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# Note: AU Introspection and Symmetry under Non-Trivial Unawareness

Yoshihiko Tada
Chuo University
Economic Reserch Institute,
Associate Researcher

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INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

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Yoshihiko Tada
Graduate School of Economics, Chuo University,
742-1, Higashinakano, Hachioji-shi, Tokyo 192-0393, Japan
yoshihiko.tada.4@gmail.com

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**Abstract**

This note discusses the relationship between AU Introspection and Symmetry for non-trivial unawareness without Negative Introspection using a set-theoretical approach in standard state-space models. Previous studies have explored the equivalence between Negative Introspection and AU Introspection, or the equivalence between Negative Introspection and Symmetry, by assuming Necessitation of the knowledge operator. As a corollary, AU Introspection is equivalent to Symmetry. However, no studies have explored the relationship between AU Introspection and Symmetry without Necessitation. Therefore, we explore this issue and show that Necessitation is not necessary to prove the equivalence of AU Introspection and Symmetry in standard state-space models. Instead, we show that both AU Introspection and Symmetry hold without crashing with non-trivial unawareness.
*Keywords*: Unawareness; Necessitation; Negative Introspection; Symmetry; AU Introspection; KU Introspection
*JEL classification*: C70; C72; D80; D83

# 1 Introduction

This note discusses the relationship between AU Introspection and Symmetry for the non-triviality of the unawareness operator without Negative Introspection using a set-theoretical approach on standard state-space models. Previous studies investigating unawareness in standard state-space models explore the relationship between Negative Introspection and Symmetry (e.g., Modica and Rustichini, 1994) or between Negative Introspection and AU Introspection (e.g., Dekel et al., 1998; Chen et al., 2012) while assuming Necessitation. Modica and Rustichini (1994) show the equivalence between Negative Introspection and Symmetry, that is, Negative Introspection is not consistent with Symmetry (Theorem 1). Dekel et al. (1998) show that, if state-space models satisfy Necessitation, Plausibility, KU Introspection, and AU Introspection, then there is no event that some agent is unaware of. Chen et al. (2012) investigate the relationship between Negative Introspection and AU Introspection. They show that Negative Introspection is equivalent to AU Introspection when assuming Necessitation (Theorem 3). From their results, it is evident that AU Introspection is equivalent to Symmetry. In fact, Chen et al. (2012) show a generalization of Deket et al. (2012) and the aforementioned equivalence (Theorem 4). However, the two properties are discussed with Necessitation. There is no study discussing the relationship between AU Introspection and Symmetry without Necessitation. Therefore, we seek to investigate the relationship between AU Introspection and Symmetry without Necessitation.

We cannot directly prove the equivalence between AU Introspection and Symmetry. To do so, we need several properties of operators that are in conjunction with the knowledge operator, that is, the KU Introspection of the unawareness operator and AA-Self Reflection of the unawareness operator. Therefore, we must prove these properties (Lemma 1, 2 and 3) before proving our main point. In their proofs, we find that Necessitation is not required, that is, an equivalence of AU Introspection and Symmetry holds without Necessitation. Therefore, when excluding Necessitation, Negative Introspection is equivalent to neither AU Introspection nor Symmetry, but AU Introspection and Symmetry are equivalent (Corollary 1). Our result implies that the non-triviality of unawareness consists of both AU Introspection and Symmetry because non-triviality is equivalent to Negative Introspection.

This note is organized as follows: The next subsection highlights related works in the literature. Section 2 introduces standard state-space models following the studies of Dekel et al. (1998) and Chen et al. (2012) and Properties of the knowledge/unawareness operator. Section 3 overviews the Triviality Theorems shown by Modica and Rustichini (1994), Dekel et al. (1998), and Chen et al. (2012). Section 4 provides and proves our main theorem that AU Introspection is equivalent to Symmetry and generalizes a proof of Triviality Theorems. The last section provides the conclusion.

### Related Literature

Pioneering works on higher-order lack of knowledge include those of Fagin and Halpern (1988) and Geanakoplos (2021). Heifetz et al. (2006) are the first to introduce unawareness structures. They assume that the family of state spaces is a lattice structure and that there is a difference in expressive power between different state spaces. Heifetz et al. (2013) and Galanis (2013, 2018) use unawareness structures and discuss and generalize Aumann's agreement theorem (Aumann, 1976) and the No-Trade Theorem (Milgrom and Stokey, 1982). Heifetz et al. (2008) propose canonical models of unawareness. Galanis (2011) considers unawareness of theorems using a logical approach, while Galanis (2013) discusses unawareness of theorems via a set-theoretical approach. Galanis (2013) provides a property named Awareness Leads to Knowledge and shows that a knowledge operator in a more expressive state-space leads to a better description of an agent's knowledge than a knowledge operator in a less expressive state space. This result means that Galanis' model allows agents to disagree on whether the opponents know about some event. Li (2009) proposes a product of the state-space model, called an information structure with unawareness. Heinsalu (2012) discusses the relationship between the works of Fagin and Halpern (1988) and Li (2009).

