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Unawareness of Actions and Myopic Discovery Process in Simultaneous-Move Games with Unawareness

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Unawareness of Actions and Myopic Discovery Process in Simultaneous-Move Games with Unawareness *

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Abstract

This study discusses the convergence to a solution in simultaneous-move games with unawareness. Since games with unawareness assume players' unawareness of actions, in some plays, players might revise their subjective views when they observe opponents' play, which they were unaware of. Recently, a model of discovery processes was proposed to analyze the revision process of subjective views. Pioneering work in discovery processes shows that the revision process that converges to subjective views is not adopted by all players. However, previous studies do not show that the players' play converges to a particular solution. This paper introduces the closedness under rational behavior (CURB) notion, that is, a set-valued solution concept, to simultaneous-move games with unawareness, and shows that the play of myopic players, who best respond to the previous play, converges to the generalized CURB set. Moreover, we propose that CURB block games, which consist only of a realizable CURB set, represent the economy of knowledge of games' structures.
JEL classification: C70; C72; D80; D83
Keywords: Game Theory; Unawareness; Discovery of Actions; Growing Awareness; Adaptive Play; Closedness under Rational Behavior

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1 Introduction

This study introduces closedness under rational behavior (CURB) notions to simultaneous-move games with unawareness, models a myopic discovery process in which every player best responds to the immediately preceding plays, and shows that myopic best responses converge to realizable CURB sets in any myopic discovery process. Games with an unawareness model lack conception of the games' situations, for example, the players' set, the actions' set, and state spaces. This means that certain players may not be able to perceive parts of the sets of players, actions, and states.[1]

Since players may be unaware of actions, some of them might revise their view when they observe opponents' play, which they were previously unaware of. Schipper (2021) discusses such revision process, namely, the discovery process, whereby players observe actions that they might have previously been unaware of, and revise their subjective views by adding the new actions to their subjective views.

A discovery process is different from a learning process. Learning is an update process of probability distributions. However, as shown by Schipper (2013), unawareness of an event is not the same as assigning the event to zero probability. Any event that the agents are unaware of is not included in the subjective state space. Hence, the agents cannot assign any probability to the event. If such an event occurs, their subjective state space must be expanded. In games with unawareness, the set of actions must also be increased when actions that players were unaware of are played. Therefore, learning cannot be applied to games with unawareness. Discovery processes are alternative models of learning processes that were proposed to avoid this very issue.

Schipper (2021) shows that when all players implement rationalizable strategies in each stage of the game, any rationalizable discovery process converges to form larger subjective views that do not need further revision[2]. However, he does not show that players' play converges toward a particular solution, although he does show that such subjective views possess the same rationalizable self-confirming equilibrium, that is, such revision processes do not converge to the equilibrium.

This paper tries to show that, through the discovery process in simultaneous-move games with unawareness, all the players' play converges to a particular solution. Our model assumes that every player best responds to the opponents' immediately preceding plays in each stage of the game. We shall call the play a "myopic play. " A myopic play is a type of Markov play. Unlike rationalizable plays, myopic plays may not be rationalizable. However, in the real world, most people are not rational and do not mutually believe in each other's rationality. Therefore, they do not play based on rationalizable strategies. Hence, an assumption of myopic play is suitable for real world analysis.

[1]Pioneering works about games with unawareness include Feinberg (2021), Čopič Galeotti (2006), Ozbay (2007), Heifetz et al. (2013b), Halpern and Rêgo (2014), Rêgo and Halpern (2012), and Grant and Quiggin (2013). For a historical survey, see Schipper (2014).

[2]Note that larger subjective games might not be common.

We consider the solution concepts in games with unawareness before we examine a convergence to a solution concept in the myopic discovery processes. There are two main notions of solution concepts in studies on games with unawareness. The first is the equilibrium notion, which has been used in Feinberg (2021), Čopič Galeotti (2006), Ozbay (2007), Halpern and Rêgo (2014), Rêgo and Halpern (2012), Grant and Quiggin (2013), and Meier and Schipper (2014). An equilibrium is interpreted as an equilibrium in beliefs. However, unlike standard game models, games with unawareness assume unawareness of actions. Therefore, some equilibria may not be steady state equilibria, which means that no rational agent deviates from the equilibrium because every equilibrium includes some action that some player is unaware of. To avoid this issue, several previous studies provide a steady state notion refined equilibrium, for example, cognitive stability notion (Sasaki, 2017) and self-confirming notion (Schipper, 2021).

The second notion is the set-wise notion, more specifically, the rationalizability notion, used by Heifetz et al. (2013b, 2021), Perea (2018), and Guarino (2020). This notion is appropriate for game situations where beliefs have not yet been formed. In addition, since the rationalizable action set is supported by all mixed-strategy equilibria, the difficulty of disproving the probability distribution in mixed action equilibria can be eliminated.

However, both the steady state and rationalizability notions have issues. Learning processes are not appropriate for games with unawareness because there could be a growing awareness of actions in the subjective game of each player in each stage of the game. Therefore, we cannot explain belief formation for steady state equilibria. Moreover, if the equilibrium is a mixed strategy, players might not be able to disconfirm the distribution of such a strategy. A rationalizability notion avoids these issues. However, the set of rationalizable actions is much larger than the support for any given equilibrium. Moreover, although the notion might be appropriate for one-shot games or games that players face at first, it is difficult to discuss the steady state using rationalizability notions.

To avoid these issues, this study introduces the notion of CURB to simultaneous-move games with unawareness. The CURB notion, proposed by Basu and Weibull (1991), is a refinement of the strict Nash equilibria and rationalizability. A CURB notion can be interpreted as a steady state, as done by Myerson and Weibull (2015), and is a set-wise notion. Furthermore, any CURB set is a subset of the rationalizable action set. Hence, CURB notions seem appropriate for games with unawareness.

Certain actions of certain players may not belong to a player's available choices, because the agent is unaware of the action. Hence, she or he cannot play the action. In other words, there exist some actions that do not belong to the subjective view. In such a scenario, a CURB set of the objective view is not included in the subjective view. Therefore, we must generalize the CURB notion to simultaneous-move games with unawareness. This paper proposes a generalized CURB set, named a "realizable CURB set."

By using a model of myopic discovery processes and realizable CURB notions, we can show that, in any myopic discovery process all the players' play converges to a realizable CURB set. More precisely, a subset of play supports converges to a realizable CURB set. Our discovery process converges to both larger subjective views and a particular solution. In some realizable CURB sets that players' play converges to, actions outside the CURB set are redundant because rational players do not deviate from the CURB set. Therefore, delineating such actions is not a problem. We can construct a game without such actions, and name the game without the redundant actions as the "CURB block game.'' A block is a Cartesian product of non-empty subsets of players' actions, and a block game is a game constructed by a block. In a simultaneous-move game with unawareness, a myopic discovery process converging to the realizable CURB set might not be a CURB in the objective game. Hence, in some CURB sets, a player who can observe a CURB set of the objective game, might think that the opponent is irrational. By contrast, in a CURB block game, the realizable CURB set must be CURB. Therefore, we can interpret constructing the CURB block game as economy of knowledge of a game's structure, and players' cognitive costs for excluding cognitive dissonance. Moreover, we can say that a myopic discovery process proposes a candidate for games in which each player economizes knowledge about actions.

In constructing models of unawareness, we used type-based approaches.[3] Although our type-based model seems similar to Harsanyi (1967), games with unawareness differ from incomplete information games, specifically, standard Bayesian games. In standard Bayesian models, a player's unawareness of actions is represented with extremely low pay-offs. In formulation, the rational participant does not perform the actions. However, if participants are irrational in models with an assumption, they might perform the actions. It means that our representation is a contradiction. In contrast to Bayesian games, unawareness of actions in games with unawareness is represented by using a (semi-) sub-lattice of game structures. The models with unawareness provide different subjective games or views of the games to each player, and assume that each player is unaware of opponents' subjective games. Thus, they cannot perform actions outside the action set in their own subjective games.

The rest of this paper is organized as follows. Section 2 provides preliminaries. Section 3 develops a generalization of CURB, and Section 4 formulates a discovery process, called the myopic discovery process. Section 5 shows the relationships with a cognitively stable, generalized Nash equilibrium. Section 6 discusses a block game, and economy of knowledge of a game's structure. Finally, Section 7 provides the discussion and the conclusion.

