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Coordination and Imitation under Unawareness

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Abstract

This study focuses on discovered versions of coordination games with unawareness, and proposes a novel solution concept under unawareness called the successful-coordination equilibrium. In games with unawareness, coordination might fail because the underlying assumption is that agents are unaware of their realized actions. When the agents observe the opponents' actions which they are unaware of, by teaching or asking their opponents how to play the opponents' actions, they might try to coordinate successfully. In this study, each player can observe the opponents' actions and imitate them. Then, all players with revised subjective games must have a successful-coordination equilibrium so that coordination is successful.
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1 Introduction

This study aims to:

- focus on coordination games with unawareness, specifically, symmetrical games with unawareness,
- introduce a successful-coordination equilibrium to coordination games with unawareness, and

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- model a discovered game with imitation and relate it to a successful-coordination equilibrium.

Games with unawareness model ignorance are those where players are unaware of each other's actions. Players might not be unaware of their own feasible actions or the opponents' actions, but their beliefs about the opponents' play are different from their actual plays. When they face each other in some coordination game with unawareness, if their beliefs are asymmetric, coordination might fail. Let us consider the following example. In a coordination game, there are two players, Alice and Bob, and two actions, X and Y. Here, suppose that Alice can play only X and Bob can play only Y. Moreover, let us suppose that both players commonly believe this. Thus, their beliefs are (X, Y), the play is a coordination failure and their beliefs are correct. Hence, the players understand that their coordination is not a success.

In this case, the failure is different from a standard coordination failure. In a standard model, coordination failure implies an equilibrium selection mistake. By contrast, in this example, coordination failure is derived from unawareness of their own actions, even though they are aware that the opponents are aware of such actions. Hence, they might be aware of their coordination failure[1]. Thus, it is natural to assume that the players will readjust by resolving their ignorance. For example, Alice may teach Bob a way to play X, or ask him about a way to play Y. This study investigates the case in which players observe the opponent's actions that they could not play and supposes that they imitate their actions. We reconstruct the game based on the above assumptions and let *imitative discovered games* be the reconstructed game.

To consider a successful coordination under unawareness, we introduce a novel equilibrium concept, *successful-coordination equilibrium*. A successful-coordination equilibrium is a specific solution concept in coordination games with unawareness, which deals only with successful coordination. The solution concept was characterized to show that there must exist a successful-coordination equilibrium in any imitative discovered game.

The rest of the study is organized as follows. In section 2, we model a coordination game (specifically, a symmetrical game) with unawareness. Next, we introduce a successful-coordination equilibrium to coordination games with unawareness and characterize it. Section 4 models imitative discovered games and shows that any imitative discovered game has a successful-coordination equilibrium. Additionally, we introduce a block game notion to coordination games with unawareness. In an initial coordination game, players exclude the redundant actions that were not played in the stage game in a coordination block game. Finally, Section 5 provides the following two discussions:

1. We discuss the relationship of successful-coordination and cognitively stable generalized Nash equilibria. A cognitively stable generalized Nash equilibrium is a generalization of a Nash equilibrium. The equilibrium

[1] As shown below, such play is possible in the Nash equilibrium based on correct beliefs.

is interpreted as equilibrium in correct beliefs. However, in some coordination games with unawareness, a cognitively stable generalized Nash equilibrium sometimes induces coordination failure. This shows that correctness of beliefs is different from the accuracy of subjective games.

Future research can compare the correctness of beliefs to the correctness of subjective games under asymmetric perception. This note provides an introduction of this comparison.

2. This note investigates the assumptions about discoveries and imitations. However, the assumptions are specific to unawareness. In this section, we examine the two assumptions in detail.

2 Preliminaries

This section proposes a coordination game with unawareness, a more strictly symmetrical game with unawareness. Let $G = (I, A, u)$ be a standard n-person coordination game. $I = \{1, \ldots, n\}$ is a finite set of players, and $I_{-i} = I \setminus \{i\}$. $A = \times_{i \in I} A_i$, where A_i is a nonempty finite set of actions of i and $A_1 = \cdots = A_n$. Let $a_i \in A_i$ be i's action. $u = (u_i)_{i \in I}$, where $u_i : A \to \mathbb{R}$ is the utility function of i. For any $a = (a_1, \ldots, a_n) \in A$, $u_1(a) = \cdots = u_n(a) > 0$ if $a_1 = \cdots = a_n$, while $u_1(a) = \cdots = u_n(a) = 0$ otherwise.