## 2 Preliminaries

Let us consider a standard state-space model like those by Dekel et al. (1998) and Chen et al. (2012), $\langle \Omega, K, U \rangle$, where

- $\Omega$ is a state space. Any $E \subseteq \Omega$ is an event, and $\neg E = \Omega \setminus E$.
- $K : 2^\Omega \to 2^\Omega$ is the knowledge operator. Given any event $E \subseteq \Omega$, a set $K(E)$ is interpreted as "the agent possessing $K$ knows that event

$E$ occurs."

- $U : 2^\Omega \to 2^\Omega$ is the unawareness operator. Given any event $E$, a set $U(E)$ is interpreted as "the agent possessing $U$ is unaware whether event $E$ occurs."

In a partitional state-space model, the knowledge operator $K$ satisfies the following properties as well-known:

K1 Necessitation: $K(\Omega) = \Omega$;

K2 Monotonicity: if $E \subseteq F$, then $K(E) \subseteq K(F)$;

K3 Truth: $K(E) \subseteq E$;

K4 Positive Introspection: $K(E) \subseteq KK(E)$; and

K5 Negative Introspection: $\neg K(E) \subseteq K\neg K(E)$.

Here, by K5, $\neg K \neg K(E) = \emptyset$ in a partitional state-space model. This means that it is impossible that an agent does not know any event, and the agent does not know that he/she does not know the event. In other words, any higher-order lack of knowledge does not hold.

Previous studies on unawareness attempt to relax the Negative Introspection and provide the following axioms of the unawareness operator:

U1 Plausibility: $U(E) = \neg K(E) \cap \neg K \neg K(E)$;

U2 KU Introspection: $KU(E) = \emptyset$;

U3 AU Introspection: $U(E) = UU(E)$; and

U4 Symmetry: $U(E) = U(\neg E)$.

U1-3 is provided by Dekel et al. (1998) and U4 is provided by Modica and Rustichini (1994).[1]

Following Chen et al. (2012), let us name and define trivial and non-trivial unawareness as follows:

[1]Specifically, U1 and U3 are not Dekel et al.'s (1998) ordinal axioms. Their ordinal Plausibility is defined as $U(E) \subseteq \neg K(E) \cap \neg K \neg K(E)$, and their ordinal AU Introspection is defined as $U(E) \subseteq UU(E)$. U1 is Modica and Rusitichini's (1994) definition of the unawareness operator rather than (ordinal) Plausibility. However, if we use not U1 and U3 but ordinal axioms, then our main result might not hold.

U5 Triviality: $\forall E \subseteq \Omega$, $U(E) = \emptyset$; and

U6 Non-Triviality: $\exists E \subseteq \Omega$ subject to $U(E) \neq \emptyset$.

**Remark 1.** Under U1, K5 if and only if (iff) U5.

Finally, we define the awareness operator as $A(E) = \neg U(E)$.

# 3 Triviality Theorems

Modica and Rustichini (1994), Dekel et al. (1998), and Chen et al. (2012) present the following theorems about trivial unawareness:

**Theorem 1.** [Modica and Rustichini (1994)]
If $\langle \Omega, K, U \rangle$ satisfies K1-4 and U1, then K5 and U4 are equivalent.

**Theorem 2.** [Dekel et al. (1998)]
If $\langle \Omega, K, U \rangle$ satisfies K1 and U1-3, then U5 is satisfied.

**Theorem 3.** [Chen et al. (2012)]
If $\langle \Omega, K, U \rangle$ satisfies K1-3 and U1, K5 iff U3

**Theorem 4.** [Chen et al. (2012)]
If $\langle \Omega, K, U \rangle$ satisfies K1-4 and U1, K5 iff U3 iff U4.

Chen et al. (2012) provide an outline of a proof of Theorem 4 as follows:

*The outline of the proof of Theorem 4.*

1. By Theorem 1, K5 and U4 are equivalent.
2. By Theorem 3, K5 and U3 are equivalent.
3. By 1 and 2, U3 and U4 are equivalent.

Hence, K5, U3, and U4 are equivalent. □

Theorem 4 is a generalization of Theorem 1 and 2. Theorem 1 suggests an equivalence between Negative Introspection and Symmetry; Theorem 3 suggests an equivalence between Negative Introspection and AU Introspection; and Theorem 4 suggests an equivalence between AU Introspection and Symmetry. In proof of Theorem 4, AU Introspection and Symmetry are not directly equivalent. This proof is related to Necessitation. However, is Negative Introspection necessary to prove the equivalence between AU Introspection and Symmetry? Can we directly prove this equivalence without Negative Introspection? We explore this issue in the next section.

