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Unawareness of Actions and Myopic Discovery Process in Simultaneous-Move Games with Unawareness

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Unawareness of Actions and Myopic Discovery Process in Simultaneous-Move Games with Unawareness *

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Abstract

The present paper introduces a generalization of a closedness under relation behavior (CURB) notion to simultaneous-move games with unawareness. The study, using a type-based approach, models a myopic discovery process in which players take the best responses to opponents' preceding plays and revise their subjective games. Mainly, the study shows that the plays of all players converge to some realizable CURB set through any myopic discovery process. Given any simultaneous-move game with unawareness, any myopic discovery process converges to some revised game in which players do not need further revision of their subjective games. Moreover, we explore the relationships of the CURB notion with other solution concepts in our model.
JEL classification: C70; C72; D80; D83
Keywords: Game Theory; Unawareness; Discovery of Actions; Growing Awareness; Closedness under Rational Behavior; Block Game

1 Introduction

In the present paper, we mainly discuss the following model and result:

- Generalizing a notion of closedness under rational behavior (CURB) to simultaneous-move games with unawareness,

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- Modeling the myopic discovery process, and
- Proving the main theorem that plays of all agents converge to some CURB set in a certain discovered game in which players do not need further revision of their subjective games.

Although a lack or abundance of knowledge influences people's decision-making, the human community is stable. Specifically, stability in people creates a specific solution (or equilibrium). However, people do not always choose a particular solution from the beginning. People need to start by recognizing the decision-making or interactive situation they are in and find a specific solution through trial and error. The main purpose of this study is to demonstrate how agents discover and play a specific solution under a lack of knowledge or understanding by using discovery processes in simultaneous-move games with unawareness.

Studies of unawareness analyze decision-making and interactive situations assuming a lack of knowledge or understanding. Models of unawareness assume that agents are unaware of events, choices, opponents' plays, and so on.[1] In games with unawareness, the outcomes of players' implementations might be different from their play expectations because the players "did" not know some of the opponents' actions. Therefore, since the players observe the opponents' play, they are surprised by the play and may realize the error of their subjective views about the game's situation. Thus, prior beliefs about the opponents' play may be different from those reformulated after the opponents' play. Consequently, the players' next-stage game might change.

Schipper (2021) proposed models of discovery processes to analyze belief revision and replay through the revision process.[2] In a discovery process, players add opponents' actions that they are unaware of in each stage game. Schipper (2013) focused on extensive-form games with unawareness and showed that a rationalizable discovery process where every player implements rationalizable actions converges to some revised game possessing a rationalizable self-confirming equilibrium.[3]

This study adopts the idea of discovery processes and discusses the convergence of plays. To discuss such convergence, we introduce and generalize a CURB notion. A CURB notion is a set-wise solution concept, that is, re-

[1]Schipper (2014) provided a historical survey. In this paper, we present related literature in Section 5.

[2]A discovery process is different from a learning process. Learning is an update process of probability distributions. However, as shown by Schipper (2013), unawareness of an event is different from assigning probability 0 to the event. Any event that agents are unaware of is not included in the subjective state space. Hence, the agent cannot assign any probability to the event. If such an event occurs, the subjective state space must be expanded. In games with unawareness, the set of actions must also be increased when actions that players were unaware of are played. Therefore, learning cannot be applied to games with unawareness. Discovery processes are alternative models of learning processes that were proposed to avoid this issue.

[3]Note that Schipper (2021) did not show that a rationalizable discovery process converges to some solution.

finements of the strict Nash equilibrium and rationalizability. It combines the characteristics of an equilibrium notion and rationalizability notion as follows.

- When players implement actions in some CURB set, they do not have inventive deviation from the CURB set. Hence, we can apply stable convention to a CURB notion.
- If players' belief formation consists of mixed actions, a CURB notion can avoid disconfirmation difficulties for probability distributions.

A CURB notion possessing the above features might be a convergence of implementations by players who best respond to opponents' immediately preceding plays. In fact, Hurkens (1995) and Young (1998) showed that in a standard game, Markov plays converge to a minimal CURB set, considering the possibility of error. Although we do not assume the possibility of error, we show myopic play, namely, that all players best respond to opponents' preceding plays and converge to some CURB set in simultaneous-move games with unawareness.

However, in some simultaneous-move games with unawareness, some players' subjective games might not have a CURB set that is on the objective game because the player is unaware of a subset of the CURB set. Hence, we try to generalize a CURB notion to simultaneous-move games with unawareness. The key here is the realizable action set, that is, the Cartesian product of the set of a player's realizable actions. We focus only on CURB sets on the realizable action set, namely, realizable CURB sets. Our main theorem is that players' myopic play converges to some realizable CURB set.[4] When myopic players implement actions on some realizable CURB set and do not deviate from the CURB set, then all players' actions in the CURB set are stable.[5]

We first formulate simultaneous-move games with unawareness in Section 2 based on a type-based approach. Additionally, in this section, we generalize a CURB notion. Section 3 models the myopic discovery processes and convergence of those processes and shows our main result. Certainly, we need to discuss the relationship of a CURB notion and other solution concepts and features of discovery processes. This discussion is provided in Section 4. Other discussions are presented in Section 5. In this section, we consider the block game notion in (simultaneous-move) games with unawareness, adaptive plays, and the limitations of this work, and review related literature. Our framework is based on Perea (2018), albeit with some issues. We point out the issues and offer a solution in the Appendix.

[4]Note that some realizable CURB set is not CURB in some player's subjective game. Hence, we have to distinguish whether the set is CURB in all subjective games or not.

[5]However, we do not show that plays converge to specific actions in the CURB set. Players might choose to alternate actions over the CURB set.

2 Preliminaries

2.1 Simultaneous-Move Games with Unawareness

This section provides a definition of simultaneous-move games with unawareness, which are type-based models, and generalizes a CURB notion to simultaneous-move games with unawareness. Let $G = (I, A, u)$ be a standard finite simultaneous-move game. I is a finite set of players and $I_{-i} = I \setminus \{i\}$. $A = \times_{i \in I} A_i$, where A_i is the non-empty finite set of i's actions, and each element on the set is $a_i \in A_i$. Let $A_{-i} = \times_{j \in I_{-i}} A_j$. $u = (u_i)_{i \in I}$, where $u_i : A \to \mathbb{R}$ is i's utility function. Denote i's mixed action on A_i by $m_i \in M(A_i)$, where $M(A_i)$ is the set of i's mixed actions, and a mixed action profile on A by $m = (m_i)_{i \in I} \in M(A) = \times_{i \in I} M(A_i)$. We denote i's expected utility for $m \in M(A)$ by $Eu_i(m)$.

First, we define simultaneous-move games with unawareness.[6] For any standard simultaneous-move game G, let $V = \times_{i \in I}(2^{A_i} \setminus \{\emptyset\})$ be the set of possible *views* of G. That is, the set of a Cartesian product of a non-empty action subset. Like most previous work, this study assumes that the set of players is commonly known and that each player's utility for each action profile is the same among all the possible views. Let $v \in V$ be a (possible) view or block,[7] and A_i^v be the set of i's actions in $v = \times_{j \in I} A_j^v$. Let $A_{-i}^v = \times_{j \in I_{-i}} A_j^v$. Here, when player i is given v, i is aware of $a \in v$ and unaware of $a \in A \setminus v$. For any $v, v' \in V$, v is contained in v' if A_i^v is a subset of $A_i^{v'}$ for any $i \in I$, that is, $A_i^v \subseteq A_i^{v'}$. Let $M(A_i^v) = \{m_i \in M(A_i) | \Sigma_{a_i \in A_i^v} m_i(a_i) = 1\}$. Given any $\delta, \delta' \in \bigcup_{v \in V} \bigcup_{X \in 2^I \setminus \{\emptyset\}} M(\times_{i \in X} A_i^v)$, $\delta \equiv \delta'$ means that δ and δ' have the same supports and probabilities. Therefore, we can say that δ and δ' are equivalent.

Let $\Gamma = (G, (T_i)_{i \in I}, (v_i)_{i \in I}, (b_i)_{i \in I})$ be a simultaneous-move game with unawareness as follows: for each $i \in I$,

- T_i is a finite and non-empty set of i's type, one of which is their actual type t_i^*.
- $v_i : T_i \to V$ is i's view function.
- $b_i : T_i \to T_{-i}$ is i's belief function, where $T_{-i} = \times_{j \in I \setminus \{i\}} T_j$. If $b_i(t_i) = (t_j)_{j \in I \setminus \{i\}}$, then for each $j \in I \setminus \{i\}$, $v_j(t_j)$ must be contained in $v_i(t_i)$. Simply put, we do not assume probabilistic beliefs.

Let us call G an objective game (in Γ). An objective game can be interpreted

[6]Our definitions are similar to those of Perea (2018). Note that there are two major differences. First, Perea's (2018) model did not fix belief hierarchies on views. We assume that the "actual type" of players is given. Second, Perea (2018) considered probabilistic beliefs on awareness, whereas our players always have point beliefs on their opponents' awareness, as is often assumed in studies of games with unawareness. Furthermore, in the Appendix, we model a probabilistic belief different from Perea's (2018) framework. He assumed that each agent assigns probabilities to all opponent types, whereas we assume that opponents' type spaces that the agent believes might be different when the agent is given different types.