We define coordination games with unawareness based on the studies by Kobayashi and Sasaki (2021) and Tada (2022), which are similar to that of Perea (2018).[2] [3] For any standard coordination game G, let $V = \times_{i \in I}(2^{A_i} \setminus \{\emptyset\})$ be the set of possible *views* of G. Similar to most previous works, we assume that the set of players is commonly known and each player's utility for each action profile does not depend on awareness. Let $v \in V$ and A_i^v be the set of actions of i in $v = \times_{j \in I} A_j^v$. Here, when player i is given v, i is aware of $a \in v$ and unaware of $a \in A \setminus v$. Let $\Gamma = (G, (T_i)_{i \in I}, (v_i)_{i \in I}, (b_i)_{i \in I})$ be a coordination game with unawareness, which is described as follows: for each $i \in I$,

- T_i is a finite and nonempty set of i's type, one of which is the type t_i^*.
- $v_i : T_i \to V$ is i's view function.

[2] There is a difference between our approach and that of Perea (2018). Perea (2018) did not fix the belief hierarchies on views, and he dealt with probabilistic beliefs on awareness. By contrast, we assume that the "actual type" of players is fixed and given, and we do not deal with probabilistic beliefs.

[3] A type-based approach under unawareness is similar to that of Harsanyi (1967). However, as pointed out by Meier and Schipper (2014) and Perea (2018), Harsanyi's (1967) framework cannot present players' unawareness of actions. In his model, a player's action that they are unaware of is assigned to extremely low payoffs. The rational player does not choose the action. However, if the player is not rational, they might play such an action. It contradicts the condition in which the player is unaware of such actions. By contrast, in a type-based approach under unawareness, each player cannot choose the actions that they are unaware of, even if they are irrational. Moreover, if the player is unaware of another player's actions, this player cannot reason that they can play the action.

- $b_i : T_i \to T_{-i}$ is the belief function of i, where $T_{-i} = \times_{j \in I \setminus \{i\}} T_j$. If $b_i(t_i) = (t_j)_{j \in I \setminus \{i\}}$, then for each $j \in I \setminus \{i\}$, $v_j(t_j) \subseteq v_i(t_i)$.

Let us call G an objective game. An objective game can be interpreted as a "true game" in Γ. i's type, t_i, describes their view of the game and belief about the opponent's types. Given t_i, $v_i(t_i) = v$ implies that i is aware of v and unaware of $A \setminus v$, and $b_i(t_i) = (t_j)_{j \in I \setminus \{i\}}$ means that at t_i, i believes that the other types are $(t_j)_{j \in I \setminus \{i\}}$. Simultaneously, i believes that each view of j is $v_j(t_j)$. Let $b_i(t_i)(j)$ be j's type in $b_i(t_i)$. Each player may be unaware of some types of players, including their own.

In contrast to the previous literature, for simplicity, we define the strategies with a focus on pure actions only. For any $i \in I$, let $s_i : T_i \to A_i$. Then, given t_i, $s_i(t_i) \in A_i^{v_i(t_i)}$ is i's local action at t_i. Let $s_i = (s_i(t_i))_{t_i \in T_i}$ be i's generalized strategy, and let $s = (s_i)_{i \in I}$ be a generalized strategy profile. For any s, $s_i(t_i^*)$ is i's actual play. The set of players' actual play $A_i^{v_i(t_i^*)}$ may be a proper subset of i's *full* action set A_i. Then, player i cannot implement $a_i \in A_i \setminus A_i^{v_i(t_i^*)}$.

3 Successful-Coordination Equilibrium

This section proposes a novel equilibrium concept in (coordination) games with unawareness, named *successful-coordination equilibrium.*

Definition 1. In a coordination game with unawareness Γ, s^* is a successful-coordination equilibrium if

1. for any $i \in I$ and $t_i \in T_i$,
$$s_i^*(t_i) \in \arg \max_{x \in A_i^{v_i(t_i)}} u_i(x, (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}});$$
2. for any $i \in I$ and $t_i \in T_i$, $s_i^*(t_i) = s_i^*(t_i^*)$; and
3. $s_1(t_1^*) = \cdots = s_n(t_n^*)$.

The first condition requires that players best respond to their beliefs about the opponent's play, and the second condition that all players' beliefs are correct.[4] The third condition requires that the coordination should be successful.

We can easily deduce the following remark and proposition[5].

Remark 1. In every coordination game *without* unawareness, any successful-coordination equilibrium is a Nash equilibrium, and vice versa.

Proposition 1. Suppose that $\bigcap_{i \in I} \bigcap_{t_i \in T_i} v_i(t_i) \neq \emptyset$ in a coordination game with unawareness Γ. If some $a \in \bigcap_{i \in I} \bigcap_{t_i \in T_i} v_i(t_i)$ is a Nash equilibrium in G, then there exists a successful-coordination equilibrium.