# 4 Main Theorem

In this section, we explore the proof of equivalence between AU Introspection and Symmetry without Negative Introspection. We show the following theorem.

**Theorem 5.** If $\langle \Omega, K, U \rangle$ satisfies K2-4 and U1, then U3 is equivalent to U4.

This theorem does not use Necessitation. In other words, Necessitation is not necessary for this theorem. Theorem 5 implies that AU Introspection is consistent with Symmetry. Put differently, Negative Introspection is not necessary under this equivalence. In other words, a pair of AU Introspection and Symmetry is not equivalent to Negative Introspection when Necessitation does not hold.

Before proving this theorem, we show the following lemmas.

**Lemma 1.** If $\langle \Omega, K, U \rangle$ satisfies K2, then

K6 $K(E \cap F) \subseteq (K(E) \cap K(F))$.

*Proof.* Suppose that $\langle \Omega, K, U \rangle$ satisfies K2. It is evident that $(E \cap F) \subseteq E$ and $(E \cap F) \subseteq F)$. By K2, $K(E \cap F) \subseteq K(E)$ and $K(E \cap F) \subseteq K(F)$. Hence, $K(E \cap F) \subseteq (K(E) \cap K(F))$. □

This property K6 is the relaxing Conjunction ($K(E \cap F) = K(E) \cap K(F)$), which is one of the standard properties of the knowledge operator. Theorem 5 needs K6, not Conjunction. See proofs of Lemma 2 and 3.

As the following proof of Lemma 2 shows, K4 is not necessary.

**Lemma 2.** If $\langle \Omega, K, U \rangle$ satisfies K2-3 and U1, then U2 is satisfied.

*Proof.* Suppose that $\langle \Omega, K, U \rangle$ satisfies K2-3 and U1. Then,

$$KU(E) = K(\neg K(E) \cap \neg K \neg K(E))$$

$$\overset{\text{K6}}{\subseteq} K \neg K(E) \cap K \neg K \neg K(E)$$

$$\overset{\text{K3}}{\subseteq} K \neg K(E) \cap \neg K \neg K(E) = \emptyset.$$

□

Lemma 2 suggests that if a standard state-space model satisfies Monotonicity, Truth, and Plausibility, then KU Introspection is satisfied.

**Lemma 3.** If $\langle \Omega, K, U \rangle$ satisfies K2-4 and U1, then it satisfies the following:

A1 AK-Self Reflection: AK(E)=A(E);

A2 AA-Self Reflection: AA(E)=A(E); and

A3 A-Introspection: KA(E)=A(E).

*Proof.* Suppose that $\langle \Omega, K, U \rangle$ satisfies K2-4 and A1.

*Proof of A1.* $AK(E) = KK(E) \cup K\neg KK(E) \overset{\text{K4}}{=} K(E) \cup K\neg K(E) = AE$.

*Proof of A3.* First, given $K(E)$, by K2 and K4, because $K(E) \subseteq A(E)$, $K(E) = KK(E) \subseteq KA(E)$ $(*)$. Next, given $K\neg K(E)$, $K\neg K(E) \subseteq A(E)$ and $K\neg K(E) \subseteq \neg K(E)$ by K3, that is, $K\neg K(E) \subseteq (\neg K(E) \cap A(E))$. Then, $K\neg K(E) \overset{\text{K4}}{=} KK\neg K(E) \overset{\text{K2}}{\subseteq} K(\neg K(E) \cap A(E))$. $K(\neg K(E) \cap A(E)) \overset{\text{K6}}{\subseteq} K\neg K(E) \cap KA(E)$. That is, $K\neg K(E) \subseteq KA(E)$. Then, $A(E) = K(E) \cup K\neg K(E) \subseteq K(E) \cup KA(E)$. Because $K(E) \subseteq KA(E)$ $(*)$, $K(E) \cup KA(E) = KA(E)$, that is, $A(E) \subseteq KA(E)$. By K3, because $KA(E) \subseteq A(E)$, $KA(E) = A(E)$.

*Proof of A2.* $AA(E) = KA(E) \cup K\neg KA(E) \overset{\text{A3}}{=} A(E) \cup K\neg A(E) = A(E) \cup KU(E) \overset{\text{U2}}{=} A(E) \cup \emptyset = A(E)$. □

A1 and A2 are proved by Modica and Rustichini (1999) and Halpern (2001), respectively, and A3 is provided by Halpern et al. (2006). Those properties can be proved in set-theoretical approaches as follows: In contrast with proofs of Lemma 1 and 2, a proof of Lemma 3 needs Positive Introspection, K4.

By the above lemmas, we can prove our main theorem.