[7]A block is a Cartesian product of non-empty subsets of actions.

as the "true game" in Γ.[8] i's type t_i describes their view about the game and belief about the opponents' types. At t_i, $v_i(t_i) = v$ means that i is aware of v and unaware of $A \setminus v$, while $b_i(t_i) = (t_j)_{j \in I \setminus \{i\}}$ means that at t_i, i believes that the others' types are $(t_j)_{j \in I \setminus \{i\}}$ and that each j's view is $v_j(t_j)$. Given $(i, t_i) \in I \times T_i$, we denote a sequence $t_{i_1}, t_{i_2}, \dots, t_{i_h}, \dots$, where $t_{i_1} = t_i$, and for any $h \geq 2$, $t_{i_h} = b_{i_{h-1}}(t_{i_{h-1}})(i_h)$. We say that t_i leads to t_j if and only if there exists a subsequence $t_{i_1}, \dots, t_{i_h}$ such that $t_{i_1} = t_i$ and $t_{i_h} = t_j$. Here, we suppose $\bigcup_{i \in I} T_i = \bigcup_{i \in I} \{t^*_{i_h}\}_{h \geq 1; t^*_{i_h} = t^*_i}$.

The set of each player's actual play $A_i^{v_i(t_i^*)}$ may be a proper subset of i's *full* action set A_i. In such a scenario, they cannot play $a_i \in A_i \setminus A_i^{v_i(t_i^*)}$. In other words, a player's realized actions exclude the non-realized actions. Let $\times_{i \in I} A_i^{v_i(t_i^*)}$ be the *realizable action set*. Some players may not perceive the realizable action set.

2.2 Closedness under Rational Behavior (CURB)

We generalize a CURB notion, which is one of the set-wise notions, to simultaneous-move games with unawareness. A CURB notion is a refinement of a strict Nash equilibrium and a rationalizable set proposed by Basu and Weibull (1991). It has the features of both an equilibrium and a rationalizability notion. A rationalizable action is a support of all mixed equilibria, whereas a CURB set is a subset of the supports of such equilibria. In a mixed equilibrium, players might not be able to disconfirm a distribution of mixed actions, whereas a CURB notion avoids the impossibility of disconfirmation. Moreover, since games with unawareness assume asymmetric subjective views, players attempt to disconfirm their beliefs on the opponents' subjective views. A CURB notion can be used for such tests.

First, we define the generalized strategies. For any $i \in I$, let $s_i : T_i \to \bigcup_{t_i \in T_i} M(A_i^{v_i(t_i)})$ with $s_i(t_i) \in M(A_i^{v_i(t_i)})$ for all $t_i \in T_i$. Then, given t_i, $s_i(t_i) \in M(A_i^{v_i(t_i)})$ is i's local action at t_i. We denote i's generalized strategy by $s_i = (s_i(t_i))_{t_i \in T_i}$, and a generalized strategy profile by $s = (s_i)_{i \in I}$. In the generalized strategy profile s, each player i's actual play is $m_i \in M(A_i)$ with $m_i \equiv s_i(t_i^*)$ and the profile is called the objective outcome induced from s.

Second, we propose a generalization of a CURB notion to simultaneous-move games with unawareness. Although Basu and Weibull (1991) first defined a CURB notion on a standard game G, this paper defines it on each view. Given any standard simultaneous-move game G, any possible view $\hat{v} \in V$, and mixed action profile $m \in M(\hat{v})$, let

$$\beta_i^{\hat{v}}(m_{-i}) = \{a_i \in A_i | a_i \in \text{supp}(m_i) \text{ be such that } m_i \in \arg\max_{x \in M(A_i^{\hat{v}})} Eu_i(x, m_{-i})\}$$

[8]The term "objective game" was used by Halpern and Rêgo (2014). Feinberg (2021) referred to such a game as the "modeler's normal-form game" and Perea (2018) called it the "base game." In this context, we follow the study by Halpern and Rêgo (2014).

be the set of i's pure-action best responses to their beliefs on $m_{-i} \in M(A^{\hat{v}}_{-i})$. For any $v \subseteq \hat{v}$, let

$$\beta^{\hat{v}}_i(A^v_{-i}) = \bigcup_{m_{-i} \in M(A^{\hat{v}}_{-i}) : m_{-i} \equiv m'_{-i} \in M(A^v_{-i})} \beta^{\hat{v}}_i(m_{-i})$$

be the set of i's optimal actions under beliefs in $M(v)$, and let $\beta^{\hat{v}}(v) = \times_{i \in I} \beta^{\hat{v}}_i(A^v_{-i})$.

Then, CURB is defined as follows:

Definition 1. Given a standard simultaneous-move game G and $\hat{v} \in V$, $C \subseteq \hat{v}$ is a CURB set on $\hat{v}$ if $\beta^{\hat{v}}(C) \subseteq C$. C is a minimal CURB set on $\hat{v}$ if C is CURB on $\hat{v}$, and every proper subset of C is not CURB on $\hat{v}$.

Basu and Weibull (1991) showed that every standard game has a minimal CURB set.

Remark 1. Given any standard game, every possible view has a minimal CURB set.

In standard games, we only need to consider a CURB set on the full action set. However, since a given possible view for each player may not be consistent with the full action set in games with unawareness, realizable CURB sets might be different for standard games and games with unawareness. Hence, we must distinguish CURB notions between the two models. In the CURB notion under unawareness, we define a CURB set on the realizable action set, called a realizable CURB set, as follows:

Definition 2. Given a simultaneous-move game with unawareness Γ, let $v^* = \times_{i \in I} A_i^{v_i(t_i^*)}$ be the realizable action set. $C \in V$ is a *realizable CURB set* if $C \subseteq v^*$ and $\beta^{v^*}(C) \subseteq C$. C is a *minimal realizable CURB set* if it is CURB on v^*, and every proper subset of C is not CURB on v^*.

Realizable CURB notions have the following property:

Lemma 1. Every simultaneous-move game with unawareness Γ has a minimal realizable CURB set; it is non-empty.

Proof. Let us construct a game $G' = (N, A', u')$ such that the following assumptions hold:

- N is common in Γ.
- $A' = \times_{i \in I} A_i^{v_i(t_i^*)}$.
- For any $i \in I$, $u'_i : A' \to \mathbb{R}$ such that $u_i(a) = u'_i(a)$ for any $a \in A'$.

Following Basu and Weibull (1991), there must be a (minimal) CURB set $C \subseteq A'$ in G'. In other words, there exists a set of each player's pure-action best response, $\beta'(C)$, such that $\beta'(C) \subseteq C$ in G. Since $\beta'(C)$ is defined on $A' = \times_{i \in I} A_i^{v_i(t_i^*)}$, C is a minimal realizable CURB set. □

Given Γ, some realizable CURB set $C \in V$ may be $C \subseteq v_i(t_i)$ for any $(i, t_i) \in I \times T_i$. However, the set might not be CURB in $v_i(t_i)$ at some t_i. Given a realizable CURB set, we distinguish between a case in which the realizable CURB set is CURB in every $v_i(t_i)$ for any $(i, t_i) \in I \times T_i$, and one in which it is not as follows:

Definition 3. In a simultaneous-move game with unawareness Γ, $C \in V$ is a *common (minimal) realizable CURB set* if for any $(i, t_i) \in I \times T_i$, C is a (minimal) realizable CURB set and $C \subseteq v_i(t_i)$. C is a *common* (*minimal*) *CURB set* if for any $(i, t_i) \in I \times T_i \setminus \{t_i^*\}$, C is CURB in $v_i(t_i)$, and for any i, $\beta_i^{v_i(t_i^*)}(A_{-i}^C) \subseteq A_i^C$.[9]

A common realizable CURB set is a subset in each subjective view, but the set might not be CURB from some view. By contrast, a common CURB set is CURB in each subjective view other than the player's actual view,[10], and each player's actions in the CURB set best respond to the opponents' actions in the CURB set in the player's actual view.

Example 1. Let us consider that two players, Alice (A) and Bob (B), face the following objective game:[11]

$v^O =$

A / B	b_1	b_2	b_3
a_1	3, 3	0, 5	0, 0
a_2	5, 0	1, 1	0, 0
a_3	0, 0	0, 0	2, 2

.

Here, if Alice is unaware of her own action a_2, then her view is as follows:

$v^A =$

A / B	b_1	b_2	b_3
a_1	3, 3	0, 5	0, 0
a_3	0, 0	0, 0	2, 2

,

If Bob is unaware of his own action b_2, then his view is as follows:

$v^B =$

A / B	b_1	b_3
a_1	3, 3	0, 0
a_2	5, 0	0, 0
a_3	0, 0	2, 2

.

Let us suppose that Alice believes that Bob's view is the same as hers, that is, they both believe that they hold the same view v^A, while Bob believes that Alice's view is the same as his, that is, they both believe that they hold the same view v^B.