[3]Type-based models were first introduced to games with unawareness by Meier and Schipper (2014). Another pioneering model was proposed by Perea (2018). However, both the models have different definitions. Meier and Schipper (2014) assume that players' types are directly associated with views of the games, whereas Perea (2018) believes that although players' types are associated with beliefs about views of the games, the types cannot be associated with the views themselves. While our formulation is similar to Perea (2018), our assumptions differ. Perea (2018) does not assume fixed belief hierarchies, but rather, probabilistic beliefs. By contrast, we assume belief hierarchies. Our formulation is a simplified version of Kobayashi et al. (2021) although their models are epistemic, and they assume probabilistic beliefs.

2 Preliminaries

2.1 Simultaneous-Move Games with Unawareness

This section provides a definition of simultaneous-move games with unawareness that are type-based models, a generalized Nash equilibrium, and rationalizability. Let $G = (I, A, u)$ be a standard finite simultaneous-move game. I is a finite set of players, and $I_{-i} = I \setminus \{i\}$. $A = \times_{i \in I} A_i$, where A_i is the non-empty finite set of i's actions, and each element on the set is $a_i \in A_i$. Let $A_{-i} = \times_{j \in I_{-i}} A_j$. $u = (u_i)_{i \in I}$, where $u_i : A \to \mathbb{R}$ is i's utility function. Denote i's mixed action on A_i by $m_i \in M(A_i)$, where $M(A_i)$ is the set of i's mixed actions, and a mixed action profile on A by $m = (m_i)_{i \in I} \in M(A) = \times_{i \in I} M(A_i)$. We denote i's expected utility for $m \in M(A)$ by $Eu_i(m)$.

First, we define simultaneous-move games with unawareness.[4] For any standard simultaneous-move game G, let $V = \times_{i \in I}(2^{A_i} \setminus \{\emptyset\})$ be the set of possible *views* of G. That is, the set of a Cartesian product of a non-empty action subset. Like most previous works, this study assumes that the set of players is commonly known, and that each player's utility for each action profile is the same among all the possible views. Let $v \in V$ be a (possible) view or a block,[5] and A_i^v be the set of i's actions in $v = \times_{j \in I} A_j^v$. Let $A_{-i}^v = \times_{j \in I_{-i}} A_j^v$. Here, when a player i is given v, i is aware of $a \in v$, and unaware of $a \in A \setminus v$. For any $v, v' \in V$, v is contained in v' if A_i^v is a subset of $A_i^{v'}$ for any $i \in I$, that is, $A_i^v \subseteq A_i^{v'}$. Let $M(A_i^v) = \{m_i \in M(A_i) | \Sigma_{a_i \in A_i^v} m_i(a_i) = 1\}$. Given any $\delta, \delta' \in \bigcup_{v \in V} \bigcup_{X \in 2^I \setminus \{\emptyset\}} M(\times_{i \in X} A_i^v)$, $\delta \equiv \delta'$ means that δ and δ' have the same supports and probabilities. Therefore, we can say that δ and δ' are equivalent.

Let $\Gamma = (G, (T_i)_{i \in I}, (v_i)_{i \in I}, (b_i)_{i \in I})$ be a simultaneous-move game with unawareness as follows: for each $i \in I$,

- T_i is a finite and non-empty set of i's type, one of which is their actual type t_i^*.
- $v_i : T_i \to V$ is i's view function.
- $b_i : T_i \to T_{-i}$ is i's belief function, where $T_{-i} = \times_{j \in I \setminus \{i\}} T_j$. If $b_i(t_i) = (t_j)_{j \in I \setminus \{i\}}$, then for each $j \in I \setminus \{i\}$, $v_j(t_j)$ must be contained in $v_i(t_i)$. Simply put, we do not assume probabilistic beliefs.

Let us call G an objective game (in Γ). An objective game can be interpreted as the "true game" in Γ.[6] i's type t_i describes their view about the game, and

[4] Our definition is essentially the same as Kobayashi et al. (2021). Note that Kobayashi et al. (2021) allow probabilistic beliefs, whereas our model assumes fixed beliefs. Our definitions are similar to that of Perea (2018). Note that there are two major differences. First, the Perea's (2018) model does not fix belief hierarchies on views. We assume that the "actual type" of the players is given. Second, Perea (2018) deals with probabilistic beliefs on awareness, whereas our players always have point beliefs on their opponents' awareness, as is often assumed in the literature on games with unawareness.

[5] A block is a Cartesian product of non-empty subsets of actions.

[6] The term "objective game" was used by Halpern and Rêgo (2014). Feinberg (2021) refers

belief about the opponents' types. At t_i, $v_i(t_i) = v$ means that i is aware of v, and unaware of $A \setminus v$, while $b_i(t_i) = (t_j)_{j \in I \setminus \{i\}}$ means that at t_i, i believes that the others' types are $(t_j)_{j \in I \setminus \{i\}}$, and that each j's view is $v_j(t_j)$. Given $(i, t_i) \in I \times T_i$, we denote a sequence $t_{i_1}, t_{i_2}, \dots, t_{i_h}, \dots$, where $t_{i_1} = t_i$, and for any $h \geq 2$, $t_{i_h} = b_{i_{h-1}}(t_{i_{h-1}})(i_h)$. We say that t_i leads to t_j if, and only if, there exists a subsequence $t_{i_1}, \dots, t_{i_h}$ such that $t_{i_1} = t_i$ and $t_{i_h} = t_j$. Here, we suppose $\bigcup_{i \in I} T_i = \bigcup_{i \in I} \{t^*_{i_h}\}_{h \geq 1; t^*_{i_h} = t^*_i}$.

The set of each player's actual play $A_i^{v_i(t_i^*)}$ may be a proper subset of i's *full* action set A_i. In such a scenario, they cannot play $a_i \in A_i \setminus A_i^{v_i(t_i^*)}$. In other words, a player's realized actions exclude the non-realized actions. Let $\times_{i \in I} A_i^{v_i(t_i^*)}$ be the *realizable action set*. Some players may not perceive the realizable action set.

2.2 Solution Concepts

In this subsection, we consider two solution concepts as equilibrium notions (Feinberg, 2021; Rêgo and Halpern, 2012; Halpern and Rêgo, 2014; Grant and Quiggin, 2013; Meier and Schipper, 2014) and set-wise notions (Heifetz et al., 2013, 2021; Perea, 2018).

Let us define the generalized strategies. For any $i \in I$, let $s_i : T_i \to \bigcup_{t_i \in T_i} M(A_i^{v_i(t_i)})$ with $s_i(t_i) \in M(A_i^{v_i(t_i)})$ for all $t_i \in T_i$. Then, given t_i, $s_i(t_i) \in M(A_i^{v_i(t_i)})$ is i's local action at t_i. We denote i's generalized strategy by $s_i = (s_i(t_i))_{t_i \in T_i}$, and a generalized strategy profile by $s = (s_i)_{i \in I}$. In the generalized strategy profile s, each player i's actual play is $m_i \in M(A_i)$ with $m_i \equiv s_i(t_i^*)$, and the profile is called the objective outcome induced from s. Let us define the belief system $\mu = (\mu_i)_{i \in I}$, where for any $(i, t_i) \in I \times T_i$, $\mu_i(t_i) \in M(A_{-i}^{v_i(t_i)})$, and $\mu_i = (\mu_i(t_i))_{t_i \in T_i}$. This study allows correlated beliefs.

2.2.1 Equilibrium Notion

First, let us consider equilibrium notions. Halpern and Rêgo (2014) introduce an equilibrium notion in games with unawareness as a generalized Nash equilibrium – in a simultaneous-move game with unawareness Γ, s^* is a generalized Nash equilibrium if there exists belief system μ such that for any $(i, t_i) \in I \times T_i$,

1. $s_i^*(t_i) \in \arg\max_{x \in M(A_i^{v_i(t_i)})} Eu_i(x, \mu_i(t_i))$, and
2. $\mu_i(t_i) \equiv (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}}$.

As they point out, a generalized Nash equilibrium is best interpreted as "an equilibrium in beliefs" (Halpern and Rêgo, 2014: 50). However, as pointed out by Schipper (2014), there exists some generalized Nash equilibrium, which consists of wrong beliefs. In that case, each player would revise their own beliefs

to such a game as the "modeler's normal-form game," and Perea (2018) calls it the "base game." In this context, we follow the study by Halpern and Rêgo (2014).

about a game's structure and the opponents' play, and may not play the same generalized Nash equilibrium.

To avoid this scenario, previous literature has discussed two approaches: equilibrium restricted by steady state notions and rationalizability. In this subsection, we focus only on steady state equilibrium notions.