[4] As described later, the first condition is a definition of generalized Nash equilibria and the second condition is a definition of cognitive stability.

[5] Proposition 1 is a special case of Sasaki (2017, Proposition 2).

Proof. Suppose that $\bigcap_{i\in I}\bigcap_{t_i\in T_i} v_i(t_i) \neq \emptyset$ in Γ and that some $a = (a_1, \ldots, a_n) \in \bigcap_{i\in I}\bigcap_{t_i\in T_i} v_i(t_i)$ is a Nash equilibrium in G. As Γ is a coordination game, the Nash equilibrium in G satisfies $a_1 = \cdots = a_n$. For any $(i, t_i) \in I \times T_i$, let $s_i(t_i) = a_i$. Then, the generalized strategy profile s satisfies the conditions of Definition 1. That is, s is a successful-coordination equilibrium. $\square$

4 Discovery and Imitation of Actions

The previous section defines a successful-coordination equilibrium. However, a successful-coordination equilibrium may not exist in some coordination games with unawareness.

Remark 2. The following example shows how a successful-coordination equilibrium may not exist.

Example 1. Consider two people, Alice, and Bob. They face the following coordination game, which is an objective game.

$v^0 =$

Alice / Bob	X	Y
X	1, 1	0, 0
Y	0, 0	1, 1

In v^0, there exist two pure Nash equilibria, (X, X) and (Y, Y).

Here, let us assume the following about Alice's belief in this game.

- Alice can implement her action X, but she cannot do the other action Y because she does not know how to play Y.
- Alice knows that Bob can choose his actions X and Y if he knows how to play them.
- Alice knows that Bob can choose only Y, and she knows that Bob does not know how to play X; hence, she knows that Bob cannot choose X.
- Alice supposes that Bob believes that it is common knowledge that Alice can choose only X, Bob can choose only Y, and the others' actions cannot be played.

In addition, let us assume the following about Bob's belief in this game.

- Bob can implement his action Y, but he cannot do the other action X because he does not know how to play X.
- Bob knows that Alice can choose her actions X and Y if she knows how to play them.
- Bob knows that Alice can choose only X, and he knows that she does not know how to play Y; hence, he knows that Alice cannot choose Y.

- Bob supposes that Alice believes that it is common knowledge that Alice can choose only X, Bob can choose only Y, and the others' actions cannot be played.

Then, Alice's first-order view of this game is as follows:

$$v^1 = \begin{array}{|c|c|c|}\hline \text{Alice / Bob} & X & Y \\ \hline X & 1,\ 1 & 0,\ 0 \\ \hline \end{array} \ ;$$

Bob's first-order view of this game is as follows:

$$v^2 = \begin{array}{|c|c|}\hline \text{Alice / Bob} & Y \\ \hline X & 0,\ 0 \\ \hline Y & 1,\ 1 \\ \hline \end{array} \ ; \text{and}$$

Both players' second- or higher-order views of this game are as follows:

$$v^3 = \begin{array}{|c|c|}\hline \text{Alice / Bob} & Y \\ \hline X & 0,\ 0 \\ \hline \end{array} \ .$$

In this example, the mathematical formulation is as follows. Denote Alice by A and Bob by B. Suppose that $T_A = \{t_A^*, t_A\}$ and $T_B = \{t_B^*, t_B\}$ such that

$v_A(t_A^*) = v^1$ and $b_A(t_A^*) = t_B$;
$v_A(t_A) = v^3$ and $b_A(t_A) = t_B$;
$v_B(t_B^*) = v^2$ and $b_B(t_B^*) = t_A$; and
$v_B(t_B) = v^3$ and $b_B(t_B) = t_A$.

Suppose that each player is rational. Each player doesn't need to believe that the opponent is rational. Alice then performs the best response to (X, Y) in v^3 as X in v^1. In addition, Bob performs the best response to (X, Y) in v^3 as Y in v^2. Their beliefs and decisions consist of the following generalized strategy profile: $s^* = ([s_A(t_A^*) = X, s_A(t_A) = X], [s_B(t_B^*) = Y, s_B(t_B) = Y])$.