[9]Tada (2020), noted that a common CURB set means only a CURB set on the full action set. By contrast, this paper generalizes this notion by focusing on a realizable action set.

[10]Here, an actual view means the view when the agent's actual type is given.

[11]This example is similar to Example 3 in Schipper (2018). An idea of our similar example is borrowed from his example.

Here, we formulate this game (with unawareness) $\Gamma = (v^O, (T_A, T_B), (v_A, v_B), (b_A, b_B))$ as follows:

$T_A = \{t_A^*, t_A\}$, and $T_B = \{t_B^*, t_B\}$;
given t_A^*, $v_A(t_A^*) = v_A$, and $b_A(t_A^*) = t_B$;
given t_A, $v_A(t_A) = v_A$, and $b_A(t_A) = t_B^*$;
given t_B^*, $v_B(t_B^*) = v_B$, and $b_B(t_B^*) = t_A$; and
given t_B, $v_B(t_B) = v_B$, and $b_B(t_B) = t_A^*$.

Since Alice's realizable actions are a_1 and a_3, while Bob's realizable actions are b_1 and b_3, the realizable action set is the following table:

$v^R =$

A / B	b_1	b_3
a_1	3, 3	0, 0
a_3	0, 0	2, 2

.

Then, there exists three CURB sets on the realizable action set, that is, three realizable CURB sets:

$C^1 = \{a_1\} \times \{b_1\}$;
$C^2 = \{a_3\} \times \{b_3\}$; and
$C^3 = \{a_1, a_3\} \times \{b_1, b_3\}$.

Here, C^3 is a maximum CURB set. Since, $C^1, C^2, C^3 \subseteq v^A$ and $C^1, C^2, C^3 \subseteq v^B$, every realizable CURB set is a common realizable CURB set. Moreover, C^2 is the only unique common CURB set because the common CURB set is CURB on v^A and v^B. □

3 Myopic Discovery Process

Standard game models study the convergence to a minimal CURB set by using a learning model or an adaptation model, for example, Hurkens (1995) and Young (1998). Previous literature on standard models shows that when all the players best respond to the opponents' preceding previous play, their plays converge to some minimal CURB set. We can also predict that in dynamics of simultaneous-move games with unawareness, all the agents' implementations converge to some generalized CURB set when they best respond to the opponents' preceding play. To prove this prediction, we model a myopic discovery process in this section.

A discovery process represents an update process by which each player revises their own belief about the game's structure and the opponents' play. This model was first introduced by Schipper (2021) to games with unawareness. He analyzed a rationalizable discovery process in extensive-form models, which is based on Heifetz et al. (2013). This study models a discovery process in simultaneous-move games with unawareness based on Perea (2018). Although our definition, at first glance, may seem different from that of Schipper (2021), both are essentially the same.

Definition 4. $\Gamma' = (G, (T'_i)_{i\in I}, (v'_i)_{i\in I}, (b'_i)_{i\in I})$ is a discovered game with $s = (s_i)_{i\in I}$ in $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$ if for any $(i, t_i) \in I \times T_i$, and any sequence of players $i_1, i_2, \ldots, i_h, \ldots$, with a sequence of types $t_{i_1}, t_{i_2}, \ldots, t_{i_h}, \ldots$, where $t_{i_1} = t_i$, there exists $t'_i \in T'_i$ and a sequence of types in Γ', $t'_{i_1}, t'_{i_2}, \ldots, t'_{i_h}, \ldots$, where $t'_{i_1} = t_i$, such that for any $h \geq 1$,

$$v'_{i_h}(t'_{i_h}) = \times_{j\in I}[A_j^{v_{i_h}}(t_{i_h}) \cup \text{supp}(s_j(t_j^*))],$$

where t_j^* is j's actual type in Γ. Note that some Γ, Γ' may be $T \not\subseteq T'$ and $T' \not\subseteq T$, or $T \cap T' = \emptyset$.

[]

Example 2. Consider the following objective game played by Colin (C) and David (D):

$v^0 =$

C / D	d_1	d_2	d_3
c_1	3, 3	0, 5	0, -1
c_2	5, 0	1, 1	1, 0
c_3	-1, 0	0, 1	2, 2

;

and two possible views as follows:

$v^1 =$

C / D	d_2	d_3
c_1	0, 5	0, -1
c_3	0, 1	2, 2

and $v^2 =$

C / D	d_1	d_3
c_1	3, 3	0, -1
c_2	5, 0	1, 0

.

Let us formulate the game with unawareness $\Gamma = (v^0, (T_C, T_D), (v_C, v_D), (b_C, b_D))$ as follows:

$T_C = \{t_C^*, t_C\}$, and $T_D = \{t_D^*, t_D\}$;
given t_C^*, $v_C(t_C^*) = v_C$, and $b_C(t_C^*) = t_D$;
given t_C, $v_C(t_C) = v_C$, and $b_C(t_C) = t_D^*$;
given t_D^*, $v_D(t_D^*) = v_D$, and $b_D(t_D^*) = t_C$; and
given t_D, $v_D(t_D) = v_D$, and $b_D(t_D) = t_C^*$.

Here, suppose that Colin and David play a generalized strategy profile:

$$s = ([s_C(t_C^*) = c_1, s_C(t_C) = c_2)], [s_D(t_D^*) = d_2, s_D(t_D) = d_3]).$$

The objective outcome is (c_1, d_2) induced by s.

Let Γ' be the discovered game with s in Γ. Then, each player's type set in Γ' is $T'_C = \{t'^*_C, t'_C\}$, and $T'_D = \{t'^*_D, t'_D\}$, where

$$b'_C(t'^*_C) = t'_D;$$

$$b'_C(t'_C) = t'^{*}_D;$$

$$b'_D(t'^{*}_D) = t'_C;$$

$$b'_D(t'_D) = t'^{*}_C;$$

$\hat{v}^1 = v'_C(t'^{*}_C) = v'_D(t'_D) =$

C / D	d_2	d_3
c_1	0, 5	0, -1
c_3	0, 1	2, 2

;

and

$\hat{v}^2 = v'_C(t'_C) = v'_D(t'^{*}_D) =$

C / D	d_1	d_2	d_3
c_1	3, 3	0, 5	0, -1
c_2	5, 0	1, 1	1, 0

.

□

Definition 5. A discovery process $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \ldots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \ldots)$, is defined as follows:

- for any λ, $\Gamma^\lambda = (G, (T_i^\lambda)_{i \in I}, (v_i^\lambda)_{i \in I}, (b_i^\lambda)_{i \in I})$,
- when $\lambda = 0$ and $s^0 = \phi$, while for any $\lambda \geq 1$, s^λ is a played generalized strategy profile in Γ^λ, and
- for any $\lambda \geq 2$, Γ^λ is a discovered game with $s^{\lambda-1}$ in $\Gamma^{\lambda-1}$.

Let us call Γ^1 the initial game with unawareness (in P).

From definition 4, definition 5 implicitly assumes perfect recall. If we exclude the assumption, some player may forget some action at λ even if they are aware of the action at $\lambda - 1$.

This study assumes that every player implements a pure action. Meanwhile, standard game models might assume that every player implements and observes a mixed action. By contrast, in games with unawareness, it does not seem appropriate that every player implements and observes a mixed action because under unawareness, the players cannot observe the frequency of their opponents' actions at each stage of the game during any discovery process.[12]

In a discovery process, cautious players might carefully revise their beliefs about the game, opponents' plays and rationalities, and pay-off uncertainty (e.g., they might play rationalizable strategies). However, in the real world, the agents must pay a higher cost for revising such beliefs and implementing rationalizable strategies. If the players are myopic, they do not pay a high cost for revising their beliefs. This section explains the myopic discovery process in which every player best responds to the opponents' previous plays.

First, we define a strategy of myopic play in a discovered game as follows:

[12] I thank an anonymous referee for pointing this out.

Definition 6. Let Γ' be a discovered game from Γ. s' is a *myopic best response* in Γ if there exists the belief system, $\mu' = (\mu'_i)_{i\in I}$ such that for any $(i, t'_i) \in I \times T'_i$,

1. $s'_i(t'_i) \in \arg\max_{x\in M(A_i^{v'_i(t'_i)})} Eu_i(x, \mu'_i(t'_i))$; and
2. $\mu'_i(t'_i) \equiv (s^*_j(t^*_j))_{j\in I_{-i}}$, where s^* is played in Γ, and for any $j \in I_{-i}$, $t^*_j \in T_j$ is j's actual type in Γ.

Second, we provide a myopic discovery process.

Definition 7. Any discovery process $P = (\langle\Gamma^1, s^0\rangle, \langle\Gamma^2, s^1\rangle, \dots, \langle\Gamma^\lambda, s^{\lambda-1}\rangle, \dots)$, is a myopic discovery process if for any $\lambda \geq 2$, s^λ is a myopic best response at λ. A discovery process P is converging to Γ if there exists h such that for any $\lambda \geq h$, $\Gamma^\lambda = \Gamma$.