First, let us consider the generalized Nash equilibrium with stable belief hierarchies, and the cognitively-stable generalized Nash equilibrium proposed by Sasaki (2017):

- The generalized Nash equilibrium s^* has stable belief hierarchies if belief system μ satisfies that for any $(i, t_i) \in I \times T_i$, $\mu_i(t_i) \equiv (s_j^*(t_j^*))_{j \in I_{-i}}$.
- The generalized Nash equilibrium s^* is cognitively stable if for any $(i, t_i) \in I \times T_i$, $s_i^*(t_i) \equiv s_i^*(t_i^*)$.

The former notion means that every player's belief about the opponents' plays is correct, while the latter notion means that each player's decision making regarding their arbitrary type is equivalent to their actual play.

Sasaki (2017) expresses the following:

Remark 1. Generalized strategy profile s^* is a generalized Nash equilibrium with stable belief hierarchies if, and only if, s^* is a cognitively-stable generalized Nash equilibrium.

Such a generalized Nash equilibrium is called a cognitively-stable generalized Nash equilibrium in this paper. A cognitively-stable generalized Nash equilibrium has the following property.

Remark 2. For any simultaneous-move game with unawareness Γ, let s^* be a cognitively-stable generalized Nash equilibrium. Then, the objective outcome $m^* \equiv (s_i^*(t_i^*))_{i \in I}$ is a Nash equilibrium on the realizable action set.

Proof. Suppose that s^* is a cognitively-stable generalized Nash equilibrium, that is, for any $(i, t_i) \in I \times T_i$, $s_i^*(t_i) \equiv m_i^*$. Then, $m^* \equiv m \in M(v_j(t_j))$ for any $(j, t_j) \in I \times T_j$. Assume that $m' \in M(\times_{i \in I} A_i^{v_i(t_i^*)})$ with $m' \equiv m^*$ is not a Nash equilibrium on the realizable action set. In other words, there exists $i \in I$ such that

$$m_i^* \equiv m_i' \notin \arg \max_{x \in A_i^{v_i(t_i^*)}} Eu_i(x, m_{-i}').$$

Then, for any $(i, t_i) \in I \times T_i$,

$$m_i^* \equiv m_i' \equiv m_i'' \notin \arg \max_{x \in A_i^{v_i(t_i)}} Eu_i(x, \mu_i(t_i)),$$

where $\mu_i \equiv m_{-i}' \equiv m_{-i}^*$. However, since s^* is a cognitively-stable generalized Nash equilibrium, this is a contradiction. Therefore, m' is a Nash equilibrium on the realizable action set. □

Next, let us consider a self-confirming equilibrium proposed by Fudenberg and Levine (1993). Schipper (2021) generalizes a rationalizable self-confirming equilibrium to include extensive-form games with unawareness. Kobayashi et al. (2021) focus on only simultaneous-move games with unawareness, and discuss a rationalizable self-confirming equilibrium by using epistemic models. We discuss the k-self-confirming equilibrium, which means that all the players in the k-th order mutually believe that all the players' beliefs are correct. The following definition of self-confirming equilibria is based on Kobayashi et al. (2021). s^* is a k-self-confirming equilibrium if there exists belief system μ such that for any $h = 1, \dots, k+1$ and $i_h \in I$, where $t_{i_1} = t_i^*$,

1. $s^*_{i_h} \in \arg\max_{x \in M(A_i^{v_{i_h}(t_{i_h})})} Eu_i(x, \mu_i(t_{i_h}))$, and
2. $\mu_{i_h} \equiv (s^*_{i_{h+1}}(t_{i_{h+1}}))_{i_{h+1} \in I_{-i_h}}$.

2.2.2 Set-Wise Notion

Next, we consider the set-wise notions. Rationalizability is the only set-wise notion discussed in previous literature on games with unawareness.

Hence, this subsection discusses only rationalizability.[7] Rationalizability is proposed by Bernheim (1984) and Pearce (1984). Heifetz et al. (2013) generalize Pearce's extensive-form rationalizability to games with unawareness. This notion is one of the approaches to avoid the issue of equilibrium notions in general. This study shows that rationalizability in simultaneous-move games with unawareness, based on Kobayashi et al. (2021), is as follows: In a simultaneous-move game with unawareness Γ, for any $t \in \times_{i \in I} T_i$, let

$$S^0(t) = \left\{ \alpha \in \bigcup_{v \in V} \times_{i \in I} M(A_i^v) \;\middle|\; \begin{array}{l} \text{for all } i \in I: \\ (1) \alpha_{-i} \equiv \alpha'_{-i} \in \times_{j \in I_{-i}} M(A_j^{v_j(b_i(t_i)(j))}); \text{and} \\ (2) \alpha_i \equiv s_i(t_i) \arg\max_{x \in M(A_i^{v_i(t_i)})} Eu_i(x, \alpha'_{-i}) \end{array} \right\}$$

, and define $S^n(t)$ for $n = 1, 2, \dots$ inductively by

$$S^n(t) = S^{n-1}(t) \bigcap_{i \in I} S^{n-1}(t_i, b_i(t_i)).$$

Then, $S(t) = \bigcap_{n=0} S^n(t)$ is the rationalizable mixed action set at t. Here, let us define the rationalizable pure action set as $R(t) = \bigcup_{\alpha \in S(t)} \mathrm{supp}(\alpha)$.

Kobayashi et al. (2021) propose a k-rationalizable self-confirming equilibrium as follows: In a simultaneous-move game with unawareness Γ, s^* is a k-rationalizable self-confirming equilibrium if it is a k-self-confirming equilibrium, and $(s_i^*(t_i^*))_{i \in I} \equiv \alpha \in S((t_i)_{i \in I})$, where for any $i \in I$, there exist $j \in I$ and $h = 1, \dots, k+1$ such that $t_i = t_{j_h}$, where $t_{j_1} = t_j^*$. As shown by Kobayashi et al. (2021), in the k-rationalizable self-confirming equilibrium, the k-th order mutual beliefs of rationality are correct. Thus, some player's $k+1$-th order belief might not be correct.

[7] Section 3 proposes a CURB notion as another set-wise notion.

3 Closedness under Rational Behavior

Before modeling a discovery process, we consider the strategy subsets closed under rational behavior (CURB sets), as proposed by Basu and Weibull (1991). In this section, we generalize a CURB notion for a simultaneous-move games with unawareness. Previous literature provides two approaches to avoid the steady state notion from not being applied to equilibrium notions, in general, in games with unawareness: a steady state equilibrium notion and a rationalizable notion. However, both notions have issues.

1. As pointed out by Schipper (2014, 2021) and Heifetz et al. (2021), we cannot apply a learning process to formulate beliefs about the opponents' plays in games with unawareness. Moreover, a steady state equilibrium might have mixed actions. Thus, if such a steady state equilibrium is unique and mixed, players might not be able to disconfirm a distribution of mixed actions.

2. Since rationalizable actions are support every mixed equilibrium, the solution is too large. Although rationalizable notions might be appropriate for one-shot games or games that players face at first, it is difficult to discuss the steady state using rationalizability.

To avoid the above issues, we use the other notion, CURB, which has characterizations of both steady state equilibrium notions and rationalizable notions. A CURB notion refines a strict Nash equilibrium and rationalizability. This notion avoids the difficulty of disconfirming mixed actions as rationalizability, and has a steady state notion as a steady state equilibrium. Hence, the CURB notion seems appropriate.

Although Basu and Weibull (1991) first define a CURB notion on a standard game, this paper defines it on each view. Given any standard simultaneous-move game G, any possible view $\hat{v} \in V$ and mixed action profile $m \in M(\hat{v})$. Let

$$\beta_i^{\hat{v}}(m_{-i}) = \{a_i \in A_i | a_i \in \text{supp}(m_i) \text{ be such that } m_i \in \arg \max_{x \in M(A_i^{\hat{v}})} Eu_i(x, m_{-i})\}$$

be the set of i's pure-action best responses to their belief about $m_{-i} \in M(A_{-i}^{\hat{v}})$. For any $v \subseteq \hat{v}$, let

$$\beta_i^{\hat{v}}(A_{-i}^{v}) = \bigcup_{m_{-i} \in M(A_{-i}^{\hat{v}}) : m_{-i} \equiv m'_{-i} \in M(A_{-i}^{v})} \beta_i^{\hat{v}}(m_{-i})$$

be the set of i's optimal actions under beliefs in $M(v)$, and let $\beta^{\hat{v}}(v) = \times_{i \in I} \beta_i^{\hat{v}}(A_{-i}^{v})$.

