r}_0\tilde{r}_1)(v - t\tilde{r}_0) < G(v - t\tilde{r}_0) < \epsilon(\tilde{r}_0 + \tilde{r}_1)(v - t\tilde{r}_0),\\ (G - (1 - q_1^*)\tilde{r}_0\tilde{r}_1)(v - t\tilde{r}_1) < G(v - t\tilde{r}_1) < \epsilon(\tilde{r}_0 + \tilde{r}_1)(v - t\tilde{r}_1).\end{aligned}$$

Thus, if δ and ϵ are sufficiently small, the values of (A16) and (A17) are negative. This shows the expansion of the DAs is not profitable.

When (A15) is satisfied for both j, if store i tries to shrink its DA, $E_j(v)+Q_j(v)$ increases, while $E_j(v-\delta)+J_j(v-\delta)$ is kept constant. Then, the situation is changed to (A14), but the price is kept constant at v. Obviously, this change decreases the profit of store i. Thus, an equilibrium is attained when (A15) is satisfied for both j, and, at the second stage, both stores set the price v. □

Proof of Lemma 16. By Lemma 2(B), $p_i^{JQ} < t\tilde{r}_i < t$, then $E_i(t)+Q_i(t) < E_i(t) + J_i(t)$ by Lemma 2(E). For a sufficiently small value of δ,

$$E_i(t) + Q_i(t) < E_i(t - \delta) + J_i(t - \delta).$$

Then, with the definition of the function $p_i^{**}(p)$,

$$E_i(p_j^{**}(t) - \delta) + J_i(p_j^{**}(t) - \delta) < E_i(t - \delta) + J_i(t - \delta),$$

which leads to $p_j^{**}(t) < t$; then $\phi_j(t) < t$, where function $\phi_j(\cdot)$ is as defined in the proof of Lemma 10. Consider a map $(f_0, f_1) = (p_0 - \phi_0(p_1), p_1 - \phi_1(p_0))$: $[0, t]^2 \to \Re^2$. Setting $I_0^- \overset{\text{def}}{=} \{(0, p_1)|p_1 \in [0, t])\}$,$I_0^+ \overset{\text{def}}{=} \{(1, p_1)|p_1 \in [0, t])\}$,

$I_1^- \stackrel{\text{def}}{=} \{(p_0, 0)|p_0 \in [0,t])\}$, nd $I_1^+ \stackrel{\text{def}}{=} \{(p_0, 1)|p_0 \in [0,t])\}$, then on I_0^-: $f_0(0, p_1) = 0 - \phi_0(p_1) < 0$, on I_0^+: $f_0(t, p_1) = t - \phi_0(p_1) \geq t - \phi_0(t) > 0$, on I_1^-: $f_1(p_0, 0) = 0 - \phi_1(p_0) < 0$, and on I_1^+: $f_1(p_0, t) = t - \phi_1(p_0) \geq t - \phi_1(t) > 0$. Thus, the conditions required for Poincaré-Miranda theorem are satisfied. Then, there exists $(p_0^*, p_1^*) \in [0, t]$ at which $(f_0, f_1) = (0, 0)$ and $\phi_i(p_j^*) = p_i^*$, the solution to Problem 1. It is obvious that $(p_0^*, p_1^*) = (t, t)$ is not the solution. If one of the prices is t, $p_i^* = t$, then $p_j^* = \phi_j(p_i^*) = \phi_j(t) < t$; however, $t = \phi_i(p_j^*) < \phi_i(t)$, a contradiction. Thus, $(p_0^*, p_1^*) < (t, t)$. By the uniqueness proven in Lemma 10, the solution here is nothing other than the solution to Problem 1.

Therefore,

$$\begin{aligned}\pi_i((p_i^*, \tilde{r}_i),(p_j^*, \tilde{r}_j)) = E_i(p_i^*) + Q_i(p_i^*) &< E_i(t) + Q_i(t) \\ &< E_i(t) + J_i(t) = t\tilde{r}_i - t\frac{\tilde{r}_i^2}{2}.\end{aligned}$$

However, given rival store's DA size $\tilde{r}_j$, store i is able to raise the price with the UPP to the ceiling reservation price v by shrinking its own DA size by Lemma 14. The profits under the solutions are

$$\pi_i((v, r_i^*), (v, \tilde{r}_j)) = E_i(v) + Q_i(v) > E_i(v) = v(1 - \tilde{r}_j) - \frac{t}{2}(1 - \tilde{r}_j)^2,$$

where r_i^* is such that $0 < r_i^* + \tilde{r}_j - 1 < \epsilon$ and ϵ is that shown to exist in Lemma 14. Because

$$v(1 - \tilde{r}_j) - \frac{t}{2}(1 - \tilde{r}_j)^2 - \left(t\tilde{r}_i - t\frac{\tilde{r}_i^2}{2}\right) = (1 - \tilde{r}_j)v - t\tilde{r}_i + \frac{t}{2}\left(\tilde{r}_i^2 - (1 - \tilde{r}_j)^2\right),$$

and setting $v^* = \max\{t\tilde{r}_i/(1 - \tilde{r}_j), t\tilde{r}_j/(1 - \tilde{r}_i)\}$, the inequalities in this Lemma hold. Note that $\tilde{r}_i > 1 - \tilde{r}_j$. □

Proof of Lemma 17. When $(\tilde{r}_0, \tilde{r}_1) = (1, 1)$ and the function $q(x)$ is symmetrical, the equations (A10) and (A11) in the proof of Remark 1 yield the symmetric solution to Problem 1 such that $(p_0^*, p_1^*) = (p^*, p^*)$ satisfy $\delta = (p^* - t)/2 + tD_1(U)$, which leads to $p^* = t(1 - 2D_1(U)) + 2\delta$. Note that $D_0(U) = 1/2$ when $q(x)$ is symmetrical. From equations (A12) and (A13) the differentials of the profits at the point are

$$
\begin{aligned}
\left.\frac{\partial Q_0(p_0^*)}{\partial \tilde{r}_0}\right|_{\substack{\tilde{r}_0=1\\ \tilde{r}_1=1}} &= \frac{1}{2}\left.\frac{\partial p_0^*}{\partial \tilde{r}_0}\right|_{\substack{\tilde{r}_0=1\\ \tilde{r}_1=1}} + q_1 \cdot (p^* - t) \\
&= \frac{4}{3}\left(\frac{1}{4}(\delta - (p^* - t)) + q_1 \cdot \left(\frac{p^*}{2} - t\right)\right) \\
&= \frac{4}{3}\left(\left(q_1 - \frac{1}{4}\right)\delta + \frac{tD_1(U)}{2} - tq_1 \cdot \left(D_1(U) + \frac{1}{2}\right)\right) \\
\left.\frac{\partial Q_1(p_1^*)}{\partial \tilde{r}_1}\right|_{\substack{\tilde{r}_0=1\\ \tilde{r}_1=1}} &= \frac{1}{2}\left.\frac{\partial p_1^*}{\partial \tilde{r}_1}\right|_{\substack{\tilde{r}_0=1\\ \tilde{r}_1=1}} + (1 - q_0) \cdot (p^* - t) \\
&= \frac{4}{3}\left(\left(q_1 - \frac{1}{4}\right)\delta + \frac{tD_1(U)}{2} - tq_1 \cdot \left(D_1(U) + \frac{1}{2}\right)\right),
\end{aligned}
$$

where q_0 and q_1 denote $q(0)$ and $q(1)$, respectively. Thus, when $q_1 = q(1) = 1 - q_0$ and δ are small enough, the differentials have positive values, which means they are locally optimized. $\square$

Proof of Lemma 18. Suppose $p_i^* < \delta$. Then, $\pi_j^{u1}(p_i^*) = 0$ because store j can avoid negative profit from negative price $p_j = p_i^* - \delta$ setting $r_j^* = 0$. Then, a strategy $s_i' = (p_i^* + \epsilon, r_i^*)$ keeps the inequality $\pi_j(s_j^*, s_i^*) \geq \pi_j^{u1} = 0$ and yields higher profit for store i for a sufficiently small value of ϵ, which contradicts $s_i^* = \underset{s_i \in S}{\operatorname{argmax}}\, \pi_i(s_i, s_j^*)$. Then, $p_i^* \in [\delta, v]$.