From the above formulations, we can show the convergence of a CURB set as follows:

Theorem 1. Given any simultaneous-move game with unawareness Γ, every myopic discovery process, P, converges to a discovered game, possessing a common realizable CURB set. Thus, a subset of the supports of all the agents' myopic best responses converges to that common realizable CURB set.

Proof. Since we consider a myopic discovery process, it is necessary to focus only on the realizable action set. For any objective outcome in the initial game $m \in M(\times_{i\in I} A_i^{v_i(t^*_i)})$, let $\beta^\lambda(m)$ be an objective outcome induced by a myopic best response on the realizable action set, and it is defined as follows: $\beta^0(m) = \text{supp}(m)$, $\beta^1(m) = \beta' \circ \beta^0(m)$, $\beta^2(m) = \beta' \circ \beta^1(m)$, $\dots$, $\beta^\lambda(m) = \beta' \circ \beta^{\lambda-1}(m)$, $\dots$. Suppose that for any CURB set on the realizable action set $C \subseteq \times_{i\in I} A_i^{v_i(t^*_i)}$, and natural number λ, $\beta^\lambda(m) \nsubseteq C$. As pointed out by Basu and Weibull (1991), since the set of the rationalizable strategy profile on $\times_{i\in I} A_i^{v_i(t^*_i)}$, $R \subseteq \times_{i\in I} A_i^{v_i(t^*_i)}$, is CURB,[13] $\beta^\lambda(m) \nsubseteq R$ for any λ, this is a contradiction. Therefore, there exists a realizable CURB set, C, and natural number, n, such that $\beta^n(m) \subseteq C$. Suppose that there exists $\lambda \geq n$, satisfying $\beta^\lambda(m) \nsubseteq C$. Simply, $\beta' \circ \cdots \circ \beta' \circ \beta^n(m) \nsubseteq C$. However, since C is CURB, that is, $\beta(v') \subseteq C$ for any $v' \subseteq C$ with $\emptyset \neq A_i^{v'} \subseteq A_i^C$, this is a contradiction. Therefore, $\beta^\lambda(m) \subseteq C$ for any $\lambda \geq n$. Since, $\beta^\lambda(m)$ supports an objective outcome induced by a myopic best response at λ and C is a realizable CURB set, the support for the objective outcome is included in the realizable CURB set. From definition 4, since CURB, C, is common, C is a common realizable CURB set. □

It is known that many intuitively appealing adjustment processes eventually settle down in a minimal CURB set (cf. Hurkens, 1995; Young, 1998). Theorem 1 adds to previous literature, highlighting the importance of the CURB

[13] Specifically, R is a maximum fixed under rational behavior (FURB) set. An action profile set, $C \in V$, is a FURB set if $\beta(C) = C$.

set. However, the process therein converges to a general CURB set and, not necessarily, a "minimal" one, such as in Hurkens (1995) and Young (1998).[14]

Example 2 (Continued)**.** Let Γ be an initial game, that is, a game at $\lambda = 1$. Then, the realizable action set is as follows:

$v^R =$

C / D	d_1	d_3
c_1	3, 3	0, -1
c_3	-1, 0	2, 2

.

v^R has three CURB sets, $C^1 = \{c_1\} \times \{d_1\}$, $C^2 = \{c_3\} \times \{d_3\}$, and $C^3 = \{c_1, c_3\} \times \{d_1, d_3\}$. Here, C^2 is a mutual CURB set.

Let us focus on two generalized strategy profiles:

$s_1 = ([s_C(t_C^*) = c_1, s_C(t_C) = c_2], [s_D(t_D^*) = d_1, s_D(t_D) = d_2])$; and
$s_2 = ([s_C(t_C^*) = c_3, s_C(t_C) = c_2], [s_D(t_D^*) = d_3, s_D(t_D) = d_3])$.

First, we focus on the former strategy profile, s_1. The objective outcome is (c_1, d_1). Since Colin is unaware of d_1, he is surprised and revises his view as follows:

$v^{1'} =$

C / D	d_1	d_2	d_3
c_1	3, 3	0, 5	0, -1
c_3	-1, 0	0, 1	2, 2

.

Then, at $\lambda = 2$, the discovered game $\Gamma' = (G, (T'_C, T'_D), (v'_C, v'_D), (b'_C, b'_D))$, where

$T'_C = \{t_C^{2*}, t_C^2\}$, and $T'_D = \{t_D^{2*}, t_D^2\}$;

$v'_C(t_C^{2*}) = v^{1'}$, and $b'_C(t_C^{2*}) = t_D^2$;

$v'_C(t_C^2) = v^2$, and $b'_C(t_C^2) = t_D^{2*}$;

$v'_D(t_D^{2*}) = v^2$, and $b'_D(t_D^{2*}) = t_C^2$; and

$v'_D(t_D^2) = v^{1'}$, and $b'_D(t_D^2) = t_C^{2*}$.

At $\lambda = 2$, when they play myopic best responses, the generalized strategy profile is

$$s_1^2 = ([s_C^2(t_C^{2*}) = c_1, s_C^2(t_C^2) = c_1], [s_D^2(t_D^{2*}) = d_1, s_D^2(t_D^2) = d_1]).$$

Neither of the players discover the opponents' actual plays. Hence, the next stage game, at $\lambda = 3$, is the same as Γ'. In Γ', the play s_1^2 is not a generalized

[14]Hurkens (1995) and Young (1998) showed the convergence of a minimal CURB set by using the adaptive plays proposed by Young (1993). This paper discusses a discovery process with adaptive plays in Section 5.

Nash equilibrium and the objective outcome is (c_1, d_1). The support of the objective outcome, $\{c_1\} \times \{d_1\}$, is a subset of a realizable CURB set, C^1.

Second, let us focus on the latter generalized Nash equilibrium s_2. The objective outcome is (c_3, d_3). Since David is unaware of c_3, he is surprised and revises his view as follows:

$v^{2'} =$

C / D	d_1	d_3
c_1	3, 3	0, -1
c_2	5, 0	1, 0
c_3	-1, 0	2, 2

.

Then, at $\lambda = 2'$, $\Gamma'' = (G, (T''_C, T''_D), (v''_C, v''_D), (b''_C, b''_D))$, where

$T''_C = \{t_C^{2'*}, t_C^{2'}\}$, and $T''_D = \{t_D^{2'*}, t_D^{2'}\}$;

$v''_C(t_C^{2'*}) = v^1$, and $b''_C(t_C^{2'*}) = t_D^{2'}$;

$v''_C(t_C^{2'}) = v^{2'}$, and $b''_C(t_C^{2'}) = t_D^{2'*}$;

$v''_D(t_D^{2'*}) = v^{2'}$, and $b''_D(t_D^{2'*}) = t_C^{2'}$; and

$v''_D(t_D^{2'}) = v^1$, and $b''_D(t_D^{2'}) = t_C^{2'*}$.

At $\lambda = 2'$, when they play myopic best responses, the generalized strategy profile is

$$s_2^{2'} = ([s_C^{2'}(t_C^{2'*}) = c_3, s_C^{2'}(t_C^{2'}) = c_3], [s_D^{2'}(t_D^{2'*}) = d_3, s_D^{2'}(t_D^{2'}) = d_3]).$$

Neither of the players discover the opponents' actual play. Hence, the next stage game, at $\lambda = 3'$, is the same as Γ''. In Γ'', the play, the generalized strategy profile is a cognitively stable generalized Nash equilibrium and a support of the objective outcome, $\{c_3\} \times \{d_3\}$, is a subset of a common CURB set, C^2 in Γ''. □

4 Discovery Process and Other Solution Concepts

4.1 Relationships among Other Solution Concepts

This section discusses the relationship between a CURB notion and other solution concepts, for example, the generalized Nash equilibrium, the self-confirming equilibrium, and rationalizability.

4.1.1 Equilibrium Notions

Previous literature discusses and proposes equilibrium notions, whereas in this study we focus on generalized Nash (Halpern and Rêgo 2021 and Sasaki 2017) and self-confirming equilibria (Schipper2021 and Kobayashi and Sasaki 2021).

Given any generalized strategy profile s, let $\mu = (\mu_i)_{i \in I}$ be the belief system in s, where for any $(i, t_i) \in I \times T_i$, $\mu_i(t_i) \in M(A_{-i}^{v_i(t_i)})$ and $\mu_i = (\mu_i(t_i))_{t_i \in T_i}$. This study allows correlated beliefs. Therefore, the generalized Nash equilibrium is defined as follows: in a simultaneous-move game with unawareness Γ, s^* is a generalized Nash equilibrium if there exists belief system μ such that for any $(i, t_i) \in I \times T_i$,

1. $s_i^*(t_i) \in \arg\max_{x \in M(A_i^{v_i(t_i)})} Eu_i(x, \mu_i(t_i))$, and
2. $\mu_i(t_i) \equiv (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}}$.

As they pointed out, a generalized Nash equilibrium is best interpreted as "an equilibrium in beliefs" (Halpern and Rêgo, 2014: 50). However, as pointed out by Schipper (2014), a generalized Nash equilibrium can consist of wrong beliefs. In that case, each player would revise their own beliefs about a game's structure and the opponents' play, and may not play the same generalized Nash equilibrium.