*Proof of Theorem 5.* Suppose that $\langle \Omega, K, U \rangle$ satisfies K2-4 and U1.

First, assume U3, that is, $U(E) = UU(E)$. Then, by a definition of the awareness operator, for any $E \subseteq \Omega$, $A(E) = AU(E) = KU(E) \cup K\neg KU(E) \overset{\text{U2}}{=} \emptyset \cup K(\neg\emptyset) = \emptyset \cup K(\Omega) = K(\Omega)$. Because $E$ is arbitrary, $A(E) = A(\neg E) = K(\Omega)$.[2] Therefore, $U(E) = \neg A(E) = \neg A(\neg E) = U(\neg E)$.

Next, assume U4, that is, $U(E) = U(\neg E)$. By Lemma 3, because A2, that is, $AA(E) = A(E)$, is satisfied, $UA(E) = U(E)$. By U4, $U(E) = UA(E) = UU(E)$. □

[2] If we use ordinal Plausibility instead of U1, $AU(E) \supseteq KU(E) \cup K\neg KU(E)$. Then, Symmetry might not hold, because $A(E) = K(\Omega)$ might not hold. If we use ordinal AU Introspection instead of U3, $A(E) \supseteq AU(E)$. Then, Symmetry also might not hold, because $A(E) = K(\Omega)$ might not hold.

By Theorem 5, we can generalize Theorem 4.

*Proof of Theorem 4.* Suppose that $\langle \Omega, K, U \rangle$ satisfies K1-4 and U1.

First, assume U4. By Theorem 5, U3 holds.

Next, assume U3. Then, $U(E) \overset{\text{U3}}{=} UU(E) = \neg KU(E) \cap \neg K \neg KU(E) \overset{\text{U2}}{=} \neg\emptyset \cap \neg K(\neg\emptyset) = \Omega \cap \neg K(\Omega) = \neg K(\Omega) \overset{\text{K1}}{=} \neg\Omega = \emptyset$. By Remark 1, K5 holds.

Finally, assume K5. By Remark 1, U5 holds, that is, $U(E) = \emptyset$ for any $E \subseteq \Omega$. Because $E$ is arbitrary, $U(E) = U(\neg E) = \emptyset$. That is, U4 holds. □

Theorem 1, 2, and 3 are evident by Theorem 4.

Note that Theorem 5 generalizes Theorems 1 and 4, but not Theorems 2 and 3. Theorems 1 and 4 require K4, whereas Theorems 2 and 3 do not require K4.

The relationship between Theorems 4 and 5 implies the following corollary.

**Corollary 1.** If $\langle \Omega, K, U \rangle$ does not satisfy K1, then K5 might not be equivalent to both U3 and U4, but U3 and U4 are equivalent.

# 5 Concluding Remarks

This note (i) shows that AU Introspection and Symmetry for unawareness are equivalent when relaxing Necessitation; and (ii) generalizes proofs of the Triviality Theorems proposed by Modica and Rustichini (1994), Dekel et al. (1998), and Chen et al. (2012).

This study has one limitation. We exclude only Necessitation, because our focus is on axioms of the knowledge operator. However, in standard information structures that may be non-partitional, Necessitation and Monotonicity are equivalent. Hence, the knowledge operator based on the standard information function or the standard possibility correspondence cannot exclude only Necessitation. In other words, an equivalence of AU Introspection and Symmetry must be equivalent to "trivial" unawareness in standard information structures. In future work, we aim to define a novel knowledge operator that excludes only Necessitation in standard information structures.

Recent studies related to the present one include those by Fukuda (2021) and Tada (2021). Fukuda (2021) proposes generalized state-space models that nest both unawareness structures and non-partitional state-space models. He posits that AU Introspection is not consistent with Necessitation, relaxes AU Introspection, and replaces AU Introspection with Reverse AU Introspection ($UU(E) \subseteq U(E)$). Tada (2021) discusses multi-attribute state spaces with complete lattices. In contrast with Heifetz et al. (2006), in which the family of spaces is a lattice structure, his state space is a lattice structure. His knowledge operator is closer to that of Heifetz et al. (2006) than to that of standard models. In his study, the symmetry of the unawareness operator is equivalent to the Necessitation of the knowledge operator; that is, if Necessitation is relaxed, Symmetry does not hold, although AU Introspection holds. He names this impossibility, Reverse Symmetry. By contrast, we show that Symmetry holds even if Necessitation is relaxed. Our result is different from his. This finding means that his models are different from standard state-space models.

This note implicitly focuses on only single-attribute state-space models. Although Fukuda (2021) discusses relationships between standard state-space models and unawareness structures, no model has been constructed discussing the relationships between standard state-space models and models of a state-space being a lattice structure. By exploring those relationships, we further characterize the knowledge operator and unawareness operator.

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