Next, suppose $\pi_j(s_j^*, s_i^*) > \pi_j^{u1}(p_i^*)$, then $p_i^* < p_j^* + \delta$ because if $p_i^* > p_j^* + \delta$, $\pi_j(s_j^*, s_i^*) = E_j(p_j^*) + J_j(p_j^*) < E_j(p_i^* - \delta) + J_j(p_i^* - \delta) \leq \pi_j^{u1}(p_i^*)$ (Lemma 7), a contradiction, and if $p_i^* = p_j^* + \delta$, $\pi_j(s_j^*, s_i^*) = E_j(p_j^*) + Q_j(p_j^*) = E_j(p_i^* - \delta) + Q_j(p_i^* - \delta) \leq \pi_j^{u1}(p_i^*)$ (Lemma 7), another contradiction. Therefore, strategy $s_i' = (p_i^* + \epsilon, r_j^*)$ satisfies the condition and yields higher profit π_i for a sufficiently small value of ϵ, which contradicts $s_i^* = \underset{s_i \in S}{\operatorname{argmax}}\, \pi_i(s_i, r_j^*)$. $\square$

Proof of Lemma 19. Suppose a combination of strategy $\langle s_0^*, s_1^* \rangle$ is the solution to Problem 3, and $r_0^* + r_1^* < 1$. Note that $\pi_i(s_i^*, s_j^*) = E_i(p_i^*)$. Assume $p_i^* > tr_i^*$; then, $s_i' = (p_i^*, r_i^* + \epsilon)$ keeps the condition $\pi_j(s_j^*, s_i'^*) \geq \pi_j^{u1}(p_i^*)$ for a sufficiently small value of ϵ because the price p_i^* is not changed, while π_j is also not changed by the effect of the blank area surrounding the stores. This strategy s_i' increases the profit of store i, but leads to a contradiction in that s_i^* maximizes the profit of store i, satisfying the inequality. Further, $p_i^* < tr_i^*$ induces another contradiction: For a sufficiently small value of ϵ, strategy $s_i'' = (p_i^*, r_i^* - \epsilon)$ provides store i with increased profit, satisfying the inequality.

Thus, $p_i^* = tr_i^*$. Then, the profit is expressed as

$$\pi_i(s_i^*, s_j^*) = \int_{A_i} (p_i^* - tx_i^*)dx = \pi_0^{u1}(p_i^* + \delta).$$

At the same time, $\pi_i(s_i^*, s_j^*) = \pi_i^{u1}(p_j^*)$ from Lemma 18. As the function π_i^{u1} is a monotonously increasing function, $p_i^* + \delta = p_j^*$. When $\delta > 0$, the equations are not consistent for both $i, j \in \{0, 1\}, i \neq j$. □

Proof of Lemma 20. Assume $p_i^* > p_j^* + \delta$, then

$$\begin{aligned}
\pi_j((p_j^*, r_j^*), s_i^*) &= E_j(p_j^*) + J_i(p_j^*) = \pi_j^{u1}(p_i^*) && \text{(Lemma 18)}\\
&\geq E_j(p_i^* - \delta) + J_j(p_i^* - \delta) && \text{(Lemma 7)}
\end{aligned}$$

This leads to $p_j^* \geq p_i^* - \delta$, a contradiction. □

Proof of Lemma 21. Under the assumption $r_0^* + r_1^* = 1$,

$$D_0^* = \int_{1-r_1^*}^{r_0^*} dx = 0, \quad \text{and} \quad D_1^* = \int_{1-r_1^*}^{r_0^*} q(x)dx = 0.$$

The conditions (6) and (21) lead to

$$p_0^* r_0^* - \frac{tr_0^{*2}}{2} = \psi^0(p_1^* - \delta) \tag{A18}$$

$$p_1^* r_1^* - \frac{tr_1^{*2}}{2} = \psi^0(p_0^* - \delta). \tag{A19}$$

Assume $\eta_0 > 0$. Then, $r_0^* = 1$ and $r_1^* = 0$ from (16). With (A19) $p_0^* = \delta \in (0, t)$, which leads to $\zeta_0 = 0$ by (10) and contradicts (4) and (8), leading, in turn, to $1 + \xi_0 = 0$. Thus, $\eta_0 = 0$. Because the same contradiction cannot be avoided as long as $r_1^* = 0$ is assumed, then $r_1^* > 0$ and $r_0^* < 1$. In the same manner, $\eta_1 = 0$, $r_0^* > 0$, and $r_1^* < 1$.

Assume $\xi_0 > 0$. Then, $p_0^* = \delta$ and $\zeta_0 = 0$, which lead to $1 - r_1^* + \xi_0 = 0$ from (4), which cannot hold, because of (33). Therefore, $\xi_0 = 0$. In the same manner, $\xi_1 = 0$.

Assume $p_1^* = \delta$. Then, $p_0^* = tr_0^*/2$ by (A18) as $r_0^* > 0$. Hence, $\zeta_1 = r_1^* > 0$ by (19), which contradicts the assumption $p_1^* = \delta$ because of (25). Thus, $p_1^* > \delta$. In the same manner, $p_0^* > \delta$.

Hereafter, the case $v > t + 3\delta$ is additionally assumed. Assume $\zeta_0 > 0$. Then, $p_0^* = v$ by (10), and the equation (A19) is transformed to $t(r_1^*)^2 - 2p_1^* r_1^* + 2v - 2\delta - t = 0$ because $v - \delta > t + 3\delta - \delta > t$ determines $\psi^0(v-\delta) = v - \delta - t/2$. Define the function $f(x) \overset{\text{def}}{=} tx^2 - 2p_1^* x + 2v - 2\delta - t$. This equation with unknown r_1^* is supposed to have solution $r_1^* \in [0,1]$. Because $f(0) = 2v - 2\delta - t > 0$ by the assumption $v > t + 3d$, the equation has a solution for $[0,1]$ in two cases. The first case is when $f(1) = 2v - 2\delta - 2p_1^* < 0$, and the second is $f(1) > 0$, $p_1^{*2} - t(2v - 2\delta - t) \geq 0$, and $0 < p_1^* < t$. The second case does not exist, because $p_1^{*2} - t(2v - 2\delta - t) < t^2 - t(2v - 2\delta - t) = 2t(t + \delta - v) < 0$. In the first case $p_1^* > v - \delta$, the conditions (A18) and (A19) are

$$v(1 - r_1^*) - \frac{t(1 - r_1^*)^2}{2} = p_1^* - \delta - \frac{t}{2} \tag{A20}$$

$$p_1^* r_1^* - \frac{t r_1^{*2}}{2} = v - \delta - \frac{t}{2}. \tag{A21}$$

Then, the condition $p_1^* > v - \delta$ with the equation (A20) requires $t(1 - r_1^*)^2 - 2v(1 - r_1^*) + 2v - 4\delta - t < 0$, which leads to $r_1^* < \left(-v + t + \sqrt{(v-t)^2 + 4\delta t}\right)/t$, while the condition $p_1^* \leq v$ by (27) with the equation (A21) requires $t r_1^{*2} - 2v r_1^* + 2v - 2\delta - t \leq 0$, which leads to $r_1^* \geq \left(v - \sqrt{(v-t)^2 + 2\delta t}\right)/t$. For these two conditions to be satisfied,

$$\frac{v - \sqrt{(v-t)^2 + 2\delta t}}{t} < \frac{-v + t + \sqrt{(v-t)^2 + 4\delta t}}{t},$$

is necessary, leading to

$$\begin{aligned}
2v - t &< \sqrt{(v-t)^2 + 2\delta t} + \sqrt{(v-t)^2 + 4\delta t} \\
&= (v - t)\left(\sqrt{1 + \frac{4\delta t}{(v-t)^2}} + \sqrt{1 + \frac{2\delta t}{(v-t)^2}}\right) \\
&< (v - t)\left(1 + \frac{4\delta t}{2(v-t)^2} + 1 + \frac{2\delta t}{2(v-t)^2}\right).
\end{aligned}$$

Then,$v - t - 3\delta < 0$, which does not hold, because $v > t + 3\delta$ is assumed. Thus, $\zeta_0 = 0$. In the same manner, $\zeta_1 = 0$ is shown.

$r_0^* > 0$, $p_0^* > \delta$, and $\xi_0 = \zeta_0 = 0$ imply $\lambda_0 > 0$ from (4), and $r_1^* > 0$, $p_1^* > \delta$, and $\xi_1 = \zeta_1 = 0$ imply $\lambda_1 > 0$ from (19). From (5), (14), and $\eta_0 = 0$, $(p_0 - tr_0) - \lambda_0 (p_1 - tr_1) \leq 0$, and from (20), (29), and $\eta_1 = 0$, $(p_1 - tr_1) - \lambda_1 (p_0 - tr_0) \leq 0$. Thus, (34) is proved.

Next, assume $p_0^* - tr_0^* < 0$. Under the assumption $v > t + 3\delta$, $\lambda_1 > 0$ and (34) lead to

$$0 \geq p_1^* - tr_1^* - \lambda_1(p_0^* - tr_0^*) > p_1^* - tr_1^*.$$

$p_0^* - tr_0^* < 0$ also implies $p_0^* < tr_0^* = \lambda_0(p_0^* - \delta)$ by (4) and $\xi_0 = \zeta_0 = 0$ because $\psi^1(p_0^* - \delta) = (p_0^* - \delta)/t$ as $p_0^* - \delta < tr_0^* - \delta < t$. Because $p_0^* > \delta$ by Lemma 21, $\lambda_0 > 1$. Hence, by (34)

$$0 \geq p_0^* - tr_0^* - \lambda_0(p_1^* - tr_1^*) > (p_0^* - tr_0^*) - (p_1^* - tr_1^*).$$

At the same time, $p_1^* - tr_1^* < 0$ implies

$$0 > (p_1^* - tr_1^*) - (p_0^* - tr_0^*),$$

by the same logic, which leads to a contradiction. Thus, $p_0^* - tr_0^* \geq 0$. In the same manner, $p_1^* - tr_1^* \geq 0$ can be ascertained. If $p_0^* - tr_0^* = 0$ and $p_1^* - tr_1^* = 0$, from (A18) and (A19), $t^2(r_0^*)^2 = (p_1^* - \delta)^2$ and $t^2(r_1^*)^2 = (p_0^* - \delta)^2$. Then, $tr_0^* = p_1^* - \delta$ and $tr_1^* = p_0^* - \delta$ because $p_i^* - \delta = tr_i^* - \delta < t$. This leads to $p_0^* = p_1^* - \delta$ and $p_1^* = p_0^* - \delta$, which is a contradiction because $\delta > 0$. Therefore, $p_0^* - tr_0^* > 0$ and/or $p_1^* - tr_1^* > 0$. □

Proof of Proposition 5. Assume $r_0^* + r_1^* = 1$ through this proof. From lemma 21

$$\begin{aligned} p_0^* - tr_0^* &\leq \lambda_0(p_1^* - tr_1^*) \leq \lambda_0\lambda_1(p_0^* - tr_0^*), \\ p_1^* - tr_1^* &\leq \lambda_1(p_0^* - tr_0^*) \leq \lambda_0\lambda_1(p_1^* - tr_1^*). \end{aligned}$$

Further, from lemma 21 $p_0^* - tr_0^* > 0$ and/or $p_1^* - tr_1^* > 0$. Then, $\lambda_0\lambda_1 \geq 1$.