Then, CURB is defined as follows.

Definition 1. Give a standard simultaneous-move game G and $\hat{v} \in V$. $C \subseteq \hat{v}$ is a CURB set on $\hat{v}$ if $\beta^{\hat{v}}(C) \subseteq C$. C is a minimal CURB set on $\hat{v}$ if C is CURB on $\hat{v}$, and every proper subset of C is not CURB on $\hat{v}$.

Basu and Weibull (1991) show that every standard game has a minimal CURB set.

Remark 3. Given any standard game, every possible view has a minimal CURB set.

In standard games, only one CURB set is necessary for the full action set. However, since a given possible view for each player may not be consistent with the full action set in games with unawareness, realized CURB sets are different for standard games, and games with unawareness. Hence, we must distinguish CURB notions between the two models. Let us discuss the CURB notion under unawareness. We define a CURB set on the realizable action set, called a realizable CURB set.

Definition 2. Given a simultaneous-move game with unawareness Γ, let $v^* = \times_{i \in I} A_i^{v_i(t_i^*)}$ be the realizable action set. $C \in V$ is a *realizable CURB set* if $C \subseteq v^*$ and $\beta^{v^*}(C) \subseteq C$. C is a *minimal realizable CURB set* if it is CURB on v^*, and every proper subset of C is not CURB on v^*.

Realizable CURB notions have the following property.

Lemma 1. Every simultaneous-move game with unawareness Γ has a minimal realizable CURB set; it is non-empty.

Proof. Let us construct a game $G' = (N, A', u')$ such that the following assumptions hold.

- N is common in Γ.
- $A' = \times_{i \in I} A_i^{v_i(t_i^*)}$.
- For any $i \in I$, $u'_i : A' \to \mathbb{R}$ such that $u_i(a) = u'_i(a)$ for any $a \in A'$.

Following Basu and Weibull (1991), there must be a (minimal) CURB set $C \subseteq A'$ in G. In other words, there exists a set of each player's pure-action best response, $\beta'(C)$, such that $\beta'(C) \subseteq C$ in G. Since $\beta'(C)$ is defined on $A' = \times_{i \in I} A_i^{v_i(t_i^*)}$, C is a minimal realizable CURB set. □

Given Γ, some realizable CURB set $C \in V$ may be $C \subsetneq v_i(t_i)$ for any $(i, t_i) \in I \times T_i$. However, the set is not CURB in $v_i(t_i)$ at some t_i. Given a realizable CURB set, we distinguish between a case wherein the realizable CURB set is CURB in every $v_i(t_i)$ for any $(i, t_i) \in I \times T_i$, and one where it is not as follows.

Definition 3. In a simultaneous-move game with unawareness Γ, $C \in V$ is a *common realizable CURB set* if for any $(i, t_i) \in I \times T_i$, C is a (minimal) realizable CURB set and $C \subseteq v_i(t_i)$. C is a *common (minimal) CURB set* if it is a common (minimal) realizable CURB set, and for any i, $\beta_i^{v^*}(A_{-i}^C) \subseteq A_i^C$. [8]

[8] In one of the previous versions of this paper, Tada (2020), notes that a common CURB set means only a CURB set on the full action set. In contrast, this paper generalizes this notion by focusing on a realizable action set.

From the definition 3, it is obvious that for any $(i, t_i) \in I \times T_i$, t_i is *not* i's actual type, and C is CURB on $v_i(t_i)$.

Common CURB notions have the following property.

Proposition 1. Any simultaneous-move game with unawareness possessing a common CURB set has a cognitively-stable generalized Nash equilibrium.

Proof. Assume that any simultaneous-move game with unawareness has a common CURB set $C \in V$. That is, for any $(i, t_i) \in I \times T_i$, C is CURB on $v_i(t_i)$. Then, following Basu and Weibull (1991), there exists a Nash equilibrium on $v_i(t_i)$, $m^* \in M(v_i(t_i))$, satisfying $m^* \equiv m' \in M(C)$. Suppose that m' is not a Nash equilibrium on v. In other words, there exists $(i, a_i) \in I \times A_i^{v_i(t_i)}$, such that $Eu_i(a_i, m'_{-i}) > Eu_i(m')$. However, since C is a common CURB set, this is a contradiction. Therefore, m' is a Nash equilibrium on C. Thus, since $(i, t_i) \in I \times T_i$ is arbitrary, $m^* \equiv m'$ is a Nash equilibrium on $v_i(t_i)$. Thus, s^* with $s_i^*(t_i) \equiv m_i^*$ for any $i \in I$ and $t_i \in T_i$ is a cognitively-stable generalized Nash equilibrium. □

Proposition 1 suggests one of the conditions for the existence of a cognitively-stable generalized Nash equilibrium in any game with unawareness. The contraposition is that if there does not exist any cognitively-stable generalized Nash equilibrium, then there does not exist any common CURB set. This means that if some players cannot perceive any CURB set in the realizable action set, then the players are surprised about a realized play because the player's belief about the opponents' plays is wrong. Our proposition suggests a condition for all players' stable plays, that is, rational players do not deviate from a specific play.[9]

The following corollary is obvious from the above proof of proposition 1: .

Corollary 1. Given any simultaneous-move game with unawareness, a common CURB set includes the support of some cognitively-stable generalized Nash equilibrium.

Remark 4. Given any simultaneous-move game with unawareness Γ, any common realizable CURB set is a support for the objective outcome, induced from a 0-rationalizable self-confirming equilibrium. Given the actual type profile, every common CURB set is included in the rationalizable action set, and the maximum CURB set is equal to the rationalizable action set.

Example 1. Let us consider that two players, Alice (A) and Bob (B), face the following objective game:

$v^O =$

A / B	b_1	b_2	b_3
a_1	3, 3	0, 5	0, 0
a_2	5, 0	1, 1	0, 0
a_3	0, 0	0, 0	2, 2

.

[9] I thank Masakazu Fukuzumi for this suggestion.

Here, if Alice is unaware of her own action a_2, then, her view is as follows:

$v^A =$

A / B	b_1	b_2	b_3
a_1	3, 3	0, 5	0, 0
a_a	0, 0	0, 0	2, 2

,

If Bob is unaware of his own action b_2, then, his view is as follows:

$v^B =$

A / B	b_1	b_3
a_1	3, 3	0, 0
a_2	5, 0	0, 0
a_3	0, 0	2, 2

.

Let us suppose that Alice believes that Bob's view is the same as hers, that is, they both believe that they hold the same view v^A, while Bob believes that Alice's view is the same as his, that is, they both believe that they hold the same view v^B.

Here, we formulate this game (with unawareness) $\Gamma = (v^O, (T_A, T_B), (v_A, v_B), (b_A, b_B))$ as follows:

$T_A = \{t_A^*, t_A\}$, and $T_B = \{t_B^*, t_B\}$;
given t_A^*, $v_A(t_A^*) = v_A$, and $b_A(t_A^*) = t_B$;
given t_A, $v_A(t_A) = v_A$, and $b_A(t_A) = t_B^*$;
given t_B^*, $v_B(t_B^*) = v_B$, and $b_B(t_B^*) = t_A$; and
given t_B, $v_B(t_B) = v_B$, and $b_B(t_B) = t_A^*$.

Since Alice's realizable actions are a_1 and a_3, and Bob's realizable actions are b_1 and b_3, the realizable action set is the following table:

$v^R =$

A / B	b_1	b_3
a_1	3, 3	0, 0
a_3	0, 0	2, 2

.

First, let us consider generalized (pure) Nash equilibria in the game. There exist four generalized pure Nash equilibria:

$s_1 = ([s_A(t_A^*) = a_1, s_A(t_A) = a_2], [s_B(t_B^*) = b_1, s_B(t_B) = b_2])$;
$s_2 = ([s_A(t_A^*) = a_1, s_A(t_A) = a_3], [s_B(t_B^*) = b_3, s_B(t_B) = b_2])$;
$s_3 = ([s_A(t_A^*) = a_3, s_A(t_A) = a_2], [s_B(t_B^*) = b_1, s_B(t_B) = b_3])$;, and
$s_4 = ([s_A(t_A^*) = a_3, s_A(t_A) = a_3], [s_B(t_B^*) = b_3, s_B(t_B) = b_3])$.

Then, the cognitively-stable generalized Nash equilibrium is only s_4.