However, each player knows that the equilibrium play is not a Nash equilibrium in each first-order subjective view. Alice is aware of the Nash equilibrium (X, X) in v^1, and Bob is aware of the Nash equilibrium (Y, Y) in v^2. In the generalized Nash equilibrium, coordination is not successful. Hence, they are aware that coordination is a failure. $\square$

The above problem can be ignored if one considers the act to be symbolic. However, given the specific situation, it is impossible to ignore. Let X be the action "going to Alice's house," and let Y be the action "going to Bob's house." In other words, assume that Example 1 is a meeting game. Then, Alice may ask Bob how to get to his house, or she may guide him about how to get to her house. Further, Bob may ask Alice how to get to her house, or he may guide Alice about how to get to his house.

To resolve the above issue, let us present a model of discoveries and imitations under unawareness based on the model of *endogenously* discovered games (Schipper 2021; Tada 2022) as follows.[6] [7]

Definition 2. $\Gamma' = (G, (T'_i)_{i \in I}, (v'_i)_{i \in I}, (b'_i)_{i \in I})$ is an *imitative discovered game* with $s = (s_i)_{i \in I}$ in $\Gamma = (G, (T_i)_{i \in I}, (v_i)_{i \in I}, (b_i)_{i \in I})$ if for any $(i, t_i) \in I \times T_i$ and any sequence of players $i_1, i_2, \ldots$, with sequence of types $t_{i_1}, t_{i_2}, \ldots$, where $t_{i_1} = t_i$, there exists $t'_i \in T'_i$ and a sequence of types in Γ', $t'_{i_1}, t'_{i_2}, \ldots$, where $t'_{i_1} = t'_i$, such that for any $h \geq 1$,

$$v'_{i_h}(t'_{i_h}) = \times_{j \in I}[A_j^{v_{i_h}(t_{i_h})} \bigcup_{k \in I} \operatorname{supp}(s_k(t_k))],$$

where t^*_k is k's actual type in Γ. Note that for some Γ, Γ' may be $T \not\subseteq T'$ and $T' \not\subseteq T$, or $T \cap T' = \emptyset$.

When all players observe each other's plays, the first condition suggests that each player not only gains knowledge of the opponents' feasible actions, but also discovers (or "learns") a way of playing such actions. The second condition suggests that as supposed by each player, every player commonly believes that all players gain knowledge of the others' feasible actions and discover a way of playing such actions.

Example 1 (Continued)**.** Suppose Alice and Bob play $s^* = ([s_A(t^*_A) = X, s_A(t_A) = X], [s_B(t^*_B) = Y, s_B(t_B) = Y])$. Then, according to the first condition, Alice adds Bob's action Y to not only Bob's choice but also Alice's choice in her subjective view v^1; and Bob adds Alice's action X to not only Alice's choice but also Bob's choice in his subjective view v^2. Moreover, as both players suppose that each of them commonly believes that they gain knowledge of each other's feasible actions and discovers a way of playing such actions according to the second condition, both players add actions X and Y to their respective choice in each other's second or any higher-order view v^3. Then, each agent's first and any higher-order view is replaced with v^0.

This imitative discovered game $\Gamma' = (G, (T'_A, T'_B), (v'_A, v'_B), (b'_A, b'_B))$ is formulated as follows:

$T_A = \{t'_A\}$ and $T_B = \{t'_B\}$;
$v_A(t'_A) = v^0$ and $b_A(t'_A) = t'_B$; and
$v_B(t'_B) = v^0$ and $b_B(t'_B) = t'_B$.

In Γ', Alice and Bob can choose two actions X and Y. $\Box$

Interestingly, any imitative discovered game has the following property:

[6]Karni and Vierø(2013, 2017) discussed cases in which agents discover their own new feasible actions. However, in their model, such actions are not *exogenously* discovered but rather *endgenously* discovered. In other words, such actions are given to agents by modelers.

[7]Unlike Schipper (2021) and Tada (2022)), we do not deal with *discovery processes*. As indicated by one of the main results, only one imitation update is required in our model.

Proposition 2. Given any n-person coordination game with unawareness and any generalized strategy profile, the imitative discovered game has a successful-coordination equilibrium.

Proof. Suppose that s^* is played by all agents. For any $i \in I$, $a_i = s_i^*(t_i^*)$ is observed and imitated by them. Hence, for some $i \in I$, $a = (a_i, \ldots, a_i)$ is a successful-coordination equilibrium. $\square$

The above proposition means that it leads to the existence of a successful-coordination equilibrium in which each player revises their subjective view with just one. In Example 1, there exist two successful coordination equilibria in Γ': $s_1' = (s_A'(t_A') = X, s_B(t_B') = X)$ and $s_2' = (s_A'(t_A') = Y, s_B(t_B') = Y)$.