There are three cases to be examined. First, $p_i^* - \delta \geq t$ for both $i \in \{0, 1\}$. Second, $p_i^* - \delta < t$ for both $i \in \{0, 1\}$. Lastly, $p_i^* - \delta \geq 0$ and $p_i^* - \delta < 0$, for $i, j \in \{0, 1\}, i \neq j$.

First, assume $p_i^* - \delta \geq t$ for both $i \in \{0, 1\}$. Then, $\psi^1(p_i^* - \delta) = 1$. From (4) and (19), $r_i = \lambda_i$ because $\xi_i = \zeta_i = 0$. This leads to $r_0^* r_1^* = \lambda_0\lambda_1 \geq 1$, and contradicts $r_i^* < 1$ from Lemma 21.

Second, assume $p_i^* - \delta < t$ for both $i \in \{0, 1\}$. Then, $\psi^0(p_i^* - \delta) = (p_i^* - \delta)^2/(2t)$ and $\psi^1(p_i^* - \delta) = (p_i^* - \delta)/t$. From (4) and (19), $\lambda_i = tr_i^*/(p_i^* - \delta)$ because $\xi_i = \zeta_i = 0$ by Lemma 21. Then,

$$\frac{t^2 r_0^* r_1^*}{(p_0^* - \delta)(p_1^* - \delta)} = \lambda_0\lambda_1 \geq 1$$

. Hence,
$$t^2 r_0^* r_1^* \geq (p_0^* - \delta)(p_1^* - \delta). \tag{A22}$$

Multiplying both sides of the equations of (A18) and (A19), and from (A22), we obtain

$$\begin{aligned}(p_0^* - \delta)^2(p_1^* - \delta)^2 &= 4t^2\psi^0(p_0^* - \delta)\psi^0(p_1^* - \delta)\\ &= 4t^2 r_0^* r_1^* \left(p_0^* - \frac{tr_0^*}{2}\right)\left(p_1^* - \frac{tr_1^*}{2}\right)\\ &\geq 4(p_0^* - \delta)(p_1^* - \delta)\left(p_0^* - \frac{tr_0^*}{2}\right)\left(p_1^* - \frac{tr_1^*}{2}\right).\end{aligned}$$

Therefore,
$$(p_0^* - \delta)(p_1^* - \delta) \geq 4\left(p_0^* - \frac{tr_0^*}{2}\right)\left(p_1^* - \frac{tr_1^*}{2}\right), \tag{A23}$$
as $p_i^* - \delta > 0$ are known through Lemma 21.

As

$$\begin{aligned}t^2 r_0^* r_1^* &\geq (p_0^* - \delta)(p_1^* - \delta) && \text{(by } (A22))\\ &\geq 4\left(p_0^* - \frac{tr_0^*}{2}\right)\left(p_1^* - \frac{tr_1^*}{2}\right) && \text{(by } (A23))\\ &= 4p_0^* p_1^* - 2tp_0^* r_1^* - 2tp_1^* r_0^* + t^2 r_0^* r_1^*,\end{aligned}$$

then
$$0 \geq 2p_0^* p_1^* - tp_0^* r_1^* - tp_1^* r_0^* = p_0^*(p_1^* - tr_1^*) + p_1^*(p_0^* - tr_0^*).$$
This means $p_i^* = tr_i^*$ for both $i \in \{0, 1\}$ because $p_i^* - tr_i^* \geq 0$ and $p_i^* > \delta > 0$, from Lemma 21. If so, the equations (A18) and (A19) lead to $tr_0^* = p_1^* - \delta$ and $tr_1^* = p_0^* - \delta$. Then, $p_0^* = p_1^* - \delta$ and $p_1^* = p_0^* - \delta$, a contradiction because $\delta > 0$.

For the last case, assume $p_0^* - \delta \geq t$ and $p_1^* - \delta < t$ without any loss of generality. Then, $\psi^0(p_0^* - \delta) = p_0^* - \delta - t/2$, $\psi^0(p_1^* - \delta) = (p_1^* - \delta)^2/t$, $\psi^1(p_0^* - \delta) = 1$, and $\psi^1(p_1^* - \delta) = (p_1^* - \delta)/t$. From (4) and (19), $\lambda_0 = r_0^*$ and $\lambda_1 = tr_1^*/(p_1^* - \delta)$. Then, $tr_0^* r_1^*/(p_1^* - \delta) = \lambda_0\lambda_1 \geq 1$, which leads to

$$tr_0^* r_1^* \geq p_1^* - \delta. \tag{A24}$$

Besides, from (A18) and $p_0^* - \delta \geq t$

$$\frac{(p_1^* - \delta)^2}{2t} = p_0^* r_0^* - \frac{t(r_0^*)^2}{2} \geq (t + \delta)\, r_0^* - \frac{t(r_0^*)^2}{2}. \tag{A25}$$

From (A24),

$$\frac{t^2(r_0^*r_1^*)^2}{2t} \geq \frac{(p_1^*-\delta)^2}{2t} \geq (t+\delta)r_0^* - \frac{t(r_0^*)^2}{2}$$

. Then,

$$\frac{t(1-r_1^*)}{2}\left(1+(r_1^*)^2\right) = \frac{tr_0^*}{2}\left(1+(r_1^*)^2\right) \geq t+\delta.$$

This inequality does not hold, because $(1-r_1^*)(1+(r_1^*)^2) \leq 1$ for $r_1^* \in [0,1]$. □

Proof of Lemma 22. Assume $r_0^* = 1$ and $r_1^* = 1$. Then, $\mu_0 = \mu_1 = 0$ from (13) and (28). Considering the case $p_0^* - \delta \geq t$ and $p_1^* - \delta \geq t$, conditions (6) and (21) are solved for p_0^* and p_1^* as follows:

$$p_0^* = \frac{2\delta(2-D_0^*) + t(1+D_0^* - 4D_1^* + 2D_0^*D_1^*)}{2(1-D_0^* + D_0^{*2})}, \tag{A26}$$

$$p_1^* = \frac{2\delta(1+D_0^*) + t(1-2D_1^* + 2D_0^{*2} - 2D_0^*D_1^*)}{2(1-D_0^* + D_0^{*2})}. \tag{A27}$$

The conditions required for the value of δ satisfying $p_i^* \in [t+\delta, v]$ are $\delta \in [\delta_i^*, \delta_i^{**}]$ for $i \in \{0,1\}$, where

$$\begin{aligned}
\delta_0^* &= t\frac{(1-D_0^*)(1-2D_0^*+2D_1^*)+2D_1^*}{2(1-D_0^{*2})}, \\
\delta_1^* &= t\frac{1-2D_0^*+2D_1^*+2D_0^*D_1^*}{2D_0^*(2-D_0^*)}, \\
\delta_0^{**} &= \frac{2v(1-D_0^*+D_0^{*2}) - t(1+D_0^*-4D_1^*+2D_0^*D_1^*)}{2(2-D_0^*)}, \\
\delta_1^{**} &= \frac{2v(1-D_0^*+D_0^{*2}) - t(1-2D_1^*+2D_0^{*2}-2D_0^*D_1^*)}{2(1+D_0^*)}.
\end{aligned}$$

Note that

$$1-2D_0^*+2D_1^* = 2\left(\int_0^1 (1-x)(1-q(x))dx\right) > 0,$$

. Then, $\delta_i^* > 0$ for $i \in \{0,1\}$. The intersection of the regions exists only when $\delta_i^{**} \geq \delta_j^*$ for any combination of $i,j \in \{0,1\}$, requiring $v \geq \max_{i\in\{0,\cdots,3\}}(\kappa_i t)$, where

$$\kappa_0 = \frac{3(1-D_0^*) + 2D_1^*(2-D_0^*)}{2\left(1-D_0^{*2}\right)}, \qquad \kappa_1 = \frac{1+2D_0^*(1-D_0^*) + 2D_1^*(1+D_0^*)}{2D_0^*\,(2-D_0^*)}$$

$$\kappa_2 = \frac{1+2D_1^*}{2D_0^*}, \qquad \kappa_3 = \frac{1-D_0^*+D_1^*}{1-D_0^*}.$$

Thus, if and only if $\max_i(\kappa_i)$ exists and $v \geq \max_i(\kappa_i)$, there exists δ, with which the solution to (6) and (21) exists. Because the function $q(x)$ is assumed such that $q:[0,1] \to (0,1)$, $D_0^* \in (0,1)$. Then, each value of κ_i and also $\max_i(\kappa_i)$ exists. In such a case, $\xi_i = 0$ for $i \in \{0,1\}$ from (7) and (22). The equations (4), (5), and (10) determine $(\lambda_0, \zeta_0, \eta_0)$, and the equations (19), (20), and (25) determine $(\lambda_1, \zeta_1, \eta_1)$.