To avoid this scenario, previous literature has discussed refinements of general equilibria as steady-state notions, namely, the stable-belief-hierarchies notion, cognitive stability notion, and self-confirming notion.

First, we consider the stable-belief-hierarchies notion and cognitive stability notion, as proposed by Sasaki (2017):

- The generalized Nash equilibrium s^* has stable belief hierarchies if belief system μ satisfies that for any $(i, t_i) \in I \times T_i$, $\mu_i(t_i) \equiv (s_j^*(t_j^*))_{j \in I_{-i}}$.
- The generalized Nash equilibrium s^* is cognitively stable if for any $(i, t_i) \in I \times T_i$, $s_i^*(t_i) \equiv s_i^*(t_i^*)$.

The former notion means that every player's belief about the opponents' plays is correct, while the latter means that each player's decision-making on their arbitrary type is equivalent to their actual play.

Sasaki (2017) expressed the following:

Remark 2. Generalized strategy profile s^* is a generalized Nash equilibrium with stable belief hierarchies if and only if s^* is a cognitively stable generalized Nash equilibrium.

Such a generalized Nash equilibrium is called a cognitively stable generalized Nash equilibrium in this paper. A cognitively stable generalized Nash equilibrium has the following property:

Remark 3. For any simultaneous-move game with unawareness Γ, let s^* be a cognitively stable generalized Nash equilibrium. Then, the objective outcome $m^* \equiv (s_i^*(t_i^*))_{i \in I}$ is a Nash equilibrium on the realizable action set.

Proof. Suppose that s^* is a cognitively stable generalized Nash equilibrium, that is, for any $(i, t_i) \in I \times T_i$, $s_i^*(t_i) \equiv m_i^*$. Then, $m^* \equiv m \in M(v_j(t_j))$ for any $(j, t_j) \in I \times T_j$. Assume that $m' \in M(\times_{i \in I} A_i^{v_i(t_i^*)})$ with $m' \equiv m^*$ is not a Nash equilibrium on the realizable action set. In other words, there exists $i \in I$ such that

$$m_i^* \equiv m_i' \notin \arg \max_{x \in A_i^{v_i(t_i^*)}} Eu_i(x, m_{-i}').$$

Then, for any $(i, t_i) \in I \times T_i$,

$$m_i^* \equiv m_i' \equiv m_i'' \notin \arg \max_{x \in A_i^{v_i(t_i)}} Eu_i(x, \mu_i(t_i)),$$

where $\mu_i \equiv m_{-i}' \equiv m_{-i}^*$. However, since s^* is a cognitively stable generalized Nash equilibrium, this is a contradiction. Therefore, m' is a Nash equilibrium on the realizable action set. □

The relationship between a CURB notion and a cognitive stability notion has the following property:

Proposition 1. Any simultaneous-move game with unawareness possessing a common CURB set has a cognitively stable generalized Nash equilibrium.

Proof. Assume that any simultaneous-move game with unawareness has a common CURB set $C \in V$. That is, for any $(i, t_i) \in I \times T_i$, C is CURB on $v_i(t_i)$. Then, following Basu and Weibull (1991), a Nash equilibrium on $v_i(t_i)$, $m^* \in M(v_i(t_i))$ exists, satisfying $m^* \equiv m' \in M(C)$. Suppose that m' is not a Nash equilibrium on v. Simply, $(i, a_i) \in I \times A_i^{v_i(t_i)}$ exists, such that $Eu_i(a_i, m_{-i}') > Eu_i(m')$. However, since C is a common CURB set, this is a contradiction. Therefore, m' is a Nash equilibrium on C. Thus, since $(i, t_i) \in I \times T_i$ is arbitrary, $m^* \equiv m'$ is a Nash equilibrium on $v_i(t_i)$. Thus, s^* with $s_i^*(t_i) \equiv m_i^*$ for any $i \in I$ and $t_i \in T_i$ is a cognitively stable generalized Nash equilibrium. □

Proposition 1 suggests one of the conditions for the existence of a cognitively stable generalized Nash equilibrium in any game with unawareness. The contraposition is that if any cognitively stable generalized Nash equilibrium does not exist, then any common CURB set does not exist. This means that if some of the players cannot perceive any CURB set in the realizable action set, then the players are surprised about a realized play because the player's belief about the opponents' plays is wrong. Our proposition suggests a condition for all players' stable plays, that is, rational players do not deviate from a specific play.[15]

The following corollary is obvious from the above proof of Proposition 1.

Corollary 1. Given any simultaneous-move game with unawareness, a common CURB set includes the support of some cognitively stable generalized Nash equilibrium.

[15] I thank Masakazu Fukuzumi for this suggestion.

Second, let us consider the self-confirming equilibrium proposed by Fudenberg and Levine (1993). Schipper (2021) generalized a rationalizable self-confirming equilibrium to include extensive-form games with unawareness. Kobayashi and Sasaki (2021) focused on only simultaneous-move games with unawareness and discussed a rationalizable self-confirming equilibrium by using epistemic models. We discuss the k-self-confirming equilibrium, which means that all the players in the k-th order mutually believe that all the players' beliefs are correct. The following definition of self-confirming equilibria is based on Kobayashi and Sasaki (2021). s^* is a k-self-confirming equilibrium if there exists belief system μ such that for any $h = 1, \dots, k+1$ and $i_h \in I$, where $t_{i_1} = t_i^*$,

1. $s_{i_h}^* \in \arg\max_{x \in M(A_i^{v_{i_h}(t_{i_h})})} Eu_i(x, \mu_i(t_{i_h}))$, and
2. $\mu_{i_h} \equiv (s_{i_{h+1}}^*(t_{i_{h+1}}))_{i_{h+1} \in I_{-i_h}}$.

4.1.2 Rationalizable Notions

Rationalizability was proposed by Bernheim (1984) and Pearce (1984). Heifetz et al. (2013b) generalized Pearce's extensive-form rationalizability to games with unawareness. This notion is one of the approaches that avoids the issue of equilibrium notions in general.

Let us give a type profile $(t_i)_{i \in I}$, let $A(h) = \times_{i_h \in I} A_{i_h}(i_h)$, and let $A_{-i_h}(i_h) = \times_{j \in I_{-i_h}} A_j^{v_j(b_{i_h}(t_{i_h})(j))}$. Here, given $A(h)$, $\beta^k(A(h)) = \underbrace{\beta \circ \cdots \circ \beta}_{k}(A(h))$. Then, this study defines rationalizability in simultaneous-move games with unawareness as follows: In a simultaneous-move game with unawareness Γ, for any $i \in I$, $R((t_{i_h})_{i_h \in I}) = \bigcap_{k=1} \beta^k(A(h))$ is a rationalizable action set at t_{i_h}, where $t_{i_1} = t_i^*$, and the rationalizable strategy $R = (R((t_{i_h})_{i_h \in I}))_{h \in N: N \text{is the set of natural numbers.}}$.[16] As pointed out by Basu and Weibull (1991), the rationalizable action set is the maximum FURB set.[17]

Kobayashi and Sasaki (2021) proposed a k-rationalizable self-confirming equilibrium as follows: In a simultaneous-move game with unawareness Γ, s^* is a k-rationalizable self-confirming equilibrium if it is a k-self-confirming equilibrium and $\text{supp}[(s_i^*(t_i^*))_{i \in I}] \subseteq R((t_{i_h})_{i_h \in I})$, where for any $i \in I$, $h = 1, \dots, k+1$ such that $t_{i_1} = t_i^*$. Kobayashi and Sasaki (2021) remarked as follows:

Remark 4. s^* is a ∞-rationalizable self-confirming equilibrium if and only if s^* is a cognitively stable generalized Nash equilibrium.

4.1.3 Example

This section compares a CURB notion with other solution concepts.

[16] As pointed out by Basu and Weibull (1991), the rationalizable action set is a maximum FURB set, we can use β to define rationalizability.

[17] Some rationalizable action set might not be the maximum CURB set. Given the rationalizable action set $X \subseteq A$ in standard game, $Y \subseteq A$ might exist such that $X \subsetneq Y$ and $\beta(Y) \subsetneq Y$.

Example 1 (Continued)**.** Let us reconsider Example 1. First, we consider generalized (pure) Nash equilibria in the game. Four generalized pure Nash equilibria exist:

$s_1 = ([s_A(t_A^*) = a_1, s_A(t_A) = a_2], [s_B(t_B^*) = b_1, s_B(t_B) = b_2]);$
$s_2 = ([s_A(t_A^*) = a_1, s_A(t_A) = a_3], [s_B(t_B^*) = b_3, s_B(t_B) = b_2]);$
$s_3 = ([s_A(t_A^*) = a_3, s_A(t_A) = a_2], [s_B(t_B^*) = b_1, s_B(t_B) = b_3]);$, and
$s_4 = ([s_A(t_A^*) = a_3, s_A(t_A) = a_3], [s_B(t_B^*) = b_3, s_B(t_B) = b_3]).$

Then, the cognitively stable generalized Nash equilibrium is only s_4.