Second, let us consider self-confirming (pure) equilibria in the game. There exist a 0-self-confirming equilibrium, and ∞-self-confirming equilibrium. The 0-self-confirming equilibrium is only

$$s_5 = ([s_A(t_A^*) = a_1, s_A(t_A) = a_1], [s_B(t_B^*) = b_1, s_B(t_B) = b_1]).$$

Note that the 0-self-confirming equilibrium is not the k-self-confirming equilibrium ($k > 1$) because in $k+1$, Alice's a_1 does not respond best to Bob's b_1, and Bob's b_1 does not respond best to Alice's a_1.

In contrast, the ∞-self-confirming equilibrium is only s_4. When comparing a cognitively-stable generalized Nash equilibrium and ∞-self-confirming equilibrium, both the equilibria are the same.[10]

Third, let us consider rationalizability. Given three tuples $t^1 = (t_A^*, t_B^*)$, $t_2 = (t_A^*, t_B)$, and $t_3 = (t_A, t_B)$, the pure rationalizable actions are as follows:

$R(t^1) = \{a_1, a_3\} \times \{b_2, b_3\}$;
$R(t^2) = \{a_2, a_3\} \times \{b_1, b_3\}$; and
$R(t^3) = \{a_1, a_3\} \times \{b_1, b_3\}$.

Then, it is obvious that 0-self-confirming equilibrium s_5 is a 0-rationalizable self-confirming equilibrium, and that ∞-self-confirming equilibrium s_4 is a ∞-rationalizable self-confirming equilibrium. Kobayashi et al. (2021) show that the ∞-rationalizable self-confirming equilibrium is a cognitively-stable generalized Nash equilibrium, and vice versa.

Finally, we consider a (realizable) CURB notion. There exist three CURB sets on the realizable action set, that is, three realizable CURB sets:

$C^1 = \{a_1\} \times \{b_1\}$;
$C^2 = \{a_3\} \times \{b_3\}$; and
$C^3 = \{a_1, a_3\} \times \{b_1, b_3\}$.

Here, C^3 is a maximum CURB set. Since, $C^1, C^2, C^3 \subseteq v^A$ and $C^1, C^2, C^3 \subseteq v^B$, every realizable CURB set is a common realizable CURB set. Moreover, C^2 is the only unique common CURB set because the common CURB set is CURB on v^A and v^B.

Let us compare the CURB notion with the other notions. First, we compare the CURB notion and equilibrium notion. By proposition 1, any common CURB set includes a support for the objective outcome, induced from cognitively-stable generalized Nash equilibrium. Since the ∞-rationalizable self-confirming equilibrium and cognitively-stable generalized Nash equilibrium are the same, C^2 includes a support for the objective outcome (a_3, b_3), induced from a cognitively-stable generalized Nash equilibrium and ∞-rationalizable self-confirming equilibrium, s^4. In contrast, a common realizable CURB set C^1, that is not a common CURB set, is a support for the objective outcome (a_1, b_1), induced from 0-rationalizable self-confirming equilibrium s^5. In C^1 or s^5, each player is rational but their 1st order belief is irrational play[11]. Moreover, they might be certain about the opponent's irrationality.

[10] This finding is similar to that of Kobayashi et al. (2021). Kobayashi et al. (2021) show that any cognitively-stable generalized Nash equilibrium is a ∞-*rationalizable* self-confirming equilibrium, and vice versa.

[11] Rational play means that players maximize their utilities.

Next, let us consider a relationship with rationalizability. Given a type t^1, $R(t^1) = C^3$. As shown by Basu and Weibull (1991), a rationalizable action set is equivalent to a maximum CURB set. In contrast, minimal CURB sets, C^1 and C^2, are subsets of the rationalizable action set, that is, any minimal CURB set is a refined notion of rationalizability. □

As shown in the above example, a realizable CURB notion is related to other solution concepts, that is, the CURB notion has similar characterizations as the other notions.

4 Myopic Discovery Process

Standard game models study a convergence to a minimal CURB set by using a learning model or an adaptation model, for example, Hurkens (1995) and Young (1998). However, in games with unawareness, some players might not notice the opponents' actions. Since the players must revise their belief about the game structure, and the opponents' views, a learning model cannot be applied to games with unawareness, as pointed out by Heifetz et al. (2021). To avoid this issue between games with unawareness and learning models, Schipper (2021) proposes a model of *discovery processes*.

A discovery process represents an update process by which each player revises their own belief about the game's structure, and the opponents' play. Schipper (2021) is the first to introduce a discovery process in extensive-form games with unawareness, which is based on Heifetz et al. (2013). This study models a discovery process in simultaneous-move games with unawareness based on Perea (2018). Although our definition, at first glance, may seem different from that of Schipper (2021), both are essentially the same.

Definition 4. $\Gamma' = (G, (T'_i)_{i\in I}, (v'_i)_{i\in I}, (b'_i)_{i\in I})$ is a discovered game with $s = (s_i)_{i\in I}$ in $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$ if for any $(i, t_i) \in I \times T_i$, there exists $t'_i \in T'_i$ such that

1. $v'_i(t'_i) = \times_{j\in I}[A_j^{v_i(t_i)} \cup \text{supp}(s_j(t^*_j))]$, where t^*_j is j's actual type in Γ; and
2. for any $(j, t_j) \in I_{-i} \times T_j$ satisfying $b_i(t_i)(j) = t_j$, there exists t'_j such that $b'_i(t'_i)(j) = t'_j$ and $v'_j(t'_j) = \times_{k\in I}[A_k^{v_j(t_j)} \cup \text{supp}(s_k(t^*_k))]$, where t^*_k is k's actual type in Γ.

Note that some Γ, Γ' may be $T \not\subseteq T'$ and $T' \not\subseteq T$, or $T \cap T' = \emptyset$.

Example 2. Consider the following objective game played by Colin (C) and David (D):

$v^0 =$

C / D	d_1	d_2	d_3
c_1	3, 3	0, 5	0, -1
c_2	5, 0	1, 1	1, 0
c_3	-1, 0	0, 1	2, 2

;

and two possible views as follows:

$v^1 =$

C / D	d_2	d_3
c_1	0, 5	0, -1
c_3	0, 1	2, 2

and $v^2 =$

C / D	d_1	d_3
c_1	3, 3	0, -1
c_2	5, 0	1, 0

.

Let us formulate the game with unawareness $\Gamma = (v^0, (T_C, T_D), (v_C, v_D), (b_C, b_D))$ as follows:

$T_C = \{t_C^*, t_C\}$, and $T_D = \{t_D^*, t_D\}$;
given t_C^*, $v_C(t_C^*) = v_C$, and $b_C(t_C^*) = t_D$;
given t_C, $v_C(t_C) = v_C$, and $b_C(t_C) = t_D^*$;
given t_D^*, $v_D(t_D^*) = v_D$, and $b_D(t_D^*) = t_C$; and
given t_D, $v_D(t_D) = v_D$, and $b_D(t_D) = t_C^*$.

Here, suppose that Colin and David play a generalized strategy profile:

$$s = ([s_C(t_C^*) = c_1, s_C(t_C) = c_2)], [s_D(t_D^*) = d_2, s_D(t_D) = d_3]).$$

The objective outcome is (c_1, d_2) induced by s.

Let Γ' be the discovered game with s in Γ. Then, each player's type set in Γ' is $T'_C = \{t'^*_C, t'_C\}$, and $T'_D = \{t'^*_D, t'_D\}$, where

$$b'_C(t'^*_C) = t'_D;$$

$$b'_C(t'_C) = t'^*_D;$$

$$b'_D(t'^*_D) = t'_C;$$

$$b'_D(t'_D) = t'^*_C;$$

$\hat{v}^1 = v'_C(t'^*_C) = v'_D(t'_D) =$

C / D	d_2	d_3
c_1	0, 5	0, -1
c_3	0, 1	2, 2

;

and

$\hat{v}^2 = v'_C(t'_C) = v'_D(t'^*_D) =$

C / D	d_1	d_2	d_3
c_1	3, 3	0, 5	0, -1
c_2	5, 0	1, 1	1, 0

.

□

A discovery process is defined as follows.

Definition 5. A discovery process $P = (\langle \Gamma^1, s^0\rangle, \langle \Gamma^2, s^1\rangle, \dots, \langle \Gamma^\lambda, s^{\lambda-1}\rangle, \dots)$, is defined as follows:

- for any λ, $\Gamma^\lambda = (G, (T_i^\lambda)_{i\in I}, (v_i^\lambda)_{i\in I}, (b_i^\lambda)_{i\in I})$,
- when $\lambda = 0$ and $s^0 = \phi$, while for any $\lambda \geq 1$, s^λ is a played generalized strategy profile in Γ^λ, and
- for any $\lambda \geq 2$, Γ^λ is a discovered game with $s^{\lambda-1}$ in $\Gamma^{\lambda-1}$.