Finally, the condition $|p_0^* - p_1^*| \leq \delta$ is checked to ascertain the existence of the solution. From (A26), (A27), and the condition $\delta \geq \delta_0^*$,

$$\begin{aligned}
\delta - (p_0^* - p_1^*) &= \frac{2\delta D_0^*(1+D_0^*) - t(4D_0^*D_1^* - 2D_1^* - 2D_0^{*2} + D_0^*)}{2(1-D_0^*+D_0^{*2})} \\
&\geq \frac{2\delta_0^* D_0^*(1+D_0^*) - t(4D_0^*D_1^* - 2D_1^* - 2D_0^{*2} + D_0^*)}{2(1-D_0^*+D_0^{*2})} \\
&= \frac{D_1^*}{1-D_0^*} > 0,
\end{aligned}$$

and from (A26), (A27), and the condition $\delta \geq \delta_1^*$,

$$\begin{aligned}
\delta + (p_0^* - p_1^*) &= \frac{2\delta(2-3D_0^*+D_0^{*2}) + t(4D_0^*D_1^* - 2D_1^* - 2D_0^{*2} + D_0^*)}{2(1-D_0^*+D_0^{*2})} \\
&\geq \frac{2\delta_1^*(2-3D_0^*+D_0^{*2}) + t(4D_0^*D_1^* - 2D_1^* - 2D_0^{*2} + D_0^*)}{2(1-D_0^*+D_0^{*2})} \\
&= \frac{1-2D_0^*+2D_1^*}{2D_0^*} > 0.
\end{aligned}$$

Thus, $\kappa = \max_i(\kappa_i)$ and $\Delta = (\max(\delta_0^*, \delta_1^*), \min(\delta_0^{**}, \delta_1^{**})]$ gives the lemma.

□

Proof of Lemma 23. Assume $r_0^* = 1$ and $r_1^* = 1$. Then, $\mu_0 = \mu_1 = 0$ by (13) and (28). Considering the case $p_0^* - \delta < t$ and $p_1^* - \delta < t$, conditions (21) and (6) are stated as follows:

$$(p_0^* - \delta)^2 - 2t(1-D_0^*)p_1^* + t^2\,(1 - 2D_0^* + 2D_1^*) = 0, \tag{A28}$$

$$(p_1^* - \delta)^2 - 2tD_0^*p_0^* + 2t^2D_1^* = 0. \tag{A29}$$

If the locus of equation (A28) and $p_0^* = p_1^*$ in the space (p_0^*, p_1^*) have one or more intersections, $(p_0^*, p_1^*) = (P, P)$, where P is the solution to the equation of P,

$$(P-\delta)^2 - 2t(1-D_0^*)P + t^2\left(1 - 2D_0^* + 2D_1^*\right) = 0.$$

As the discriminant of the quadratic equation is $t^2(D_0^{*2} - 2D_1^*) + 2t\delta(1-D_0^*)$, when $\delta < (2D_1^* - D_0^{*2}) / (2(1-D_0^*))$, they do not have any intersection. In this case, the locus of the equation (A28) is limited in the region $p_1^* > p_0^*$. Note that, because $2D_1([1-r_1, r_0]) - D_0([(1-r_1, r_0])^2 = 0$ when $r_0 + r_1 = 1$, and the differentiation by r_0 gives $2q(r_0)r_0 - 2D_0 q(r_0) = 2q(r_0)(r_0 - D_0) > 0$, then $2D_1([1-r_1, 1]) - D_0([1-r_1, 1])^2 > 0$ for any r_1. Hence, $2D_1^* - D_0^{*2} > 0$.

In the same manner, when $\delta < (2D_1^* - D_0^{*2}) / (2D_0^*)$, the locus of equation (A29) is limited in region $p_1^* < p_0^*$. Therefore, when

$$\delta < \delta^*, \text{ where } \delta^* \stackrel{\text{def}}{=} \min\left(\frac{2D_1^* - D_0^{*2}}{2(1-D_0^*)}, \frac{2D_1^* - D_0^{*2}}{2D_0^*},\right)$$

the equations (A28) and (A29) have no solution.

The solution (p_0^*, p_1^*) that satisfies (A28) and (A29) must also satisfy (4), (5),(19), and (20). They are stated as

$$tD_0^* - \lambda_0(p_0^* - \delta) = 0 \tag{A30}$$
$$p_0^* - t - \lambda_0 p_1^* \geq 0, \tag{A31}$$
$$t\left(1 - D_0^*\right) - \lambda_1(p_1^* - \delta) = 0 \tag{A32}$$
$$p_1^* - t - \lambda_1 p_0^* \geq 0 \tag{A33}$$

because, if $\xi_0 > 0$, then $p_0^* = \delta$ and $\zeta_0 = 0$, which lead to $D_0^* + \xi_0 = 0$ from (4), contradicting $D_0^* > 0$ and $\xi_0 \geq 0$, and also because $\xi_1 > 0$ leads to the same contradiction. (A30) and (9) lead to $\lambda_0 \geq 0$. With (A31), $p_0^* - t \geq 0$. In the same manner, (A32), (A33), and (24) lead to $p_1^* - t \geq 0$. □

Proof of Lemma 24. Assume $r_0^* = 1$ and $r_1^* = 1$. Then, $\mu_0 = \mu_1 = 0$ by (13) and (28). Considering the case $p_0^* - \delta < t$ and $p_1^* - \delta \geq t$ without any loss of generality, the condition (6) is stated as

$$p_0^* D_0^* - tD_1^* - \left(p_1^* - \delta - \frac{t}{2}\right) = 0.$$

Then, with $p_1^* - \delta \geq t$,

$$t \leq p_1^* - \delta = \frac{t}{2} + p_0^* D_0^* - tD_1^*.$$

From this inequality and $p_0^* < t+\delta$,

$$t+\delta > p_0^* \geq \frac{\frac{t}{2}+tD_1^*}{D_0^*} > t.$$

Note that

$$\frac{1}{2}+D_1^*-D_0^* = \int_0^1 (1-x)\,(1-q(x))\,dx > 0.$$

For p_0^*, the condition

$$\delta > \frac{\frac{t}{2}+tD_1^*}{D_0^*} - t > 0$$

is necessary. Thus, setting the central term of this inequality as δ^{**}, if $\delta < \delta^{**}$, no solution exists that satisfies the UPE. □

Proof of Proposition 7. Assume $p_0^* = p_1^* = v$ for the solution to the UPE. From (6) and (21),

$$\begin{aligned}
&v(r_1^* - D_0^*) + \frac{t}{2}(r_1^{*2} - 2r_1^*) + tD_1^* = \delta,\\
&v(1 - r_1^* + D_0^*) + \frac{t}{2}(r_1^{*2} - 1) - tD_0^* + tD_1^* = \delta.
\end{aligned}$$

Then,

$$D_0^* = r_1^* - \frac{1}{2}, \quad D_1^* = \frac{t(2r_1^* - r_1^{*2}) - (v - 2\delta)}{2t}.$$

$D_1^* > 0$ requires $v - 2\delta < t \max_{r_1^* \in [0,1]} (2r_1^* - r_1^{*2}) = t$. When $v > t + 2\delta$, this condition is not satisfied. □

Proof of Lemma 25. Considering the case $p_0^* - \delta > t$ and $p_1^* - \delta > t$, conditions (6) and (21) are solved for p_0^* and p_1^* as

$$\begin{aligned}
p_0^* = \Big(&2\delta(1 + r_1^* - D_0^*) - t\Big(r_1^{*3} \qquad\qquad \text{(A34)}\\
&-(1 + D_0^*)r_1^{*2} + 2(D_0^* + D_1^*)r_1^* + 2(1 - D_0^*)D_1^* - 2D_0^* - 1\Big)\Big)/K,\\
p_1^* = \Big(&2\delta(2 - r_1^* + D_0^*) - t\Big(-r_1^{*3} + (2 + D_0^*)r_1^{*2} \qquad\qquad \text{(A35)}\\
&-(1 - 2D_0^* + 2D_1^*)r_1^* + 2D_1^*(2 + D_0^*) - 2D_0^{*2} - 3D_0^* - 1\Big)\Big)/K,
\end{aligned}$$

where $K \stackrel{\text{def}}{=} 2(1 - r_1^* + D_0^* + (r_1^* - D_0^*)^2)$.