After players discover revised subjective games in the imitative discovered game, which set of actions do the players pay attention to? It seems to be redundant that a player rationalizes actions based on their subjective view. Let us consider the following example:

Example 2. Consider the following objective game played by Colin (C) and David (D):

$v_O =$

Colin / David	α	β	γ	δ
α	1, 1	0, 0	0, 0	0, 0
β	0, 0	1, 1	0, 0	0, 0
γ	0, 0	0, 0	1, 1	0, 0
δ	0, 0	0, 0	0, 0	1, 1

.

Here, suppose that Colin believes the following view is a common belief:

$v_C =$

Colin / David	α	β
α	1, 1	0, 0

.

By contrast, David believes the following view as a common belief:

$v_D =$

Colin / David	γ	δ
β	0, 0	0, 0
δ	0, 0	1, 1

.

Let us formulate this game $\Gamma = (G, (T_C, T_D), (v_C, v_D), (b_C, b_D))$ as follows:

$T_C = \{t_C^*, t_C\}$ and $T_D = \{t_D^*, t_D\}$;
given t_C^*, $v_C(t_C^*) = v_C$ and $b_C(t_C^*) = t_D$;
given t_C, $v_C(t_C) = v_D$ and $b_C(t_C) = t_D^*$;
given t_D^*, $v_D(t_D^*) = v_D$ and $b_D(t_D^*) = t_C$; and
given t_D, $v_D(t_D) = v_C$ and $b_D(t_D) = t_C^*$.

Suppose that both players implement a generalized strategy profile $s^* = ([s_C(t_C^*) = \alpha, s_C(t_C) = \beta], [s_D(t_D^*) = \gamma, s_D(t_D) = \alpha])$. In the strategy profile, the actual play is (α, γ). Then, the imitative discovered game $\Gamma' = (G, (T_C', T_D'), (v_C', v_D'), (b_C', b_D'))$ is formulated as follows:

$T'_C = \{t'_C, t''_C\}$ and $T'_D = \{t'_D, t''_D\}$;
given t'_C, $v'_C(t'_C) = v'_C$ and $b'_C(t'_C) = t''_D$;
given t''_C, $v'_C(t''_C) = v'_D$ and $b'_C(t''_C) = t'_D$;
given t'_D, $v'_D(t'_D) = v'_D$ and $b'_D(t'_D) = t''_C$; and
given t''_D, $v'_D(t''_D) = v'_C$ and $b'_D(t''_D) = t'_C$, where

$v'_C =$

Colin / David	α	β	γ
α	1, 1	0, 0	0, 0
γ	0, 0	0, 0	1, 1

, and

$v'_D =$

Colin / David	α	γ	δ
α	1, 1	0, 0	0, 0
β	0, 0	0, 0	0, 0
γ	0, 0	1, 1	0, 0
δ	0, 0	0, 0	1, 1

.

Then, Colin knows that there exist two Nash equilibria in v'_C, (α, α) and (γ, γ), whereas David knows that there exist three Nash equilibria in v'_D, (α, α), (γ, γ), and (δ, δ). Note that Colin is not aware that David's view is v'_D and that David is not aware that Colin's view is v'_C.

Each player must select one of those equilibria in each other's view. Here, let us focus on David. Although he knows that there are three equilibria, it seems odd that he includes all equilibria in his choices because δ is played by neither Colin nor David. $\square$

After the players imitate the opponents' plays and revise their views, they might exclude redundant actions that nobody plays. Then, the players might reconstruct their subjective views to exclude such actions. To provide such representation, we use the *block game* notions proposed by Myerson and Weibull (2015). A block is a Cartesian product of nonempty subsets of players' actions. Let us first focus on actions that each player observes and imitates, and then let us define a coordination block game as follows.[8]

Definition 3. Given any coordination game without unawareness $G = (I, A, u)$ and any block $T = \times_{i \in I} T_i \in V$, $G^T = (I, T, u^T)$ is a *coordination block game* if

1. $T_i = \cdots = T_n$; and
2. $u^T = (u_i^T)_{i \in I}$, where $u_i^T(a) = u_i(a)$ for any $i \in I$ and $a \in T$.

Here, T is called a coordination block.

Thus, the following proposition holds.

[8] Tada (2022) proposed a similar notion as a realizable CURB block game. In a realizable CURB block game, the block is CURB.

Proposition 3. Given any n-person coordination game Γ and any generalized strategy profile s, let $T \in V$ be a block such that for any $i \in I$, $A_i^T = \bigcup_{j \in I} s_j(t_j^*)$. Then, the block game $G^T = (I, T, u^T)$ is a coordination block game.