Let us call Γ^1 the initial game with unawareness (in P).

By definition 4, definition 5 implicitly assumes perfect recall. If we exclude the assumption, some player may forget some action at λ even if they are aware of the action at $\lambda - 1$.

This study assumes that every player implements a pure action. Meanwhile, standard game models might assume that every player implements and observes a mixed action. In contrast, in games with unawareness, it does not seem appropriate that every player implements, and observes a mixed action because under unawareness, players cannot observe the frequency of their opponents' actions at each stage of the game during the course of any discovery process.[12]

Schipper (2021) models rationalizable discovery processes. We formulate rationalizable discovery processes based on Perea (2018) as follows.

Definition 6. A discovery process $P = (\langle \Gamma^1, s^0\rangle, \langle \Gamma^2, s^1\rangle, \dots, \langle \Gamma^\lambda, s^{\lambda-1}\rangle, \dots)$ is a rationalizable discovery process if for any λ and $t \in \times_{i\in I} T_i$, $(s_i^\lambda(t_i))_{i\in I} \in S^\lambda(t)$, where $S^\lambda(t)$ is the rationalizable mixed action set on Γ^λ.

In a discovery process, cautious players might carefully revise their beliefs about the game, the opponents' plays and rationalities, and pay-off uncertainty. However, in the real world, The agents must pay a higher cost for revising such beliefs, and implementing rationalizable strategies. If players are myopic, they do not pay a high cost for revising their beliefs. This section explains the myopic discovery process in which every player responds best to the opponents' previous plays.

First, we define a strategy of myopic play in a discovered game as follows.

Definition 7. Let Γ' be a discovered game from Γ. s' is a *myopic best response* in Γ if there exists the belief system, $\mu' = (\mu_i')_{i\in I}$ such that for any $(i, t_i') \in I \times T_i'$,

1. $s_i'(t_i') \in \arg\max_{x\in M(A_i^{v_i'(t_i')})} Eu_i(x, \mu_i'(t_i'))$; and
2. $\mu_i'(t_i') \equiv (s_j^*(t_j^*))_{j\in I_{-i}}$, where s^* is played in Γ, and for any $j \in I_{-i}$, $t_j^* \in T_j$ is j's actual type in Γ.

Next, we provide a myopic discovery process.

[12] I thank an anonymous referee for pointing this out.

Definition 8. Any discovery process $P = (\langle\Gamma^1, s^0\rangle, \langle\Gamma^2, s^1\rangle, \dots, \langle\Gamma^\lambda, s^{\lambda-1}\rangle, \dots)$, is a myopic discovery process if for any $\lambda \geq 2$, s^λ is a myopic best response at λ.

Some important questions arise here. Does any myopic discovery process converge to a discovered game in which the players cannot revise each other's views? What solution concept does the discovered game have? The following theorem answers the above questions.

Theorem 1. Given any simultaneous-move game with unawareness Γ, every myopic discovery process, P, converges to a discovered game, possessing a common realizable CURB set. Thus, a subset of the supports of all the agents' myopic best responses converges to common realizable CURB set.

Proof. Since we consider a myopic discovery process, it is necessary to focus only on the realizable action set. For any objective outcome in the initial game $m \in M(\times_{i\in I} A_i^{v_i(t_i^*)})$, let $\beta^\lambda(m)$ be an objective outcome induced by a myopic best response on the realizable action set, and it is defined as follows: $\beta^0(m) = \text{supp}(m)$, $\beta^1(m) = \beta' \circ \beta^0(m)$, $\beta^2(m) = \beta' \circ \beta^1(m)$, $\dots$, $\beta^\lambda(m) = \beta' \circ \beta^{\lambda-1}(m)$, $\dots$. Suppose that for any CURB set on the realizable action set $C \subseteq \times_{i\in I} A_i^{v_i(t_i^*)}$, and natural number λ, $\beta^\lambda(m) \not\subseteq C$. As pointed out by Basu and Weibull (1991), since the set of the rationalizable strategy profile on $\times_{i\in I} A_i^{v_i(t_i^*)}$, $R \subseteq \times_{i\in I} A_i^{v_i(t_i^*)}$, is CURB,[13] $\beta^\lambda(m) \not\subseteq R$ for any λ, this is a contradiction. Therefore, there exists a realizable CURB set, C, and natural number, n, such that $\beta^n(m) \subseteq C$. Suppose that there exists $\lambda \geq n$, satisfying $\beta^\lambda(m) \not\subseteq C$. In other words, $\beta' \circ \cdots \circ \beta' \circ \beta^n(m) \not\subseteq C$. However, since C is CURB, that is, $\beta(v') \subseteq C$ for any $v' \subseteq C$ with $\emptyset \neq A_i^{v'} \subseteq A_i^C$, this is a contradiction. Therefore, $\beta^\lambda(m) \subseteq C$ for any $\lambda \geq n$. Since, $\beta^\lambda(m)$ supports an objective outcome induced by a myopic best response at λ, and C is a realizable CURB set, the support for the objective outcome is included in the realizable CURB set. By definition 4, since CURB, C, is common, C is a common realizable CURB set. $\square$

It is known that many intuitively appealing adjustment processes eventually settle down in a minimal CURB set (cf., Hurkens, 1995; Young, 1998). Theorem 1 adds to the previous literature, highlighting the importance of the CURB set. However, the process therein converges to a general CURB set, And, not necessarily, a "minimal" one, such as in Hurkens (1995) and Young (1998).

Example 2 (Continued.) Let Γ be an initial game, that is, a game at $\lambda = 1$. Then, the realizable action set is as follows:

$v^R =$

C / D	d_1	d_3
c_1	3, 3	0, -1
c_3	-1, 0	2, 2

.

[13] Specifically, R is a maximum tight CURB set. An action profile set, $C \in V$, is a tight CURB set if $\beta(C) = C$.

v^R has three CURB sets, $C^1 = \{c_1\} \times \{d_1\}$, $C^2 = \{c_3\} \times \{d_3\}$, and $C^3 = \{c_1, c_3\} \times \{d_1, d_3\}$. Here, C^2 is a mutual CURB set.

In Γ, there exist two generalized Nash equilibria:

$$s_1 = ([s_C(t_C^*) = c_1, s_C(t_C) = c_2], [s_D(t_D^*) = d_1, s_D(t_D) = d_2]); \text{ and}$$
$$s_2 = ([s_C(t_C^*) = c_3, s_C(t_C) = c_2], [s_D(t_D^*) = d_3, s_D(t_D) = d_3]).$$

Obviously, both equilibria are cognitively unstable.

First, let us focus on the former equilibrium, s_1. The objective outcome is (c_1, d_1). Since Colin is unaware of d_1, he is surprised and revises his view as follows:

$v^{1'} =$

C / D	d_1	d_2	d_3
c_1	3, 3	0, 5	0, -1
c_3	-1, 0	0, 1	2, 2

.

Then, at $\lambda = 2$, the discovered game $\Gamma' = (G, (T'_C, T'_D), (v'_C, v'_D), (b'_C, b'_D))$, where

$T'_C = \{t_C^{2*}, t_C^2\}$, and $T'_D = \{t_D^{2*}, t_D^2\}$;

$v'_C(t_C^{2*}) = v^{1'}$, and $b'_C(t_C^{2*}) = t_D^2$;

$v'_C(t_C^2) = v^2$, and $b'_C(t_C^2) = t_D^{2*}$;

$v'_D(t_D^{2*}) = v^2$, and $b'_D(t_D^{2*}) = t_C^2$; and

$v'_D(t_D^2) = v^{1'}$, and $b'_D(t_D^2) = t_C^{2*}$.

At $\lambda = 2$, when they play myopic best response, the generalized strategy profile is

$$s_1^2 = ([s_C^2(t_C^{2*}) = c_1, s_C^2(t_C^2) = c_1], [s_D^2(t_D^{2*}) = d_1, s_D^2(t_D^2) = d_1]).$$

Both players do not discover the opponents' actual plays. Hence, the next stage game, at $\lambda = 3$, is the same as Γ'. In Γ', the play s_1^2 is not a generalized Nash equilibrium, and the objective outcome is (c_1, d_1). The support of the objective outcome, $\{c_1\} \times \{d_1\}$, is a subset of a realizable CURB set, C^1.