The conditions required for the value of δ satisfying $p_i^* > t + \delta$ are $\delta > \delta_i^+$ for $i \in \{0, 1\}$, where

$$\delta_0^+ \stackrel{\text{def}}{=} t\frac{2r_1^{*2} - 2r_1^* - 2D_0^* + 4D_1^* + 1}{2(2 - r_1^* + D_0^*)} + t\frac{(1 - r_1^* + D_0^*)((1 - r_1^*)^2 + 2D_1^*)}{2(r_1^* - D_0^*)(2 - r_1^* + D_0^*)},$$

$$\delta_1^+ \stackrel{\text{def}}{=} t\frac{2r_1^{*2} - 2r_1^* - 2D_0^* + 4D_1^* + 1}{2(1 + r_1^* - D_0^*)} + t\frac{(r_1^* - D_0^*)(r_1^{*2} - 2D_0^* + 2D_1^*)}{2(1 - r_1^* + D_0^*)(1 + r_1^* - D_0^*)}$$

Note that $2r_1^{*2} - 2r_1^* - 2D_0^* + 4D_1^* + 1 > 0$ because setting $f_0(r_0^*, r_1^*) \stackrel{\text{def}}{=} 2r_1^{*2} - 2r_1^* - 2D_0^* + 4D_1^* + 1$, $\partial f_0/\partial r_0^* = 2q(r_0^*)(2r_0^* - 1)$ and $\partial f_0/\partial r_1^* = 2(1 - q(1 - r_1^*))(2r_1^* - 1)$, $f_0(r_0^*, r_1^*) \geq f_0(1/2, 1/2) = 1/2 > 0$. Further, $r_1^{*2} - 2D_0^* + 2D_1^* > 0$ because setting $f_1(r_0^*, r_1^*) \stackrel{\text{def}}{=} r_1^{*2} - 2D_0^* + 2D_1^*$, $\partial f_1/\partial r_0^* = -2q(r_0^*)(1 - r_0^*) < 0$ and $\partial f_1/\partial r_1^* = 2r_1^*(1 - q(1 - r_1^*)) > 0$. Then, $f_1(r_0^*, r_1^*) \geq f_1(1, 0) = 0$. Thus, $\delta_i^+ > 0$ for $i \in \{0, 1\}$. Although δ_i^+s are functions of r_i^*s, there is no case such that either δ_i^+ converges to zero, as the first terms have positive minimum values. Therefore, when $\delta < \min(\delta_0^+, \delta_1^+)$, no solution exists that satisfies (6) and (21) such that $p_i > t + \delta$ for both of i, $i \in \{0, 1\}$.

Next, consider the case where $p_0^* - \delta \leq t$ and $p_1^* - \delta > t$. The condition (6) leads to

$$\begin{aligned} t\left(D_1^* + \frac{(1 - r_1^*)^2}{2}\right) &= p_0^* \left(1 - r_1^* + D_0^*\right) - \left(p_1^* - d - \frac{t}{2}\right) \\ &< (t + \delta)\left(1 - r_1^* + D_0^*\right) - \left(t - \frac{t}{2}\right), \end{aligned}$$

which requires the condition

$$\delta > \frac{\frac{r_1^{*2}}{2} + D_1^* - D_0^*}{1 - r_1^* + D_0^*}.$$

The numerator of the right-hand side takes its minimum value as zero when $r_0^* = 1$ and $r_1^* = 0$, as is shown above ($f_1(r_0^*, r_1^*)/2$). When δ takes a value sufficiently small, the values of r_0^* and r_1^* converge to 1 and 0, respectively, requiring p_0^* to converge to δ by (21), as both D_0^* and D_1^* converge to zero. Then, with equation (6), p_1^* converges to 2δ, which contradicts the assumption $p_1^* > \delta + t$ because δ is assumed to take a sufficiently small value. Thus, in this case as well, there exists a value of δ^* such that no solution exists that satisfies (6) and (21) when $\delta < \delta^*$. In the other case, where $p_0^* - \delta > t$ and $p_1^* - \delta \leq t$, the same is proved in the same manner. $\square$

Proof of Lemma 26. As $p_i^* - \delta < t$, $\psi_0(p_i^* - \delta) = (p_i^* - \delta)^2/(2t)$. Then, (35) and (36) are directly given from (21) and (6), respectively.

$r_0^* + r_1^* > 1$, and (13) leads to $\mu_0 = 0$. By the same manner, (28) leads to $\mu_1 = 0$. $1 > r_0^*$, and (16) leads to $\eta_0 = 0$. $1 > r_1^*$, and (31) lead to $\eta_1 = 0$. $p_0^* < t + \delta < v$, and (10) leads to $\zeta_0 = 0$. $p_1^* < t + \delta < v$, and (25) leads to $\zeta_1 = 0$.

Assume $\xi_0 > 0$. (7) leads to $p_0^* = \delta$. Then, $\psi^1(p_0^* - \delta) = 0$. Thus, (4) indicates $1 - r_1^* + D_0^* + \xi_0 = 0$. However, this contradicts with $\xi_0 > 0$ because $1 - r_1^* + D_0^* > 0$ for $1 - r_1^* > 0$ and $r_0^* + r_1^* > 1$. Therefore, $\xi_0 = 0$ and $p_0^* - \delta > 0$. In the same manner, $\xi_1 = 0$ and $p_1^* - \delta > 0$.

Thus, $\mu_i = \eta_i = \zeta_i = \xi_i = 0$. Then, (4) and (5) mean

$$1 - r_1^* + D_0^* - \lambda_0 \frac{p_0^* - \delta}{t} = 0, \quad (p_0^* - tr_0^*) - \lambda_0(p_1^* - t(1 - r_0^*)) = 0,$$

respectively, because $\psi^1(p_0^* - \delta) = (p_0^* - \delta)/t$ for $p_0^* - \delta < t$, and $q(r_0^*) > 0$ for $r_0^* \in (0, 1)$ by Assumption 2. Eliminating λ_0 from these equations gives (37). In the same manner, (19) and (20) lead to

$$r_1^* - D_0^* - \lambda_1 \frac{p_1^* - \delta}{t} = 0, \quad (p_1^* - tr_1^*) - \lambda_1(p_0^* - t(1 - r_1^*)) = 0,$$

which gives (38). □

Proof of Lemma 27. The proof consists of three parts. First, the map M_0,

$M_0 : U^2 \to \Re^{++2}, M_0(r_0^*, r_1^*) = (p_0^*, p_1^*), (p_0^*, r_0^*, p_1^*, r_1^*)$ satisfies (35) and (36),

is analyzed. Next, the map M_1,

$M_1 : \Re^{++2} \to U^2, M_1(p_0^*, p_1^*) = (r_0^*, r_1^*), (p_0^*, r_0^*, p_1^*, r_1^*)$ satisfies (37) and (38)

, is analyzed. Then, the map $M_0 * M_1 : U^2 \to U^2$ is shown as contraction mapping. Thus, the map $M_0 * M_1$ has a fixed point, which is the solution to the equations (35) $\sim$ (38).

1. Map $M_0(r_0^*, r_1^*) = (p_0^*, p_1^*)$ by (35) and (36).

Substituting variables by $(\rho_0, \rho_1) = ((p_0^* + p_1^*)/2 - \delta, (p_0^* - p_1^*)/2)$, (35) and (36) are transferred to

$$\left(\rho_0 - \frac{t}{2}\right)^2 + (\rho_1 - r_2 t)^2 = t\delta + W \tag{A36}$$

$$\left(\rho_0 + \frac{t}{2}\right)(\rho_1 + r_2 t) = t(t - \delta) r_2, \tag{A37}$$

where

$$W(r_0^*, r_1^*) \stackrel{\text{def}}{=} t^2 \left(D_0^{*2} + 2D_0^* - 2r_1^* D_0^* - 2D_1^* \right), \text{ and } r_2 \stackrel{\text{def}}{=} \frac{1}{2} - r_1^* + D_0^*.$$

The equations (A36) and (A37) are $((35) + (36))/2$ and $((35) - (36))/4$, respectively. Hereafter, in this proof, $W(r_0^*, r_1^*)$ is simply shown as W when no confusion is expected. The locus (ρ_0, ρ_1) that satisfies (A36) is a circle with the center $(t/2, r_2 t)$ and radius $\sqrt{t\delta + W}$ if $t\delta + W > 0$, while the locus (A37) is hyperbolic. If $t\delta + W > 0$, $(\rho_0, \rho_1) = (t/2, r_2 t \pm \sqrt{t\delta + W})$ are two ends of a diameter of the circle. Therefore, if $t\delta + W > 0$ and

$$f_0 \left(\frac{t}{2}, r_2 t + \sqrt{t\delta + W} \right) \cdot f_0 \left(\frac{t}{2}, r_2 t - \sqrt{t\delta + W} \right) < 0,$$

where $f_0(\rho_0, \rho_1) \stackrel{\text{def}}{=} (\rho_0 + t/2)(\rho_1 + r_2 t) - r_2 t(t - \delta)$, (A36) and (A37) have the solution (ρ_0^*, ρ_1^*) such that $\rho_0^* \in (t/2, t/2 + \sqrt{t\delta + W}]$.

