Discussion Paper No.343

Discovery Process in Normal-Form Games with Unawareness: Cognitive Stability and Closedness under Rational Behavior

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February 2021

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Discovery Process in Normal-Form Games with Unawareness: Cognitive Stability and Closedness under Rational Behavior *

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February 6, 2021

Abstract

This study examines how each player chooses her/his optimal action in "normal-form games with unawareness" by applying a "discovery process" to them. We show that if each player implements a best response to the opponents' immediately preceding plays, then any discovery process converges to a set closed under rational behavior (CURB) on the realizable action set. Moreover, in the objective game in any initial normal-form game with unawareness, when every CURB set on the realizable action set is mutually known, every discovery process converges to a discovered game possessing a cognitively stable generalized Nash equilibrium. It is not necessary that each player must be aware of the opponents' utilities in our results.

JEL classification: C72; D83

Keywords: Game Theory; Unawareness; Discovery; Generalized Nash Equilibrium; Cognitive Stability; Closed under Rational Behavior

1 Introduction

This study analyzes a discovery process named the *discovery process with Markov best responses* and convergence. Games with unawareness assume that some player may be unaware of certain actions and their opponents' actual subjective games.[1] In games with unawareness, some players' beliefs about their opponents' play may be wrong in a play. When some player produces an action that

*This article is based on the author's master's thesis, Tada (2018). An earlier version of this article was presented at the 2018 Japanese Economic Association Autumn Meeting at Gakushuin University, September 8–9, 2018, Tokyo. The author thanks Norimasa Kobayashi, Yasuo Sasaki, Toichiro Asada and Hirokazu Takizawa for insightful comments on this study. I would like to thank Editage (www.editage.com) for English language editing.

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[1] Schipper (2014) provides a historical survey on "unawareness" and "games with unawareness."

another player is unaware of, another player is surprised at the action and would revise her belief about the game's structure. Then, we must revise the original game with unawareness.

Schipper (2018) first introduces and formulates the revision process known as a *discovery process*.[2] A study of discovery processes analyzes a convergence of each player's update to her subjective game and belief hierarchy. In the process, each player updates her higher-order subjective games by adding played actions that she is unaware of. Through the revision process, each player arrives at some situation improving each player's unawareness of the opponents' actions and replays the situation.

Schipper (2018) shows that in any extensive-form game with unawareness, if every player implements a rationalizable strategy, then the (rationalizable) discovery process converges to some *discovered game*, in which each player revises her beliefs about her subjective view and her opponents' play, possessing a self-confirming equilibrium[3]. The present study provides another discovery process. Simply, we focus on only a normal-form game with unawareness and assume that our player implements a best response to an immediately preceding play, called a *Markov best response*. In contrast to Schipper (2018), although our player's strategy might not be rationalizable, our discovery process under some condition converges to some discovered game possessing a cognitively stable generalized Nash equilibrium.

It might not be possible to apply the analysis in this study to normal-form games with unawareness in general, because some game with unawareness might not have a cognitively stable generalized Nash equilibrium and each player might not play her own action that some opponent is unaware of. Then, no player can discover any novel action that they are unaware of and thus, we cannot update to the (normal-form) game with unawareness. To resolve this issue, we focus on *closedness under rational behavior* (*CURB*). Basu and Weibull (1991) propose a CURB set that is a closed set of best responses to the opponents' actions in a standard model and refinement of rationalizability (Bernheim 1984; Pearce 1984). Tada (2020) is the first to apply CURB to static games with unawareness. He shows that if a static game with unawareness has a CURB set that is commonly known (*common CURB set*), then there is a cognitively stable generalized Nash equilibrium. He focuses on CURB sets only on an objective game in a normal-form game with unawareness; our study, meanwhile, focuses on CURB sets not only on an objective game but also on the realizable action set that is the Cartesian product of non-empty sets of each player's own actions that she is aware of.

Our result shows that any discovery process with Markov best responses converges to a discovered game in which the support of the *objective outcome*

[2] A discovery process is different to a learning process. Roughly, in a learning process, each player learns if she replaces previous probabilities with novel probabilities about the same event when the novel event that some player is unaware of does not occur; meanwhile, in a discovery process, each player discovers if she adds novel events that she does not know about and assigns probabilities to the events.

[3] Schipper (2018) calls the game a "rationalizable self-confirming game."

induced from Markov best responses is a subset of some CURB set on the realizable action set (Theorem 1). Moreover, for any initial normal-form game with unawareness, when there exists a CURB set in the objective game which is mutually known, called mutual CURB set, some discovery process with Markov best responses converges to some discovered game possessing a non-empty common CURB set, called a cognitively stable game under CURB (Theorem 2). Then, the cognitively stable game under CURB has a cognitively stable generalized Nash equilibrium by Tada (2020) (Corollary 1). Furthermore, if every CURB set on realizable action set is a mutual CURB set in any initial normal-form game with unawareness, then every discovery process with Markov best responses converges to a cognitively stable game under CURB (Corollary 2).