Next, let us focus on the latter generalized Nash equilibrium s_2. The objective outcome is (c_3, d_3). Since David is unaware of c_3, he is surprised and revises his view as follows:

$v^{2'} =$

C / D	d_1	d_3
c_1	3, 3	0, -1
c_2	5, 0	1, 0
c_3	-1, 0	2, 2

.

Then, at $\lambda = 2'$, $\Gamma'' = (G, (T''_C, T''_D), (v''_C, v''_D), (b''_C, b''_D))$, where

$T''_C = \{t_C^{2'*}, t_C^{2'}\}$, and $T''_D = \{t_D^{2'*}, t_D^{2'}\}$;

$v''_C(t_C^{2'*}) = v^1$, and $b''_C(t_C^{2'*}) = t_D^{2'}$;

$v''_C(t_C^{2'}) = v^{2'}$, and $b''_C(t_C^{2'}) = t_D^{2'*}$;

$v''_D(t_D^{2'*}) = v^{2'}$, and $b''_D(t_D^{2'*}) = t_C^{2'}$; and

$v''_D(t_D^{2'}) = v^1$, and $b''_D(t_D^{2'}) = t_C^{2'*}$.

At $\lambda = 2'$, when they play myopic best response, the generalized strategy profile is

$$s_2^{2'} = ([s_C^{2'}(t_C^{2'*}) = c_3, s_C^{2'}(t_C^{2'}) = c_3], [s_D^{2'}(t_D^{2'*}) = d_3, s_D^{2'}(t_D^{2'}) = d_3]).$$

Both players do not discover the opponents' actual play. Hence, the next stage game, at $\lambda = 3'$, is the same as Γ''. In Γ'', the play, the generalized strategy profile is a cognitively-stable generalized Nash equilibrium, and a support of the objective outcome, $\{c_3\} \times \{d_3\}$, is a subset of a common CURB set, C^2 in Γ''. □

5 Relationships with Cognitively-Stable Generalized Nash Equilibria

The previous section focused only on CURB notions. However, steady state notions are not only CURB notions, but also equilibrium notions.[14] Below, we consider the convergence to discovered games in a discovery process, possessing a steady state equilibrium, by associating it with CURB notions.

First, we provide a mutual CURB notion that each player's actual view has the same CURB set.

Definition 9. In any simultaneous-move game with unawareness Γ, $C \in V$ is a *mutual CURB set* if for any $i \in I$, C is a non-empty CURB set in $v_i(t_i^*)$.

A mutual CURB notion has the following property.

Lemma 2. Every mutual CURB set is a realizable CURB set.

Proof. Given any mutual CURB set, $C \in V$, $C \subseteq v_i(t_i^*)$ for any $i \in I$. Suppose that C is not a realizable CURB set, that is, there exists some i, such that

[14] Kobayashi et al. (2021) consider a rationalizable self-confirming equilibrium. They try to apply steady state notions to rationalizability by associating it with equilibrium notions.

$\beta_i^*(A_{-i}^C) \not\subseteq A_i^C$ in the realizable action set. Since the realizable action set is defined by $\times_{i\in I} A_i^{v_i(t_i^*)}$, $\beta_i^{v_i(t_i^*)}(A_{-i}^C) \not\subseteq A_i^C$. This contradicts that C is a mutual CURB set. Hence, C is a realizable CURB set. □

Lemma 3. In a simultaneous-move game with unawareness, if a mutual CURB set is present in every view, then there exists a common CURB set.

Proof. Suppose that a mutual CURB set, C, is present in every view in a simultaneous-move game with unawareness. Suppose that for some $(i, t_i) \in I \times T_i$, C is not CURB in $v_i(t_i)$. Since for some $j \in I$ $v_i(t_i) \subseteq v_j(t_j^*)$, where t_j^* is j's actual type and t_j^* leads to t_i, C is not CURB in $v_i(t_i^*)$. This is a contradiction. Therefore, the mutual CURB set is the CURB in every view in the game. Then, by lemma **??**, since the mutual CURB set is a realizable CURB set, the set is a common CURB set. □

When relating mutual CURB notions with steady state equilibrium notions, we can show the condition for converging a discovered game possessing some steady state equilibrium. Moreover, we can show the condition for converging a game, such that every equilibrium is a steady state equilibrium. This is proved in the following theorems.

Proposition 2. In any simultaneous-move game with unawareness, if there exists a mutual CURB set, such that the CURB set is CURB in every view in the game with unawareness, then there exists a cognitively-stable generalized Nash equilibrium.

Proof. Suppose that some mutual CURB set is present in every view in a simultaneous-move game with unawareness. By lemma 3, the mutual CURB set is a common CURB set. Then, by proposition 1, there exists a cognitively-stable generalized Nash equilibrium. □

Theorem 2. Suppose a simultaneous-move game with unawareness, Γ, has a mutual CURB set. Then, there exists a myopic discovery process converging to a discovered game possessing a cognitively-stable generalized Nash equilibrium.

Proof. Suppose that there exists a mutual CURB set, C, in Γ. By lemma 2, C is a realizable CURB set. By theorem 1, there exists a myopic discovery process, P, converging to C. Since the C is a common realizable CURB set by lemma 3, by proposition 2, there exists a cognitively-stable generalized Nash equilibrium. □

Corollary 2. Suppose that every realizable CURB set is a mutual CURB set in Γ. Then, every myopic discovery process converges to a discovered game possessing a cognitively-stable generalized Nash equilibrium.

Note that the process considered in the present study starts from an arbitrary generalized strategy profile. Our convergence result holds even if the starting point is not, necessarily, a generalized Nash equilibrium.[15]

Next, let us consider a relationship with a Nash equilibrium in an objective game. Sasaki (2017) discusses relationships between a cognitively-stable generalized Nash equilibrium and a Nash equilibrium in an objective game in any simultaneous-move game with unawareness. The researcher shows the following proposition.

Proposition 3. Given any simultaneous-move game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, where t_i^* is i's actual type, then every cognitively-stable generalized Nash equilibrium induces an objective outcome to be a Nash equilibrium in an objective game G.

Proof. Given any simultaneous-move game with unawareness Γ, suppose that for any $i \in I$, $A_i^{v_i(t_i^*)} = A_i$, where t_i^* is i's actual type. Suppose that generalized strategy profile, s^*, is a cognitively-stable generalized Nash equilibrium. By remark 2, the objective outcome induced from a cognitively-stable generalized Nash equilibrium is a Nash equilibrium of the realizable action set. Since every player is aware of their own actions in the objective game, the realizable action set is equivalent to the action set of the objective game. Hence, since every Nash equilibrium of the realizable action set is a Nash equilibrium in the objective game, the support of the objective outcome, induced by a cognitively-stable generalized Nash equilibrium, is a Nash equilibrium in the objective game. □

By theorem 1 and proposition 3, we show the following theorem.

Theorem 3. In any simultaneous-move game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, where t_i^* is i's actual type, then any myopic discovery process converges to a discovered game, such that any cognitively-stable generalized Nash equilibrium induces an objective outcome to be a Nash equilibrium in an objective game G.

6 A CURB Block Game and Economy of Cognitive Costs

Discovery processes represent convergence of all the agents' subjective game, where players need not revise their subjective game. In other words, these processes are aimed toward at searching for a game's true structure. Schipper (2021) states that a rationalizable discovery process is a process where each

[15] Tada (2018) discusses a revision process in which players play a generalized Nash equilibrium in each round, and conjectures that the process converges to a cognitively-stable generalized Nash equilibrium, if there is any. However, the conjecture is wrong in assuming that players play a generalized Nash equilibrium in each round. This study yields a result in the same spirit as that, under another condition, in which players play myopic best responses.

player searches for their larger subjective game, and shows that every rationalizable discovery process converges to a discovered game possessing a rationalizable self-confirming equilibrium. Our myopic discovery process converges to a discovered game possessing a common realizable CURB set, and in the myopic discovery process, supports of plays converge to the common realizable CURB set.

In our model, as shown in Section 3, some myopic discovery processes do not converge to a discovered game possessing a common CURB set. Some players may be certain of the opponents' irrationality. However, by using the block game notion (e.g., Myerson and Weibull, 2015) of a smaller game than each player's subjective game, players can reconstruct a block game possessing a common CURB set from a discovered game that a myopic discovery process converges to, and they can be certain about each other's rationality.