The latter condition is equivalent to:

$$\begin{aligned} &\left(t(t+\delta) r_2 + t\sqrt{t\delta + W} \right) \cdot \left(t(t+\delta) r_2 - t\sqrt{t\delta + W} \right) \\ &= t^2 \left\{ r_2^2 (t+\delta)^2 - (t\delta + W) \right\} < 0. \end{aligned} \tag{A38}$$

If the inequality (A38) is satisfied, the former condition $t\delta + W > 0$ is also satisfied, so that, for the solution of (A36) and (A37), $\rho_0^* \in (t/2, t/2 + \sqrt{t\delta + W}]$.

Here, condition $(r_0^*, r_1^*) \in \Omega_0 \subset U^2$ is shown as a sufficient condition for the inequality (A38), where

$$\Omega_0 \stackrel{\text{def}}{=} \left\{ (r_0^*, r_1^*) \,\middle|\, r_0^* < \frac{1}{2} + \frac{\sqrt{t\delta}}{2t}, r_1^* < \frac{1}{2} + \frac{\sqrt{t\delta}}{2t}, r_0^* + r_1^* > 1. \right\} \tag{A39}$$

This is proved below. Differentiating W by r_1^* for a given r_0^*,

$$\frac{\partial W}{\partial r_1^*} = -2t^2 q_1^* D_0^* < 0,$$

where $q_1^* \stackrel{\text{def}}{=} 1 - q(1 - r_1^*)$. Therefore, $W(r_0^*, r_1^*) < W(r_0^*, 1 - r_0^*) = 0$ because $D_i^*([r_0^*, 1 - r_0^*]) = 0$. Then, regarding the inequality (A38),

$$\begin{aligned} t\delta + W - r_2^2(t+\delta)^2 &= t\delta + t^2(D_0^{*2} + 2D_0^* - 2r_1^* D_0^* - 2D_1^*) - r_2^2(t+\delta)^2 \\ &> t\delta + (t+\delta)^2(D_0^{*2} + 2D_0^* - 2r_1^* D_0^* - 2D_1^*) - r_2^2(t+\delta)^2 \\ &= t\delta + (t+\delta)^2 \left(D_0^* - 2D_1^* - \left(\frac{1}{2} - r_1^* \right)^2 \right), \end{aligned}$$

and denoting the value of the last expression as $f_1(r_0^*, r_1^*)$, the differentiations are

$$\frac{\partial f_1}{\partial r_0^*} = (t+\delta)^2(1-2r_0^*)q_0^*, \quad \frac{\partial f_1}{\partial r_1^*} = (t+\delta)^2(1-2r_1^*)q_1^*,$$

where $q_0^* \stackrel{\text{def}}{=} q(r_0^*)$. Then, considering the closure,$\overline{\Omega}_0$, of region Ω_0, $f_1(r_0^*, r_1^*)$ takes its minimum value in either $\{(r_0^*, r_1^*)|\ r_0^* + r_1^* = 1\}$ or $\left(1/2 + \sqrt{t\delta}/(2t), 1/2 + \sqrt{t\delta}/(2t)\right)$. When $t > \delta$ is assumed,

$$\begin{aligned}
&\min_{(r_0^*, r_1^*)\in\Omega_0} f_1(r_0^*, r_1^*) \geq \min_{(r_0^*, r_1^*)\in\overline{\Omega}_0} f_1(r_0^*, r_1^*) \\
&= \min\left\{\min_{r_1^*\in\left[\frac{1}{2}-\frac{\sqrt{t\delta}}{2t}, \frac{1}{2}+\frac{\sqrt{t\delta}}{2t}\right]} f_1(1-r_1^*, r_1^*), f_1\left(\frac{1}{2}+\frac{\sqrt{t\delta}}{2t}, \frac{1}{2}+\frac{\sqrt{t\delta}}{2t}\right)\right\} \\
&= \min\left\{f_1\left(\frac{1}{2}\pm\frac{\sqrt{t\delta}}{2t}, \frac{1}{2}\mp\frac{\sqrt{t\delta}}{2t}\right), t\delta + (t+\delta)^2\left(D_0^* - 2D_1^* - \frac{\delta}{4t}\right)\right\} \\
&= \min\left\{\frac{\delta(3t+\delta)(t-\delta)}{4t}, \frac{\delta(3t+\delta)(t-\delta)}{4t}\right. \\
&\qquad\qquad \left. +(t+\delta)^2\int_{\frac{1}{2}-\frac{\sqrt{t\delta}}{2t}}^{\frac{1}{2}+\frac{\sqrt{t\delta}}{2t}} (1-2x)\, q(x)dx\right\} \\
&\geq \frac{\delta(3t+\delta)(t-\delta)}{4t} + \min\left\{0, (t+\delta)^2\int_{\frac{1}{2}-\frac{\sqrt{t\delta}}{2t}}^{\frac{1}{2}+\frac{\sqrt{t\delta}}{2t}} (1-2x)\, q\left(\frac{1}{2}\right) dx\right\} \\
&\quad\left((1-2x)q(x) \geq (1-2x)q\left(\frac{1}{2}\right) \quad \because q'(x) < 0, \right) \\
&= \frac{\delta(3t+\delta)(t-\delta)}{4t} + \min\left\{0, (t+\delta)^2 q\left(\frac{1}{2}\right)\int_{\frac{1}{2}-\frac{\sqrt{t\delta}}{2t}}^{\frac{1}{2}+\frac{\sqrt{t\delta}}{2t}} (1-2x)\, dx\right\} \\
&= \frac{\delta(3t+\delta)(t-\delta)}{4t} > 0.
\end{aligned}$$

Thus, $t\delta + W > r_2^2(t+\delta)^2$. Here, $t > \delta$ is assumed. In the last part of this proof, it is shown that condition $\delta < \min(t/9, \delta^{**})$ is sufficient for the existence of the solution, where the definition of δ^{**} is given. Meanwhile, $\delta < t$ is assumed.

Next, check the range (p_0^*, p_1^*) mapped from Ω_0 by (35) and (36). The locus of the hyperbolic (A37) on the space (ρ_0, ρ_1) with horizontal axis ρ_0 and vertical axis ρ_1 is monotonously decreasing (increasing) when $r_2 > 0$ ($r_2 < 0$) at $\rho_0 > t/2$. Besides, it is easy to check that the points $(t/2, -r_2\delta)$ and

$(t/2+\sqrt{t\delta}, -r_2\sqrt{t\delta})$ are on the locus of the hyperbolic (A37), while they are inside and outside of the circle (A36). Then, for the intersection of the circle (A36) and the hyperbolic (A37) that is denoted as (ρ_0^*, ρ_1^*),we have

$$\begin{aligned} \frac{t}{2} < \rho_0^* < \frac{t}{2} + \sqrt{t\delta}, \\ -r_2\sqrt{t\delta} \le \rho_1^* \le -r_2\delta, \ \text{ when } r_2 \ge 0, \\ -r_2\delta < \rho_1^* < -r_2\sqrt{t\delta} \ \text{ when } r_2 < 0. \end{aligned}$$

These conditions lead to

$$2\delta + t < p_0^* + p_1^* < 2\delta + t + 2\sqrt{t\delta}. \tag{A40}$$

Further, because $D_0^*([1-r_1^*, r_0^*]) \in (0, r_0^* + r_1^* - 1)$,

$$r_2 = \frac{1}{2} - r_1^* + D_0^* \in \left(\frac{1}{2} - r_1^*, r_0^* - \frac{1}{2}\right) \subset \left(-\frac{\sqrt{t\delta}}{2t}, \frac{\sqrt{t\delta}}{2t}\right),$$

when $(r_0^*, r_1^*) \in \Omega_0$. Then,

$$\begin{aligned} -\frac{\delta}{2} < -r_2\sqrt{t\delta} \le \rho_1^* \le -r_2\delta \le 0, \text{ when } r_2 \ge 0 \\ 0 < -r_2\delta < \rho_1^* < -r_2\sqrt{t\delta} < \frac{\delta}{2}, \text{ when } r_2 < 0. \end{aligned}$$

Thus,

$$|\rho_1^*| < \frac{\delta}{2},$$

which leads to

$$|p_0^* - p_1^*| < \delta. \tag{A41}$$

(A40) and (A41) limit range $M_0(\Omega_0)$ as

$$M_0(\Omega_0) \subset \Omega_1 \stackrel{\text{def}}{=} \left\{ (p_0^*, p_1^*) \;\middle|\; \begin{array}{l} t + 2\delta < p_0^* + p_1^* < t + 2\delta + 2\sqrt{t\delta}, \\ |p_0^* - p_1^*| < \delta. \end{array} \right\} \tag{A42}$$

Check that, if $(p_0^*, p_1^*) \in \Omega_1$,

$$p_i^* \in \left(\frac{t+\delta}{2}, \frac{t+3\delta}{2} + \sqrt{t\delta}\right). \tag{A43}$$

2. Map $M_1(p_0^*, p_1^*) = (r_0^*, r_1^*)$ by (37) and (38).

In this part, $\exists(r_0^*, r_1^*) \in \Omega^*,\ \forall(p_0^*, p_1^*) \in \Omega_1\ (r_0^*, p_0^*, r_1^*, p_1^*)$ satisfy (37) and (38) is proved, where

$$\Omega^* \stackrel{\text{def}}{=} \{(r_0^*, r_1^*) |\ r_0^* < 1, r_1^* < 1, r_0^* + r_1^* > 1\}\,.$$

For this purpose, assuming $(p_0^*, p_1^*) \in \Omega_1$ and $(r_0^*, r_1^*) \in \Omega^*$, the existence of such (r_0^*, r_1^*) is shown. Then, in the remaining part of section 2, $(p_0^*, p_1^*) \in \Omega_1$, and $(r_0^*, r_1^*) \in \Omega^*$ is assumed when no confusion is expected.