Second, we consider self-confirming (pure) equilibria in the game. A 0-self-confirming equilibrium and ∞-self-confirming equilibrium exist. The 0-self-confirming equilibrium is only

$$s_5 = ([s_A(t_A^*) = a_1, s_A(t_A) = a_1], [s_B(t_B^*) = b_1, s_B(t_B) = b_1]).$$

Note that the 0-self-confirming equilibrium is not the k-self-confirming equilibrium ($k > 1$) because in $k+1$, Alice's a_1 does not respond best to Bob's b_1, and Bob's b_1 does not respond best to Alice's a_1.

By contrast, the ∞-self-confirming equilibrium is only s_4. When comparing a cognitively stable generalized Nash equilibrium with an ∞-self-confirming equilibrium, both the equilibria are the same.

Third, we consider rationalizability. Given three tuples $t^1 = (t_A^*, t_B^*)$, $t_2 = (t_A^*, t_B)$, and $t_3 = (t_A, t_B^*)$, the rationalizable strategy is $R = (R(t_1), R(t_2), R(t_3))$, and the pure rationalizable actions at t_2 and t_3 are as follows:

$R(t^2) = \{a_1, a_3\} \times \{b_2, b_3\}$; and
$R(t^3) = \{a_2, a_3\} \times \{b_1, b_3\}$.

Then, at t_1,

$$R(t^1) = \beta_A^{v_i(t_A^*)}(\{b_2, b_3\}) \times \beta_B^{v_i(t_B^*)}(\{a_2, a_3\}) = \{a_1, a_3\} \times \{b_1, b_3\}.$$

Here, it is obvious that 0-self-confirming equilibrium s_5 is a 0-rationalizable self-confirming equilibrium and that ∞-self-confirming equilibrium s_4 is an ∞-rationalizable self-confirming equilibrium.

Let us compare a CURB notion with the other notions. First, we compare a CURB notion with equilibrium notions. From Proposition 1, any common CURB set includes a support for the objective outcome, induced from the cognitively stable generalized Nash equilibrium. Since the ∞-rationalizable self-confirming equilibrium and cognitively stable generalized Nash equilibrium are the same, C^2 includes a support for the objective outcome (a_3, b_3), induced from a cognitively stable generalized Nash equilibrium and ∞-rationalizable self-confirming equilibrium, s^4. By contrast, a common realizable CURB set C^1, which is not a common CURB set, is a support for the objective outcome (a_1, b_1), induced from 0-rationalizable self-confirming equilibrium s^5. In C^1 or

s^5, each player is rational but their first-order belief is irrational play.[18] Moreover, they might be certain about the opponent's irrationality.

Next, we consider a relationship with rationalizability. Given a type t^1, $R(t^1) = C^3$. As shown by Basu and Weibull (1991), a rationalizable action set is equivalent to a maximum CURB set. By contrast, minimal CURB sets, C^1 and C^2, are subsets of the rationalizable action set, that is, any minimal CURB set is a refined notion of rationalizability. □

As shown in the above example, a realizable CURB notion is related to other solution concepts, that is, a CURB notion has similar characterizations to the other notions.

4.2 Discovery and Equilibrium Notions

This section discusses the relationships between discovery processes and equilibrium notions.

4.2.1 Rationalizable Discovery Process and Self-Confirming Equilibrium

Schipper (2021) modeled rationalizable discovery processes in which all players implement rationalizable actions in each stage game. We formulate rationalizable discovery processes based on Perea (2018) as follows:

Definition 8. A discovery process $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \dots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \dots)$ is a rationalizable discovery process if for any λ, R^λ is the set of rationalizable strategy.

Schipper (2021) modeled a rationalizable discovery process based on Heifetz et al. (2013b) and showed that every rationalizable discovery process (in any extensive-form game with unawareness) converges to a self-confirming game, in which no rational player needs further revision and possesses some (0-)rationalizable self-confirming equilibrium.

However, in our framework, we do not show Schipper's (2021) result.

Example 3. Let us consider two agents Elena and Filip and assume they face a zero-sum game:

$v =$

E / F	f_1	f_2
e_1	1, -1	-1, 1
e_2	-1, 1	1, -1

.

Here, suppose that in a zero-sum game with unawareness, there exists a view as

$v' =$

E / F	f_1	f_2
e_1	1, -1	-1, 1

,

$T_E = \{t_e^*, t_e, t_e'\}$ and $T_f = \{t_f^*, t_f\}$,

[18] Rational play means that the players maximize their utilities.

given t_e^*, $v_e(t_e^*) = v$ and $b_e(t_e^*) = t_f$,
given t_e, $v_e(t_e) = v'$ and $b_e(t_e) = t_f^*$,
given t_e', $v_e(t_e') = v'$ and $b_e(t_e') = t_f^*$,
given t_f^*, $v_f(t_f^*) = v'$ and $b_f(t_f^*) = t_e'$, and
given t_f, $v_f(t_f) = v$ and $b_f(t_f) = t_e$.

Then, a strategy profile that is uniquely rationalizable is

$$s = ([s_e(t_e^*) = e_1, s_e(t_e) = e_2, s_e(t_e') = e_1], [s_f(t_f^*) = f_2, s_f(t_f) = f_1]),$$

that is, the actual play is (e_1, f_2). In the play, Filip's belief is correct, whereas Elena's belief is wrong because she predicts that Filip plays f_1. However, both players are aware of the actual play, that is, discoveries do not occur. Hence, in the next-stage game, they play the same s. Then, there is no n-rationalizable self-confirming equilibrium for any natural number n, that is, the 0-rationalizable self-confirming equilibrium does not exist because in Elena's actual subjective view v, a unique self-confirming equilibrium is that both players assign probability $\frac{1}{2}$ to each of their actions, while in Filip's actual subjective view v', the unique self-confirming equilibrium is (e_1, f_2). □

4.2.2 Myopic Discovery Process and Cognitive Stability

In this section, we consider cognitively stable generalized Nash equilibria in myopic discovery processes. First, we provide a mutual CURB notion that each player's actual view has the same CURB set.

Definition 9. In any simultaneous-move game with unawareness Γ, $C \in V$ is a *mutual CURB set* if for any $i \in I$, C is a non-empty CURB set in $v_i(t_i^*)$.

A mutual CURB notion has the following property.

Lemma 2. Every mutual CURB set is a realizable CURB set.

Proof. Given any mutual CURB set, $C \in V$, $C \subseteq v_i(t_i^*)$ for any $i \in I$. Suppose that C is not a realizable CURB set, that is, there exists some i, such that $\beta_i^*(A_{-i}^C) \not\subseteq A_i^C$ in the realizable action set. Since the realizable action set is defined by $\times_{i \in I} A_i^{v_i(t_i^*)}$, $\beta_i^{v_i(t_i^*)}(A_{-i}^C) \not\subseteq A_i^C$. This contradicts that C is a mutual CURB set. Hence, C is a realizable CURB set. □

Lemma 3. In a simultaneous-move game with unawareness, if a mutual CURB set is present in every view, then a common CURB set exists.

Proof. Suppose that a mutual CURB set, C, is present in every view in a simultaneous-move game with unawareness. Suppose that for some $(i, t_i) \in I \times T_i$, C is not CURB in $v_i(t_i)$. Since for some $j \in I$ $v_i(t_i) \subseteq v_j(t_j^*)$, where t_j^* is j's actual type and t_j^* leads to t_i, C is not CURB in $v_i(t_i^*)$. This is a contradiction. Therefore, the mutual CURB set is CURB in every view in the game. Then, from Lemma 2, since the mutual CURB set is a realizable CURB set, the set is a common CURB set. □

When relating mutual CURB notions with steady-state equilibrium notions, we can show the condition for converging a discovered game possessing some steady-state equilibrium. Moreover, we can show the condition for converging a game such that every equilibrium is a steady-state equilibrium. This is proved in the following theorems:

Proposition 2. In any simultaneous-move game with unawareness: If there exists a mutual CURB set such that the CURB set is CURB in every view in the game with unawareness, then there exists a cognitively stable generalized Nash equilibrium.

Proof. Suppose that some mutual CURB set is present in every view in a simultaneous-move game with unawareness. From Lemma 3, the mutual CURB set is a common CURB set. Then, from Proposition 1, a cognitively stable generalized Nash equilibrium exists. □

Theorem 2. Suppose a simultaneous-move game with unawareness, Γ, has a mutual CURB set. Then, a myopic discovery process converging to a discovered game possessing a cognitively stable generalized Nash equilibrium exists.

Proof. Suppose that a mutual CURB set, C, in Γ exists. From Lemma 2, C is a realizable CURB set. From Theorem 1, a myopic discovery process, P, converging to C exists. Since C is a common realizable CURB set from Lemma 3, from Proposition 2, a cognitively stable generalized Nash equilibrium exists. □

Corollary 2. Suppose that every realizable CURB set is a mutual CURB set in Γ. Then, every myopic discovery process converges to a discovered game possessing a cognitively stable generalized Nash equilibrium.