Let us consider a case where a discovered game possesses a realizable CURB set. When all the players implement a generalized strategy profile so that the objective outcome is in the realizable CURB set, if they are rational, they do not perform actions outside the realizable CURB set. Thus, all the actions in the complement set of the realizable CURB set are redundant for them. Therefore, each player excludes the actions in the complement set to economize cognitive costs of the true structure of the game. If they economize the cognitive costs, their subjective games are the smallest games in which the action set is a common realizable CURB set. The following definition represents the "economy of knowledge" about a game's structure.

Definition 10. Given any game with unawareness, $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$, and any common realizable CURB set, $C \in V$ in Γ, $\Gamma' = (G, (T'_i)_{i\in I}, (v'_i)_{i\in I}, (b'_i)_{i\in I})$ is an *economized game* by C in Γ, if for any $(i, t_i) \in I \times T_i$, there exists $t'_i \in T'_i$ so that

- $v'_i(t'_i) = C$; and
- for any $(j, t_j) \in I_{-i} \times T_j$ with $b_j(t_j)(j) = t_j$, there exists $t'_j \in T'_j$ so that $b'_j(t'_j)(j) = t'_j$, and $v'_j(t'_j) = C$.

Then, $G^C = (I, C, u^C)$ is called a *realizable CURB block game* with C, where $u^C = (u_i)^C_{I\in I}$, and $u^C_i : C \to \mathbb{R}$ so that for any $a \in C$, $u^C_i(a) = u_i(a)$.

In the example 2, when Colin and David play s_1 in the initial game, and s^2_1 in the next stage of the game, since the objective outcome induced by s^2_1 is (c_1, d_1), the realizable CURB block game with C^1 is $G^{C^1} = (I, C^1, (u^{C^1}_C, u^{C^1}_D))$. Thus, in the economized game, Γ^{C^1}, all subjective games are G^{C^1}.

The following remark is obvious.

Remark 5. Every economized game, Γ' by C, in Γ has a cognitively-stable generalized Nash equilibrium.

In Γ^{C^1} in the example 2, there exists a unique generalized Nash equilibrium such that Colin and David play c_1 and d_1 in each subjective game, respectively.

Thus, by the definition of cognitive stability, the generalized Nash equilibrium is cognitively stable.

When Γ is a discovered game that a myopic discovery process converges to, every subjective game is a realizable CURB block game with a CURB set such that supports of players' actual actions converge in the process. Hence, a rationalizable discovery process is a search process for larger subjective games, whereas our myopic discovery process is a search process for common, smaller subjective games, that is, realizable CURB block games.

7 Discussion

7.1 Adaptive Play

This study considers myopic agents and myopic play. In the model, each player responds best to the opponents' strategies in the previous stage of the game. However, a bounded agent may not be able to provide their best response to the opponents' strategies. Young (1993) provides adaptive play models that allow participants to not provide best responses to previous plays. This subsection discusses a generalization of the adaptive plays to simultaneous-move games with unawareness.

First, we provide a definition of adaptive plays in a discovered game.

Definition 11. Let Γ' be a discovered game from Γ, and let $\varepsilon > 0$ be an error rate such that ε is sufficiently small. Generalized strategy profile, s', is an adaptive play in Γ', if for any $(i, t'_i) \in I \times T_i$, with probability $1 - \varepsilon$, player i chooses a best response to i's beliefs $\mu'_i(t'_i) \equiv (s^*_j(t^*_j))_{j \in I_{-i}}$ such that s^* is a generalized strategy profile played in Γ, and t^*_j is j's actual type in Γ; further, with probability ε, i chooses an action in $A_i^{v'_i(t'_i)}$ at random.

We propose a discovery process with an adaptive play as follows, based on the definition 11.

Definition 12. Any discovery process, $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \ldots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \ldots)$, is an *adaptive discovery process*, if for any $\lambda \geq 2$, s^λ is an adaptive play profile at λ.

In a game without unawareness, Hurkens (1995) and Young (1998) use an adaptive play notion, and show a convergence to a minimal CURB set. Our proof of the theorem 1 focuses on only the realizable action set. Additionally, we can conjecture the following.

Conjecture 1. Given any simultaneous-move game with unawareness, in any adaptive play, supports of the objective outcome, induced by adaptive plays, converge to a common minimal realizable CURB set.

Although we omit a proof of this conjecture, we show an informal proof as follows.

The informal proof. Given any simultaneous-move game with unawareness, and any adaptive discovery process.

1. It is necessary to focus on only the realizable action set as per the theorem 1.

2. Based on Hurkens (1995) and Young (1998), adaptive plays converge to a minimal CURB set of the realizable action set.

3. Then, the set is a common minimal realizable CURB set. □

□

7.2 Growing Awareness

A study of discovery processes entails an analysis of growing awareness or updating awareness. Karni and Vierø(2013, 2017) discuss decision-making under unawareness, and propose a reverse Bayesian model. As pointed out by Schipper (2013), an agent who is unaware of an event is different from an agent who assigns probability zero to the event. This means that an unaware agent cannot assign a probability to an event that they are unaware of. Given such an event, Karni and Vierø's (2013, 2017) model discusses the methods to revise such agents' beliefs.

Galanis and Kotoronis (2021) provide generalizations of the results of Genakoplos and Polemarchakis (1982) and Ostrovsky (2012). They suppose that updating awareness is minimal, and a true state is never excluded. Traders eventually agree on the price of the security. Moreover, if the security is separable, traders agree on the correct price and there is information aggregation.

7.3 Limitations

Our research has the following limitations.

1. In a game with unawareness, in a generalized Nash equilibrium or under a rationalizable strategy, each player may be convinced that they are playing a higher-order subjective game, or that the opponents are unaware of certain actions. However, in certain plays, each player may discover actions that they were unaware of, which may confirm that the players' subjective game was wrong. Here, the question arises, why was the player convinced that their higher-order subjective game was correct in the initial game with unawareness? In the example 2, there exist two cognitively-unstable generalized Nash equilibria, s_1 and s_2, in the initial game. This study does not yield any appropriate answer to the question as to which equilibrium Colin and David play when they both implement a generalized Nash equilibrium play.

   Our discovery process, and that of the previous works, explain how to build each player's subjective game under unawareness, however, they do not explain how to do so in an initial game with unawareness. This issue is an subject for future research on games with unawareness.

2. Each player pays attention to the opponents' subjective games in the initial game with unawareness, however, they do not pay attention to them in a discovered game. We do not have any appropriate answer to why each player ceases to pay attention.

3. Models of discovery processes suppose that each player recognizes the opponents' plays and actions that they were previously unaware of. However, the assumptions may be too strict. For example, most children of preschool age would not be able to understand conversations among adults, or, at least, cannot have the same conversations. In further research, we aim to relax this assumption, and reconstruct the models of discovery processes.

7.4 Related Literature

Unawareness in General

The first motivation of studies on unawareness is overcoming the No-Trade Theorem presented by Milgrom and Stokey (1982). Previous works about unawareness, that address this issue, had two approaches. One was a non-partitional state space model, for example, Geankoplos (2021); and the other was an unawareness structure model, for example, Heifetz et al. (2006, 2013a), and Galanis (2013, 2018).

Interpretations about unawareness under the two approaches are different. The former corresponds to a lack of knowledge, that is, an agent does not know an event, and does not know that he or she does not have that knowledge. The other literature on this approach includes Samet (1990), Shin (1993), and Ewerhart (2001). However, in (non-)partitional models, several assumptions lead to trivial unawareness, that is, an agent cannot be unaware of any event; see Modica and Rustichini (1994, 1999), Dekel et al. (1998), and Chen et al. (2012). The latter model is proposed to avoid this issue. Unawareness structures first formulate the family of state spaces, and give different state spaces to different agents. Players' unawareness is represented by different subjective games. Other literature on unawareness structures or similar structures includes Li (2009) and Heinsalu (2012). In a recent study, Fukuda (2021) compared the two approaches.

CURB Notions

Basu and Weibull (1991) first introduce CURB notions to standard game models. CURB notions in dynamic models are discussed by Hurkens (1995), Young (1998), and Grandjean et al. (2017). Voorneveld et al. (2005) discuss the axiom and properties of minimal CURB sets. Pruzhansky (2003) shows that in extensive games with perfect information and finite horizon, there exists only one minimal CURB set. Benisch et al. (2010) provide algorithms for computing CURB sets. Asheim et al. (2016) discuss epistemic robustness of CURB in epistemic models.

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