First, define a function $f_2(r_0)$ and $\check{r}_0$ as

$$f_2(r_0) \stackrel{\text{def}}{=} (p_0^* - \delta)(p_0^* - tr_0) - tr_0(p_1^* - t + tr_0),\ \ f_2(\check{r}_0) = 0,\ \ \check{r}_0 \in (0, 1).$$

Hereafter, in this part of the proof, $t > 9\delta$ is assumed. Under this assumption,

$$p_0^* > \frac{t+\delta}{2} > 5\delta > \delta.\ (\because (A43)). \tag{A44}$$

When $(p_0^*, p_1^*) \in \Omega_1$,

$$t - p_0^* > t - \left(\frac{t}{2} + \frac{3\delta}{2} + \sqrt{t\delta}\right) = \frac{(\sqrt{t} - 3\sqrt{\delta})(\sqrt{t} + \sqrt{\delta})}{2} > 0. \tag{A45}$$

Then, $\check{r}_0$ exists because $f_2(0) = p_0^*(p_0^* - \delta) > 0$ and $f_2(1) = (p_0^* - \delta)(p_0^* - t) - tp_1^* < 0$. Consider a space of (r_0^*, r_1^*) with a horizontal axis r_0^* and a vertical axis r_1^*. It is easily checked that the point $(\check{r}_0, 1 - \check{r}_0)$ is on l_0, where l_0 is defined as the locus (r_0^*, r_1^*) of (37). Because of $D_0^*([1 - r_1^*, r_0^*]) > 0$ and (A44),

$$\frac{(p_0^* - tr_0^*)(p_1^* - t + tr_0^*)}{(p_0^* - tr_0^*)^2} = \frac{p_0^* - \delta}{t(1 - r_1^* + D_0^*)} > 0$$

from (37). Note that $1 - r_1^* + D_0^* > 0$, when $(r_0^*, r_1^*) \in \Omega^*$. If $p_0^* < tr_0^*$ and $p_1^* < t(1 - r_0^*)$, then $p_0^* + p_1^* < t$, which contradicts the assumption $(p_0^*, p_1^*) \in \Omega_1$. Thus,

$$p_0^* > tr_0^* \text{ and } p_1^* > t(1 - r_0^*). \tag{A46}$$

Differentiating (37), we obtain

$$\left.\frac{dr_0^*}{dr_1^*}\right|_{(r_0^*, r_1^*) \in l_0} = \frac{q_1^*\,(p_1^* - t + tr_0^*)}{(p_0^* - \delta) + q_0^*(p_1^* - t + tr_0^*) + t(1 - r_1^* + D_0^*)} > 0$$

because of (A44) and (A46).

The value of the left-hand side of equation (37) takes the following form:

$$\begin{array}{ll} (p_0^* - \delta)p_0^* > 0, & \text{when } (r_0^*, r_1^*) = (0,1), \\ (p_0^* - \delta)(p_0^* - t) - tD_0([0,1])p_1^* < 0, (\because (A45)) & \text{when } (r_0^*, r_1^*) = (1,1). \end{array}$$

Thus, the locus l_0 is an increasing curve on space (r_0^*, r_1^*) that passes the point $(\check{r}_0, 1-\check{r}_0)$ and a point on a segment $r_1^* = 1,\ r_0^* \in (\check{r}_0, 1)$.

In the same manner, defining a function $f_3(r_1)$ and $\check{r}_1$ as

$$f_3(r_1) \overset{\text{def}}{=} (p_1^* - \delta)(p_1^* - tr_1) - tr_1(p_0^* - t + tr_1),\ f_3(\check{r}_1) = 0,\ \check{r}_1 \in (0,1), \quad \text{(A47)}$$

it is checked that the point $(1-\check{r}_1, \check{r}_1)$ is on l_1, where l_1 is defined as the locus (r_0^*, r_1^*) of (38). In the process,

$$t > p_1^* > \delta,\ \ p_1^* - tr_1^* > 0,\ \ p_0^* - t + tr_1^* > 0 \quad \text{(A48)}$$

are proved.

$$\left.\frac{dr_0^*}{dr_1^*}\right|_{(r_0^*, r_1^*) \in l_1} > 0$$

is proved as well. Then, locus l_1 is an increasing curve that passes the point $(1-\check{r}_1, \check{r}_1)$ and a point on a segment $r_0^* = 1,\ r_1^* \in (\check{r}_1, 1)$.

Next, $\check{r}_0 + \check{r}_1 > 1$ is checked. First, evaluating

$$\begin{aligned} f_3\left(\frac{p_1^* - \delta}{t}\right) &= (p_1^* - \delta)\delta - (p_1^* - \delta)(p_0^* + p_1^* - t - \delta) \\ &= -(p_1^* - \delta)(p_0^* + p_1^* - t - 2\delta) < 0, \end{aligned}$$

it is known that

$$(p_1^* - \delta)/t > \check{r}_1, \quad \text{(A49)}$$

because $f_3(r_1)$ is decreasing for $r_1 > 0$, as $f_3'(r_1) = -t\,(p_0 + p_1 - t - \delta + 2tr_1) < 0$. Further, evaluating

$$\begin{aligned} f_2\,(1-\check{r}_1) &= (p_0^* - \delta)(p_0^* - t(1-\check{r}_1)) - t(1-\check{r}_1)(p_1^* - t\check{r}_1) \\ &= (p_0^* - \delta)\frac{(p_1^* - \delta)(p_1^* - t\check{r}_1)}{t\check{r}_1} - t(1-\check{r}_1)(p_1^* - t\check{r}_1)\ (\because (A47)) \\ &= \frac{(p_1^* - t\check{r}_1)}{t\check{r}_1}\big((p_0^* - \delta)(p_1^* - \delta) - t^2\check{r}_1(1-\check{r}_1)\big) \\ &= \frac{(p_1^* - t\check{r}_1)}{t\check{r}_1}\left((p_0^* - \delta)(p_1^* - \delta) + (p_1^* - \delta)(p_1^* - t\check{r}_1) - t\check{r}_1 p_0^*\right) \\ &\qquad \because t^2\check{r}_1(1-\check{r}_1) = -(p_1^* - \delta)(p_1^* - t\check{r}_1) + tp_0^*\check{r}_1 \text{ by } (A47) \\ &= \frac{(p_1^* - t\check{r}_1)}{t\check{r}_1}(p_0^* + p_1^* - \delta)(p_1^* - \delta - t\check{r}_1) > 0\ (\because (A48)(A49)) \end{aligned}$$

$1-\check{r}_1 < \check{r}_0$ is known.

The increasing locus l_0 connects $(\check{r}_0, 1-\check{r}_0)$ and a point on a segment of line $r_1^* = 1,\ r_0^* \in (\check{r}_0, 1)$. Further, increasing locus l_1 connects $(1-\check{r}_1, \check{r}_1)$ and segment $r_0^* = 1,\ r_1^* \in (\check{r}_1, 1)$. Therefore, for all $(p_0^*, p_1^*) \in \Omega_1$, there exists a unique intersection of loci (37) and (38) in region $\{(r_0^*, r_1^*) \mid r_i^* \in (\check{r}_i, 1),\ i \in \{0,1\}\}$.

This region of (r_0^*, r_1^*) is further limited. From (37) and (38), eliminating term D_0^*,

$$(r_0^* - \alpha_0)(r_1^* - \alpha_1) = \alpha_2 \tag{A50}$$

$$\text{where } \alpha_0 = \frac{(p_0^* - p_1^*)(p_0^* + p_1^* - \delta) + t(t-\delta)}{t(p_0^* + p_1^* + t - 2\delta)}$$

$$\alpha_1 = \frac{(p_1^* - p_0^*)(p_0^* + p_1^* - \delta) + t(t-\delta)}{t(p_0^* + p_1^* + t - 2\delta)}$$

$$\alpha_2 = \frac{(p_0^* - \delta)(p_1^* - \delta)(p_0^* + p_1^* - t)^2}{t^2(p_0^* + p_1^* + t - 2\delta)^2}$$

is given. Assume $(p_0^*, p_1^*) \in \Omega_1$; by (A42) and (A48), $\alpha_2 > 0$. Moreover, because

$$\begin{aligned} f_2(\alpha_0) &= (p_0^* - \delta)(p_0^* - t\alpha_0) - t\alpha_0(p_1^* - t(1-\alpha_0)) \\ &= \frac{(p_0^* - \delta)(p_0^* + p_1^* - t)}{(p_0^* + p_1^* + t - 2\delta)^2} \\ &\quad \times \left((t + p_1^* - \delta)(p_0^* + p_1^* - t - 2\delta) + (p_1^* - p_0^* + t)(p_0^* + p_1^* + t - \delta)\right), \end{aligned}$$

here $p_1^* - p_0^* + t > p_0^* - \delta - p_0^* + t > 0$ and (A44). Then, $f_2(\alpha_0) > 0$ for $(p_0^*, p_1^*) \in \Omega_1$, which leads to $\alpha_0 < \check{r}_0$, as $t > 9\delta$ is assumed here. In the same way, $\alpha_1 < \check{r}_1$ is given. Therefore, the locus (r_0^*, r_1^*) of a hyperbolic (A50) with asymptotes $r_0^* = \alpha_0$ and $r_1^* = \alpha_1$ is monotonously decreasing in region $r_0^* \in (\check{r}_0, \infty)$. The line $r_0^* = \check{r}_0$ intersects the hyperbolic (A50) at $(\check{r}_0, r_1^s)$, where r_1^s is defined as satisfying

$$(p_1^* - \delta)(p_1^* - tr_1^s) - t(1-\check{r}_0)(p_0^* - t + tr_1^s) = 0. \tag{A51}$$