Proof. It is obvious. □

In the case of Example 2, as Colin and David play (α, γ), they focus on α and γ. That is, David excludes δ from his choices. Then, the coordination block is $\{\alpha, \gamma\} \times \{\alpha, \gamma\}$, and the coordination block game is

$v_T =$

Colin / David	α	γ
α	1, 1	0, 0
γ	0, 0	1, 1

.

Hence, David can restrict his choices. Then, in their equilibrium selection, both players focus on α and γ.

In some coordination games (with unawareness), some players are unaware of some of the choices, and the set of choices might be too large. Hence, in the first play, players might not be able to select some specific (successful coordination) equilibrium, or they might not be able to restrict action sets to a coordination block. However, by discovering and imitating only the actions taken in the first play, players can restrict their actions to some specific coordination blocks.

5 Discussion

5.1 Relationship with Generalized Nash Equilibrium

A self-confirming equilibrium is related to a generalized Nash equilibrium. This section considers relationships among a self-confirming equilibrium, generalized Nash equilibrium, and cognitive stability. We first define the generalized pure Nash equilibrium proposed by Halpern and Rêgo (2014) as follows.

Definition 4. s^* is a generalized (pure) Nash equilibrium if, for any $i \in I$ and $t_i \in T_i$,

$$s_i^*(t_i) \in \arg \max_{x \in A_i^{v_i(t_i)}} u_i(x, (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}}).$$

A generalized Nash equilibrium is interpreted as an equilibrium in beliefs. However, as shown by Schipper (2014), because games with unawareness assume the unawareness of players' actions, a generalized Nash equilibrium might consist of wrong beliefs. Then, players who have such wrong beliefs might revise their subjective games, and they might choose different actions from the immediately preceding stage in the game. To avoid such issues, Sasaki (2017) proposed a notion of cognitive stability or stable belief hierarchies. This notion represents that in an equilibrium satisfying cognitive stability or stable belief hierarchies, all participants' beliefs about the opponents' plays are correct. Although Sasaki (2017) distinguished a notion of stable belief hierarchies from a notion of cognitive stability, he showed that the two notions are equivalent. Let us define cognitive stability as follows.

Definition 5. A generalized Nash equilibrium s^* is cognitively stable if for any $i \in I$ and $t_i \in T_i$

$$s_i^*(t_i) = s_i^*(t_i^*).$$

Cognitive stability represents whether all players' beliefs about the opponents' play are correct. In a cognitively stable generalized Nash equilibrium, all players' local actions are the same. It means that each player's belief is correct. To facilitate comparison with a successful-coordination equilibrium, the definition of a cognitively stable generalized Nash equilibrium is rewritten as follows:

Remark 3. s^* is a cognitively stable generalized Nash equilibrium if for any $i \in I$ and $t_i \in T_i$,

1. $s_i^*(t_i) \in \arg\max_{x \in A_i^{v_i(t_i)}} u_i(x, (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}})$; and
2. $s_i^*(t_i) = s_i^*(t_i^*)$.

It is evident that two conditions of the definition of a cognitively stable generalized Nash equilibrium are similar in the first and second condition of Definition 1. Hence, the following remark is true.

Remark 4. Under unawareness, every successful-coordination equilibrium is a cognitively stable generalized Nash equilibrium.

This is clear from Definition 1 and Remark 4. However, the opposite does not hold true, that is, in some cognitively stable generalized Nash equilibrium, coordination might be a failure. As seen in example 1, a generalized strategy profile $s^* = ([s_A(t_A^*) = X, s_A(t_A) = X], [s_B(t_B^*) = Y, s_B(t_B) = Y])$ satisfies a definition of a cognitively stable generalized Nash equilibrium, but the strategy profile does not satisfy a definition of a successful-coordination equilibrium.

Cognitive stability is the concept of checking the correctness of beliefs in a played equilibrium. In any cognitively stable generalized Nash equilibrium, all players confirm that their beliefs are correct; while in cognitively unstable generalized Nash equilibrium, some players confirm that their beliefs are incorrect. In many classes of games with unawareness, it seems that in a cognitive stable generalized Nash equilibrium all players need not revise their subjective views, and in a cognitively unstable generalized Nash equilibrium some players need to revise their subjective view. In other words, cognitive stability seems to be a concept of the stability of subjective views.

However, in coordination games with unawareness, since coordination might be a failure, players who play with a cognitively stable equilibrium might revise their subjective views as shown in example 1. Therefore, we need to distinguish between stability of beliefs and stability of subjective views in a coordination game.

Next, we examine the mathematical relationships between successful-coordination equilibrium and cognitively stability. This relationship has the following properties.

Proposition 4. In a coordination game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, then every cognitively stable generalized Nash equilibrium is a successful-coordination equilibrium and vice versa.