Note that the process considered in the present study starts from an arbitrary generalized strategy profile. Our convergence result holds even if the starting point is not necessarily a generalized Nash equilibrium.[19]

Next, let us consider a relationship with a Nash equilibrium in an objective game. Sasaki (2017) discussed the relationships between a cognitively stable generalized Nash equilibrium and a Nash equilibrium in an objective game in any simultaneous-move game with unawareness. The researcher showed the following proposition:

Proposition 3. Given any simultaneous-move game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, where t_i^* is i's actual type, then every cognitively stable generalized Nash equilibrium induces an objective outcome to be a Nash equilibrium in objective game G.

[19]Tada (2018) discussed a revision process in which players play a generalized Nash equilibrium in each round and conjectured that the process converges to a cognitively stable generalized Nash equilibrium, if there is any. However, the conjecture is wrong in assuming that players play a generalized Nash equilibrium in each round. This study yields a result in the same spirit as that, under another condition, in which players play myopic best responses.

Proof. Given any simultaneous-move game with unawareness Γ, suppose that for any $i \in I$, $A_i^{v_i(t_i^*)} = A_i$, where t_i^* is i's actual type. Suppose that the generalized strategy profile, s^*, is a cognitively stable generalized Nash equilibrium. From Remark 3, the objective outcome induced from a cognitively stable generalized Nash equilibrium is a Nash equilibrium of the realizable action set. Since every player is aware of their own actions in the objective game, the realizable action set is equivalent to the action set of the objective game. Hence, since every Nash equilibrium of the realizable action set is a Nash equilibrium in the objective game, the support of the objective outcome, induced by a cognitively stable generalized Nash equilibrium, is a Nash equilibrium in the objective game. □

From Theorem 1 and Proposition 3, we show the following theorem:

Theorem 3. In any simultaneous-move game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, where t_i^* is i's actual type, then any myopic discovery process converges to a discovered game such that any cognitively stable generalized Nash equilibrium induces an objective outcome to be a Nash equilibrium in objective game G.

5 Discussion

5.1 A CURB Block Game and Economy of Cognitive Costs

In our model, some myopic discovery processes do not converge to a discovered game possessing a common CURB set. Some of the players may be certain of the opponents ’ irrationality. However, by using the block game notion (e.g., Myerson and Weibull, 2015) of a smaller game than each player ’ s subjective game, the players can reconstruct a block game possessing a common CURB set from a discovered game that a myopic discovery process converges to and they can then ascertain each other's rationality.

Let us consider a case in which a discovered game possesses a realizable CURB set. When all the players implement a generalized strategy profile so that the objective outcome is in the realizable CURB set, if they are rational, they do not perform actions outside the realizable CURB set. Thus, all the actions in the complementary set of the realizable CURB set are redundant for them. Therefore, each player excludes the actions in the complementary set to economize the cognitive costs of the true structure of the game. If they economize these cognitive costs, their subjective games are the smallest games in which the action set is a common realizable CURB set. The following definition represents the "economy of knowledge" about a game's structure.

Definition 10. Given any game with unawareness, $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$, and any common realizable CURB set, $C \in V$ in Γ, $\Gamma' = (G, (T_i')_{i\in I}, (v_i')_{i\in I}, (b_i')_{i\in I})$ is an *economized game* by C in Γ, if for any $(i, t_i) \in I \times T_i$, there exists $t_i' \in T_i'$ so that

- $v_i'(t_i') = C$; and

- for any $(j, t_j) \in I_{-i} \times T_j$ with $b_j(t_j)(j) = t_j$, there exists $t'_j \in T'_j$ so that $b'_j(t'_j)(j) = t'_j$, and $v'_j(t'_j) = C$.

Then, $G^C = (I, C, u^C)$ is called a *realizable CURB block game* with C, where $u^C = (u_i)^C_{I \in I}$, and $u_i^C : C \to \mathbb{R}$ so that for any $a \in C$, $u_i^C(a) = u_i(a)$.

In Example 2, when Colin and David play s_1 in the initial game and s_1^2 in the next stage of the game, since the objective outcome induced by s_1^2 is (c_1, d_1), the realizable CURB block game with C^1 is $G^{C^1} = (I, C^1, (u_C^{C^1}, u_D^{C^1}))$. Thus, in the economized game, Γ^{C^1}, all the subjective games are G^{C^1}.

The following remark is obvious:

Remark 5. Every economized game, Γ' by C, in Γ has a cognitively stable generalized Nash equilibrium.

In Γ^{C^1} in Example 2, there exists a unique generalized Nash equilibrium such that Colin and David play c_1 and d_1 in each subjective game, respectively. Thus, from the definition of cognitive stability, the generalized Nash equilibrium is cognitively stable.

When Γ is a discovered game that a myopic discovery process converges to, every subjective game is a realizable CURB block game with a CURB set such that supports for players' actual actions converge in the process. Hence, a rationalizable discovery process is a search process for larger subjective games, whereas our myopic discovery process is a search process for common, smaller subjective games, that is, realizable CURB block games.

5.2 Adaptive Play

This study considers myopic agents and myopic play. In the model, each player responds best to the opponents' strategies in the previous stage of the game. However, a bounded agent may not be able to provide their best response to the opponents' strategies. Young (1993) provided adaptive play models that do not allow participants to provide their best responses to previous plays. This subsection discusses a generalization of the adaptive plays to simultaneous-move games with unawareness.

First, we provide a definition of adaptive plays in a discovered game.

Definition 11. Let Γ' be a discovered game from Γ and let $\varepsilon > 0$ be an error rate such that ε is sufficiently small. The generalized strategy profile, s', is an adaptive play in Γ', if for any $(i, t'_i) \in I \times T_i$, with probability $1 - \varepsilon$, player i chooses a best response to i's beliefs $\mu'_i(t'_i) \equiv (s^*_j(t^*_j))_{j \in I_{-i}}$ such that s^* is a generalized strategy profile played in Γ, and t^*_j is j's actual type in Γ; further, with probability ε, i chooses an action in $A_i^{v'_i(t'_i)}$ at random.

We propose a discovery process with an adaptive play as follows, based on Definition 11.

Definition 12. Any discovery process, $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \dots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \dots)$, is an *adaptive discovery process* if for any $\lambda \geq 2$, s^λ is an adaptive play profile at λ.

In a game without unawareness, Hurkens (1995) and Young (1998) used an adaptive play notion and showed the convergence to a minimal CURB set. Our proof of Theorem 1 focuses on only the realizable action set. Additionally, we can conjecture the following:

Conjecture 1. Given any simultaneous-move game with unawareness, in any adaptive play, The supports of the objective outcome induced by adaptive plays converge to a common minimal realizable CURB set.

Informal proof. Given any simultaneous-move game with unawareness and any adaptive discovery process, it is necessary to focus on only the realizable action set, as per Theorem 1. Based on Hurkens (1995) and Young (1998), adaptive plays converge to a minimal CURB set of the realizable action set. Then, the set is a common minimal realizable CURB set.[20] □

5.3 Limitations

Our research has the following limitations:

1. In a game with unawareness, in a generalized Nash equilibrium or under a rationalizable strategy, each player may be convinced that they are playing a higher-order subjective game or that the opponents are unaware of certain actions. However, in certain plays, each player may discover actions that they were unaware of, which may confirm that the players' subjective game was wrong. Here, the question arises, why was the player convinced that their higher-order subjective game was correct in the initial game with unawareness? In Example 2, two cognitively unstable generalized Nash equilibria, s_1 and s_2, exist in the initial game. This study does not yield any appropriate answer to the question as to which equilibrium Colin and David play when they both implement a generalized Nash equilibrium play.

   Our discovery process, and that of previous works, explains how to build each player's subjective game under unawareness; however, it does not explain how to do so in an initial game with unawareness. This issue is a subject for future research on games with unawareness.

2. Each player pays attention to the opponents' subjective games in the initial game with unawareness; however, they do not pay attention to them in a discovered game. We do not have any appropriate answer to why each player ceases to pay attention.

[20]The exact proofs are beyond the capabilities of the author and are omitted.

3. Models of discovery processes suppose that each player recognizes the opponents' plays and actions that they were previously unaware of. However, the assumptions may be too strict. For example, most children of preschool age would not be able to understand conversations among adults, or, at least, cannot have the same conversations. In further research, we aim to relax this assumption and reconstruct the models of discovery processes.

4. In Section 4, we showed that Schipper's (2021) result might not hold in our framework. Specifically, in our model, some rationalizable discovery process might not converge to any (simultaneous-move) game with unawareness possessing a self-confirming equilibrium. This paper proposes an open question as follows: what are the conditions for satisfying the result of Schipper (2021) in our framework?

5.4 Related Literature

Games with Unawareness

Pioneering works about games with unawareness include Feinberg (2021), Čopič Galeotti (2006), Ozbay (2007), Heifetz et al. (2013b), Halpern and Rêgo (2014), Rêgo and Halpern (2012), and Grant and Quiggin (2013). The majority of the literature has modeled extensive-form games with unawareness. Studies of normal-form models (or, more strictly, simultaneous-move models) are Čopič Galeotti (2006), Sasaki (2017), Perea (2018), and Kobayashi and Sasaki (2021). Meier and Schipper (2014) modeled Bayesian games with unawareness and discussed relationships with standard Bayesian games.