This is because, if $(\check{r}_0, r_1^s)$ is on the hyperbolic, $(r_0^*, r_1^*) = (\check{r}_0, r_1^s)$ satisfies both (37) and (38). Then,

$$(p_0^* - \delta)(p_0^* - t\check{r}_0) - t(1 - r_1^s + D_0^*)(p_1^* - t + t\check{r}_0) = 0 \tag{A52}$$

$$and (p_1^* - \delta)(p_1^* - tr_1^s) - t(r_1^s - D_0^*)(p_0^* - t + tr_1^s) = 0. \tag{A53}$$

Equation (A52) and equation $f_2(\check{r}_0) = 0$ that define $\check{r}_0$ lead to $r_1^s - D_0^* = 1 - \check{r}_0$. This relationship and equation (A53) yield equation (A51). In the same manner, line $r_1^* = \check{r}_1$ intersects the hyperbolic (A50) at $(r_0^s, \check{r}_1)$, where r_0^s is defined as satisfying

$$(p_0^* - \delta)(p_0^* - tr_0^s) - t(1 - \check{r}_1)(p_1^* - t + tr_0^s) = 0. \tag{A54}$$

The loci l_0 and l_1 satisfy (37) and (38), respectively. Then, the intersection of l_0 and l_1 is on the hyperbolic (A50) because he (r_0^*, r_1^*) that satisfies (37) and (38) should also satisfy (A50). See the figure 4. The increasing curve l_0 passes point $(\check{r}_0, 1 - \check{r}_0)$; another increasing curve l_1 passes point $(1 - \check{r}_1, \check{r}_1)$. They intersect on the hyperbolic (A50). Then, the intersection of l_0 and l_1 is on the segment of the hyperbolic between point $(\check{r}_0, r_1^s)$ and point $(r_0^s, \check{r}_1)$.

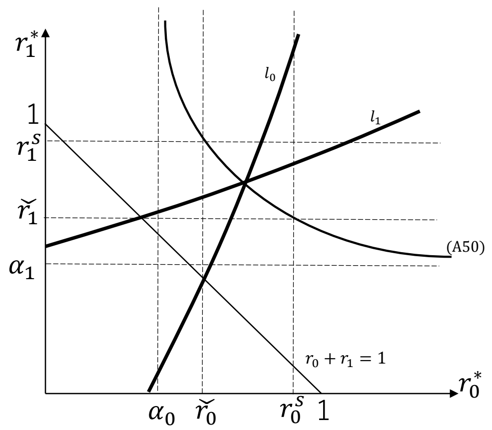

FIGURE 4: Lemma 27 part 2.

Thus, under the assumption $t > 9\delta$, the range $M_1(\Omega_1)$ is limited to

$$M_1(\Omega_1) \subset \Omega_2 \stackrel{\text{def}}{=} \{(r_0^*, r_1^*) \,|\check{r}_0 < r_0^* < r_0^s,\ \check{r}_1 < r_1^* < r_1^s\}. \tag{A55}$$

Because $\check{r}_0 + \check{r}_1 > 1$, $\check{r}_0 < r_0^s$ and $\check{r}_1 < r_1^s$,

$$\Omega_2 \subset \{(r_0^*, r_1^*) \,|r_0^* < r_0^s, r_1^* < r_1^s, r_0^* + r_1^* > 1\}. \tag{A56}$$

3. Mapping $M_1(M_0(r_0^*, r_1^*)) = (r_0^*, r_1^*)$ by (35) $\sim$ (38).

First, it is proved that, when $(p_0^*, p_1^*) \in \Omega_1$, there exists a positive value of δ for given value of t, say δ^*, such that if δ is less than the value, then

$r_i^s < 1/2 + \sqrt{t\delta}/(2t)$. By definition, (A54) is

$$r_0^s = \frac{p_0^*(p_0^* - \delta) + t(t - p_1^*)(1 - \check{r}_1)}{t(p_0^* - \delta) + t^2(1 - \check{r}_1)}.$$

Then,

$$\frac{\partial r_0^s}{\partial \check{r}_1} = \frac{(p_0^* - \delta)(p_0^* + p_1^* - t)}{\left(p_0^* - \delta + t(1 - \check{r}_1)\right)^2} > 0. \quad (\because (A44), (p_0^*, p_1^*) \in \Omega_1).$$

Therefore,

$$\begin{aligned} r_0^s &< \frac{p_0^*(p_0^* - \delta) + t(t - p_1^*)\left(1 - (p_1^* - \delta)/t\right)}{t(p_0^* - \delta) + t^2\left(1 - (p_1^* - \delta/)t\right)} \quad (\because (A49)) \\ &= \frac{(u_1 + t)^2 + (u_0 - t)(u_0 - t - 2\delta)}{2t(u_1 + t)}, \end{aligned} \tag{A57}$$

where $u_0 \stackrel{\text{def}}{=} p_0^* + p_1^*$ and $u_1 \stackrel{\text{def}}{=} p_0^* - p_1^*$. Denote the right-hand side of the inequality (A57) as $f_4(u_0, u_1)$. Obviously, the function is an increasing function of u_0 and a convex function of u_1 because $u_1 + t > -\delta + t > 0$ from (A42) when $t > \delta$ is assumed. Then,

$$r_0^s < \max\left(f_4(\max(u_0), \max(u_1)), f_4(\max(u_0), \min(u_1))\right).$$

From (A42),

$$\begin{aligned} f_4(\max(u_0), \max(u_1)) &< \frac{1 + 6\tau + 4\tau^{3/2} + \tau^2}{2(1 + \tau)} \\ f_4(\max(u_0), \min(u_1)) &< \frac{1 + 2\tau + 4\tau^{3/2} + \tau^2}{2(1 - \tau)}, \end{aligned}$$

where τ denotes δ/t. Then, calculate the differences between the values of these equations and $1/2 + \sqrt{t\delta}/(2t) = 1/2 + \sqrt{\tau}/2$:

$$\begin{aligned} \frac{1}{2} + \frac{\sqrt{\tau}}{2} - f_4(\max(u_0), \max(u_1)) &= \frac{\sqrt{\tau} - 5\tau - 3\tau^{3/2} - \tau^2}{2(1 + \tau)} \\ \frac{1}{2} + \frac{\sqrt{\tau}}{2} - f_4(\max(u_0), \min(u_1)) &= \frac{\sqrt{\tau} - 3\tau - 5\tau^{3/2} - \tau^2}{2(1 - \tau)}. \end{aligned}$$

It is obvious that both values are positive when $\tau = \delta/t$ is less than a certain positive number, respectively. Numerically, they are $0.032\cdots$ and $0.056\cdots$, respectively. Denote such a positive number as δ^{**}. Thus, when δ is less than $t\delta^{**}$, r_0^s is less than $1/2+\sqrt{t\delta}/(2t)$. In the same manner, when δ is less than $t\delta^{**}$, r_1^s is proved to be less than $1/2+\sqrt{t\delta}/(2t)$.

From (A42), $M_0(\Omega_0) \subset \Omega_1$, and from (A55), $M_1(\Omega_1) \subset \Omega_2$ when $\delta < t/9$. Then, by (A56) and $r_i^s < 1/2+\sqrt{t\delta}/(2t)$ for $i \in \{0,1\}$, for $\delta < t\delta^{**}$

$$M_1(M_0(\Omega_0)) \subset \Omega_2 \subset \{(r_0^*, r_1^*)\,|r_0^* < r_0^s, r_1^* < r_1^s, r_0^* + r_1^* > 1\} \subset \Omega_0,$$

for $\delta < \min(t/9, t\delta^{**})$. Because Ω_0 is a compact convex set and $M_0 * M_1$ is continuous, the map $M_0 * M_1$ has a fixed point in Ω_0 under Brouwer's fixed point theorem. Denote the fixed point (r_0^{**}, r_1^{**}). Denote also $M_0(r_0^{**}, r_1^{**})$ as (p_0^{**}, p_1^{**}). Then, $(r_0^*, p_0^*, r_1^*, p_1^*) = (r_0^{**}, p_0^{**}, r_1^{**}, p_1^{**})$ is the solution to the equation system (35) $\sim$ (38). Thus, the equation system has an interior solution when $\delta < \min(t/9, t\delta^{**})$. Here, the tentative assumption $t > \delta$ is validated. $\square$

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