Before proving this proposition, we refer to Sasaki's (2017) proposition. Although his proposition includes a generalized mixed strategy profile, we restrict his result to pure strategies.

Proposition 5 (Sasaki 2017)**.** In a simultaneous move game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, then in any cognitively stable generalized Nash equilibrium, the actual plays of all players are Nash equilibria in G.

Proof. Suppose that for any $i \in I$, $A_i^{v_i(t_i^*)} = A_i$. A cognitively stable generalized Nash equilibrium s^* is given. That is, for any $(i, t_i) \in I \times T_i$, $u_i(s_i^*(t_i), (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}}) = u_i(s_i^*(t_i), (s_j^*(t_j^*))_{j \in I_{-i}}) = u_i((s_j^*(t_j^*))_{j \in I})$, that is, every participant's actual play best responds to the others' actual play. Suppose $((s_j^*(t_j^*))_{j \in I})$ is not a Nash equilibrium in G, that is, there exist $(i, a_i) \in I \times A_i$ such that $a_i \neq s_i^*(t_i^*)$ and $u_i(a_i, (s_j^*(t_j^*))_{j \in I_{-i}}) > u_i((s_j^*(t_j^*))_{j \in I})$. However, since s^* is a cognitively stable generalized Nash equilibrium, it is a contradiction. Hence, $((s_j^*(t_j^*))_{j \in I})$ is a Nash equilibrium in G. Here, because t_i^* denotes i's actual type, $((s_j^*(t_j^*))_{j \in I})$ refers to all players' actual plays. □

Proof of Proposition 4. Suppose that for any $i \in I$, $A_i^{v_i(t_i^*)} = A_i$. A coordination game with unawareness is a special case of static game with unawareness. Therefore, by Proposition 5, in every cognitively stable generalized Nash equilibrium s^*, the actual play of all players $(s_i^*(t_i^*))_{i \in I}$ is a Nash equilibrium in the standard coordination game G. In any standard coordination game, a Nash equilibrium $a^* = (a_1^*, \ldots a_n^*)$ satisfies $a_1^* = \cdots = a_n^*$. By definition of an actual play, for any $i \in I$, since $s_i(t_i^*) = a_i^*$, $s_1^*(t_1^*) = \cdots = s_n^*(t_n^*)$. Therefore, s^* satisfies every condition of Definition 1.

The opposite clearly holds true. 4. □

5.2 Unawareness versus Covery

Studies on unawareness distinguish between two approaches; one is lack of conception (e.g., Heifetz, Meier, and Schipper 2006); and the other is lack of knowledge (e.g., Geanakoplos 2021).

1. *Lack of conception*: The cholera bacterium was discovered by Koch in 1884. However, the cholera bacteria itself had existed before 1884, but people were unaware of its existence. Therefore, prior to 1884, people infected with cholera bacteria battled the disease without realizing they were infected.

2. *Lack of knowledge*: As was recently discovered, some COVID-19-infected patients do not show symptoms. Asymptomatic patients like them remain unaware that they are asymptomatically infected unless they undergo PCR testing, even when they learn that there are asymptomatic patients with COVID-19.

Games with unawareness model lack of conception in terms of awareness of the games' situation. When players are unaware that they might face a coordination game, they believe that they face a standard game with unawareness excluding coordination games. However, games with unawareness model lack of knowledge in terms of awareness of the action set. Previous studies have interpreted unawareness of action sets as a lack of knowledge, but not as a lack of conception. For example, if the meeting place is a well-known place like the Big Ben, people can get there without needing directions. However, if it is not a widely known place, people may not know how to get there, even if they are told how to. Many people may then ask the opponents to change the meeting place to a more recognizable location. Knowing the choices influences decision-making, but whether the choices are understandable and feasible also influence decision-making. From the above perspective, it is important how accurately players can observe the opponents' actions.

This study assumes that if the agents observe the opponent's actions that they were unaware of, then they can understand and imitate those actions. However, as shown above, it is not necessarily that players understand the opponent's actions. When the opponents choose unnoticed actions that players were unaware of, their play can be classified and discussed as follows.

Unawareness that the opponents have already made a decision

When the opponents implement actions that the agent is unaware of, they cannot recognize such actions. Then, there may be two cases, as enumerated below:

- Games are not completed: Shiso Kanakuri, who was a marathon runner in the 1912 Summer Olympics, fell sick with sunstroke during the competition. He did not wake up until the day after the competition, and so, he abstained. However, his decision to abstain was not communicated to the Olympic Committee. Shiso's competition record was not stopped until March 21, 1967, when he officially reached the finish line. In other words, his decisions were not tied to the outcome of the game.