As mentioned in the text, there are two main solution concepts; one is equilibrium notions (Feinberg 2021, Čopič Galeotti 2006, Ozbay 2007, Halpern and Rêgo 2014, Rêgo and Halpern 2012, Grant and Quiggin 2013, and Meier and Schipper 2014). Halpern and Rêgo (2014) generalized a Nash equilibrium. Rêgo and Halpern (2012) and Grant and Quiggin (2013) discussed a sequential equilibrium in extensive-form games. Meier and Schipper (2014) proposed an unawareness perfect equilibrium. Sasaki 2017, Schipper 2021, and Kobayashi and Sasaki (2021) discussed refinements of equilibria notions and corrected beliefs on an equilibrium.

The other notion is the set-wise notion, more specifically, the rationalizability notion, used by Heifetz et al. (2013b, 2021), Perea (2018), and Guarino (2020). This notion is appropriate for game situations where beliefs have not yet been formed. In addition, since the rationalizable action set is supported by all mixed-strategy equilibria, the difficulty of disproving the probability distribution in mixed action equilibria can be eliminated.

Growing Awareness

A study of discovery processes entails an analysis of growing awareness or updating awareness. Karni and Vierø(2013, 2017) discussed decision-making under unawareness and proposed a reverse Bayesian model. As pointed out by Schipper (2013), an agent who is unaware of an event is different from an agent who assigns probability 0 to the event. This means that an unaware agent cannot assign a probability to an event that they are unaware of. Given such an event, Karni and Vierø's (2013, 2017) model discusses the methods to revise such agents' beliefs.

Galanis and Kotoronis (2021) modeled that all agents announce prices to each other and provided generalizations of the results of Genakoplos and Polemarchakis (1982) and Ostrovsky (2012). They supposed that updating awareness is minimal and a true state is never excluded. Traders eventually agree on the price of the security. Moreover, if the security is separable, traders agree on the correct price and aggregate information.

Unawareness in General

The first motivation of studies of unawareness is overcoming the No-Trade Theorem presented by Milgrom and Stokey (1982). Previous works of unawareness that address this issue have adopted two approaches. One was a non-partitional state space model, for example, Geankoplos (2021); and the other was an unawareness structure model, for example, Heifetz et al. (2006, 2013a), and Galanis (2013, 2018).

Interpretations of unawareness under the two approaches are different. The former corresponds to a lack of knowledge, that is, an agent does not know an event and does not know that they do not have that knowledge. The other strand of the literature on this approach includes Samet (1990) and Shin (1993). However, in (non-)partitional models, several assumptions lead to trivial unawareness, that is, an agent cannot be unaware of any event; see Modica and Rustichini (1994, 1999), Dekel et al. (1998), and Chen et al. (2012). The latter model is proposed to avoid this issue. Unawareness structures first formulate the family of state spaces and give different state spaces to different agents. Players' unawareness is represented by different subjective games. Other literature on unawareness structures or similar structures includes Li (2009) and Heinsalu (2012). In a recent study, Fukuda (2021) compared the two approaches.

CURB Notions

Basu and Weibull (1991) first introduced CURB notions to standard game models. CURB notions in dynamic models are discussed by Hurkens (1995), Young (1998), and Grandjean et al. (2017). Voorneveld et al. (2005) discussed the axiom and properties of minimal CURB sets. Pruzhansky (2003) showed that in extensive games with perfect information and finite horizons, there exists only one minimal CURB set. Benisch et al. (2010) provided algorithms for computing CURB sets. Asheim et al. (2016) discussed the epistemic robustness of CURB in epistemic models.

Appendix A

To discuss the ignorance, as a lack of conception, of a structure of a game, this paper uses a model of games with unawareness, but not Bayesian games. Bayesian games represent ignorance of actions as extremely low payoffs assigned to the actions that a player does not know (Harsanyi). In this formulation, rational players do not choose such actions. However, irrational players might choose those actions. Moreover, since each player's belief about the opponents' actions is a probability distribution on all the actions in the game, each player assigns some probability to all the actions even if modelers suppose that the player does not know some of the actions. As pointed out by Schipper (2013), since the unawareness of an event is different from a probability of 0 being assigned to the event, we cannot consider ignorance as a lack of conception. Hence, games with unawareness are appropriate for discussing the ignorance of actions as a lack of conception.

This paper uses a non-probabilistic type-based approach and models simultaneous-move games with unawareness. In simultaneous-move games with unawareness, type-based approaches are adapted by Meier and Schipper (2014) and Perea (2018). Meier and Schipper (2014) modeled Bayesian games with unawareness based on Heifetz et al.'s (2013a) probabilistic unawareness structures. By contrast, Perea's (2018) type spaces are similar to Harsanyi's (1967) type spaces. Note that crucial differences between Meier and Schipper (2014) and Perea (2018) exist. The former assumed that players' types are directly associated with views of the games, whereas the latter believed that although players' types associated with beliefs about views of the games, the types cannot be associated with the views themselves.

However, Perea (2018) had a problem similar to that encountered by Harsanyi (1967). He assumed probabilistic beliefs in his type-based approach. Thus, his probability distribution about the opponents' types, that is, the opponents' type spaces, is mutually known. Specifically, his framework cannot distinguish between unawareness and assigning some probability. This study uses Perea's (2018) style. Although we ignore this issue, we do not solve it. Can we model games with unawareness by using similar models to the Harsanyi–Perea style? This study's answer is that we can represent Harsanyi–Perea-style's games with unawareness by making an additional assumption.

Given standard game G, let V be the set of possible views. Then, a probabilistic version of simultaneous-move games with unawareness is defined as follows:

Definition 13. Given any standard game G, let $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$ be a probabilistic version of a simultaneous-move game with unawareness as follows: for each $i \in I$,

- T_i is a finite and non-empty set of i's type. This study assumes that one of the types is i's actual type, denoted by t_i^*.
- $v_i : T_i \to V$ is i's view function.

- $b_i : T_i \to \bigcup_{T'_{-i} \in \times_{j \in I_{-i}} (2^{T_j} \setminus \{\emptyset\})} \Delta(T'_{-i})$ is i's belief function, where for all $t_i \in T_i$, $T'_{-i} \in \times_{j \in I_{-i}} (2^{T_j} \setminus \{\emptyset\})$ exists such that $b_i(t_i) \in \Delta T'_{-i}$. Here, $\Delta(T'_{-i})$ is the set of probability measures over T'_{-i}, that is, $b_i(t_i)$ is a probability measure over some subset of T_{-i}. For each $t_i \in T_i$ and $t_{-i} = (t_j)_{j \in I_{-i}} \in T_{-i}$, $b_i(t_{-i}|t_i) \geq 0$ implies that $v_j(t_j) \subseteq v_i(t_i)$ for all $j \in I_{-i}$, where $b_i(t_{-i}|t_i)$ is the probability that $b_i(t_i)$ assigns to t_{-i}. Given any $t_i \in T_i$ and $t_j \in T_j$, denote by $b_i(t_j|t_i)$ to be the probability that $b_i(t_i)$ assigns to t_j.

A player is aware of a subset of the opponents' types if and only if for any opponents' type in the subset, the player assigns some probability to the type.

This definition is different from Perea's (2018). His model assigned some probability to every type on the type set. By contrast, any probability might not be assigned to some type. Simply, in our model, we can represent the unawareness of types. However, under this definition, we cannot distinguish between the unawareness of the opponents' types and probability 0 assigned to the types because we do not define relationships among types. To distinguish unawareness from probability 0, we need an additional assumption. Given $(i, t_i) \in I \times T_i$, let $t_{i_1}, t_{i_2}, \ldots, t_{i_h}, \ldots$ be a sequence of types, where $t_{i_1} = t_i$ and for any $h \geq 2$, $t_{i_h} = t_j$, where $(j, t_j) \in I \times T_j$ and $b_i(t_j|t_{i_{h-1}}) \geq 0$.

Assumption 1. Given Γ, for any $(i, t_i) \in I \times T_i$, any t_{i_h} and t_{i_k} $(h \leq k)$, where $i_1 = i_h = i_k = i$, and the two opponents' type subsets $T_{-i_h}, T_{-i_k} \subseteq T_{-i}$, if $b_{i_h}(t_{i_h}) \in \Delta(T_{-i_h})$ and $b_{i_k}(t_{i_k}) \in \Delta(T_{-i_k})$, then $T_{-i_k} \subseteq T_{-i_h}$ and for any $t_{-i} \in T_{-i_h} \setminus T_{-i_k}$, any probability must not be assigned to t_{-i}.

This assumption specifies the relationships among the types. When player i is given two types, if one t_i leads to the other t'_i, then the opponents' type set T_{-i} at t_i is a superset of the set T'_{-i} at t'_i. Then, when T'_{-i} is a proper subset of T_{-i}, any probability is assigned to every type $t''_i \in T_{-i} \setminus T'_{-i}$ at t'_i. Hence, under this assumption, we can distinguish between unawareness and probability 0.

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