- Not realizing being in a game situation: Companies advertise and consumers decide whether to buy products based on the advertisements. However, companies may use subliminal effects in their advertising. The consumers ignorant of advertising strategies aimed at subliminal effects may not realize that they are being put in a game situation with the company.

Awareness of the opponents' decision making, but unawareness about what the opponents have decided

Players are aware that their opponents are making a decision when the opponents take actions that they are unaware of. However, it is not always possible to know exactly how the opponents have played.

- Misrecognition: Let us assume there are three entrances: East, West, and South. Out of the three, you know the East and West entrances, but do not know the South entrance. If I go to the South entrance and you go to the East entrance, we cannot meet. Then, you may be misled into thinking that I went to the West entrance, because you do not know the South entrance.
- Not recognized as symbols: Famous sites such as the White House and Big Ben sound familiar even to first-time visitors to the area. However, if you have never heard of a company or a niche restaurant, you may not know where to find it even by looking at a map. For example, Lake Kawaguchi, located at the foot of Mt. Fuji, is one of the most famous lakes in Japan, but you might not know of it, even if you knew where Mt. Fuji was. Then, you may not understand if I were at Lake Kawaguchi.

Awareness about what the opponents have played, but unawareness of the way of play

Even if we know exactly what the opponent's choices are, we may not be able to imitate them. We may never get there, even in case of famous places like the White House, Big Ben, and Mt. Fuji. When watching a game of soccer or baseball, only a limited number of people can imitate the players' moves. For us to imitate the opponents' behavior, we also need to recognize how they did it.

However, it is not necessary to recognize it in an exact manner. If it is a tall building like the Big Ben, it will stand out and we can go there. In other words, the ability to imitate the behavior of others depends on the ease of imitation.

Of the three types described above, the first and second are caused by lack of conception; the third is due to lack of knowledge. The study of discovery processes is clearly the third[9]. As you can see from the three types above, there is a very strong assumption under unawareness that they know exactly what the opponents are doing and imitate it.

As pointed out by Schipper (2014), unawareness means lack of conception rather than lack of knowledge. By contrast, in almost all the games with unawareness, unawareness of actions means lack of knowledge rather than lack of

[9]Moreover, most games with unawareness seem to assume the third type.

conception. Hence, most previous works seem to discuss games with "covery" rather than games with unawareness.[10]

5.3 Related Literature

Feinberg (2021), Heifetz, Meier, and Schipper (2013), Halpern and Rêgo (2014), and Meier and Schipper (2014) are pioneering works on games with unawareness. Specifically, Heifetz, Meier, and Schipper (2013) and Halpern and Rêgo (2014) formulated extensive games with unawareness.

Meier and Schipper (2014) formulated Bayesian games with unawareness that are a generalization of Bayesian games. Another type-based model was proposed by Perea (2018) as a special case of the model by Meier and Schipper (2014). Note that the two models have different specifications. Meier and Schipper (2014) assumed that player's types are directly associated with beliefs about the structure of a game, whereas Perea (2018) did not make such an assumption. In his model, player's types are associated with beliefs about the structure of a game, but the types cannot be associated with the structure of the game itself.

The main solution concepts in games with unawareness have two approaches: equilibrium notions (Feinberg 2021; Čopič and Galeotti 2006; Ozbay 2007; Halpern and Rêgo 2014; Rẽgo and Halpern 2012; Grant and Quiggin 2013; Meier and Schipper 2014 Sasaki 2017; Schipper 2021; Kobayashi and Sasaki (2021)) and rationalizability notions (Heifetz, Meier, and Schipper 2013, 2021; Perea 2018; Guarino 2020). Recently, Tada (2022) proposed a generalization of the closedness-under-rational-behavior (CURB) notion proposed by Basu and Weibull (1991), which is one of the set-wise notions.

A split of our model is based on the literature about growing awareness, updating awareness, and discoveries, such as Karni and Vierø(2013, 2017), Schipper (2021), Galanis and Kotronis (2021), and Tada (2022). Studies in the literature indicate that agents additionally know information about states, events, consequences, actions, and so on, which they *were* previously unaware of. Note that our model refers to the growing awareness of the ways in which opponents play their game rather than the growing awareness of opponents' plays.

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[10]The spirit of the lack of conception of actions can be seen in several prior studies, for instance, Heifetz, et al. (2013) and Halpern and Rêgo (2014). They discuss awareness of unawareness of actions. In their frameworks, players know that the opponents have some action, but they do not know what the opponents' action is.

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