This rest of this paper organized as follows. Section 2 provides preliminaries. Section 3 defines a discovery process and Section 4 formulates one of discovery processes, called a discovery process with Markov best responses. Finally, Section 5 shows the mathematical results.

2 Preliminaries

2.1 Normal-Form Games with Unawareness

Let $G = (I, A, u)$ be a finite normal-form game. I is a finite set of players and $I_{-i} = I \setminus \{i\}$. $A = \times_{i \in I} A_i$, where A_i is the non-empty finite set of i's actions and each element on the set is $a_i \in A_i$. $u = (u_i)_{i \in I}$, where $u_i : A \to \mathbb{R}$ is i's utility function. Denote i's mixed action on A_i by $m_i \in M(A_i)$, where $M(A_i)$ is the set of i's mixed actions, and a mixed action profile on A by $m = (m_i)_{i \in I} \in M(A) = \times_{i \in I} M(A_i)$. We denote i's expected utility for $m \in M(A)$ by $Eu_i(m)$.

First, we define normal-form games with unawareness.[4] For any standard normal-form game G, let $V = \times_{i \in I}(2^{A_i} \setminus \{\emptyset\})$ be the set of possible *views* of the G. Like most previous works, this study assumes that the set of players is commonly known and that each player's utility for each action profile does not depend on awareness. Let $v \in V$ and A_i^v be the set of i's actions in $v = \times_{j \in I} A_j^v$. Here, when player i is given v, i is aware of $a \in v$ and unaware of $a \in A \setminus v$. For any $v, v' \in V$, v is contained in v' if A_i^v is a subset of $A_i^{v'}$ for any $i \in I$, that is, $A_i^v \subseteq A_i^{v'}$. Let $M(A_i^v) = \{m_i \in M(A_i) | \Sigma_{a_i \in A_i^v} m_i(a_i) = 1\}$.

Let $\Gamma = (G, (T_i)_{i \in I}, (v_i)_{i \in I}, (b_i)_{i \in I})$ be a static game with unawareness as follows: for each $i \in I$,

- T_i is a finite and non-empty set of i's type, one of which is her actual type t_i^*.
- $v_i : T_i \to V$ is i's view function.

[4] Our definition is similar to that by Perea (2018). Note that there are two differences. First, his model does not fix belief hierarchies on views, whereas we do so. We assume that the "actual type" of the players is given. Second, he deals with probabilistic beliefs on awareness, whereas we do not. Our players always have point beliefs on their opponents' awareness, as is often assumed in the literature of games with unawareness.

- $b_i : T_i \to T_{-i}$ is i's belief function, where $T_{-i} = \times_{j\in I\setminus\{i\}} T_j$. If $b_i(t_i) = (t_j)_{j\in I\setminus\{i\}}$, then for each $j \in I \setminus \{i\}$ must be contained in $v_i(t_i)$.

Let us call G an objective game (in Γ). An objective game can be interpreted as the "true game" in Γ.[5] i's type t_i describes her view about the game and belief about the opponents' types. At t_i, $v_i(t_i) = v$ means that i is aware of v and unaware of $A \setminus v$; while $b_i(t_i) = (t_j)_{j\in I\setminus\{i\}}$ means that at t_i, i believes that the others' types are $(t_j)_{j\in I\setminus\{i\}}$ and that each j's view is $v_j(t_j)$. We can write j's type in $b_i(t_i)$ as $b_i(t_i)(j)$. In a normal-form game with unawareness, each player may be unaware of some types of players, including her own.

For any $i \in I$, let $s_i : T_i \to M(A_i)$. Then, given t_i, $s_i(t_i) \in M(A_i^{v_i(t_i)})$ is i's local action at t_i. We denote i's generalized strategy by $s_i = (s_i(t_i))_{t_i\in T_i}$, and a generalized strategy profile by $s = (s_i)_{i\in I}$. In a generalized strategy profile s, each player i's actual play is $m_i \in M(A_i)$ with $m_i = s_i(t_i^*)$, and then the profile is called the objective outcome induced from s. The set of each player's actual play $A_i^{v_i(t_i^*)}$ may be a proper subset of i's *full* action set A_i. Then, she cannot play $a_i \in A_i \setminus A_i^{v_i(t_i^*)}$. Let $\times_{i\in I} A_i^{v_i(t_i^*)}$ be the realizable action set.

Halpern and Rêgo (2014) introduce a solution concept in games with unawareness as generalized Nash equilibrium: s^* is a generalized Nash equilibrium if for any $i \in I$ and $t_i \in T_i$,

$$s_i^*(t_i) \in \arg\max_{x\in M(A_i^{v_i(t_i)})} Eu_i(x, (s_j^*(b_i(t_i)(j))_{j\in I\setminus\{i\}})).$$

As they point out, a generalized Nash equilibrium is best interpreted as "an equilibrium in beliefs." (Halpern and Rêgo 2014: 50) However, as pointed out by Schipper (2014), there exists some generalized Nash equilibrium, which consists of wrong beliefs. Then, each player would revise her own beliefs about a game's structure and the opponents' play and they might not play the same generalized Nash equilibrium. To avoid this case, Sasaki (2017) proposes cognitive stability about generalized Nash equilibrium. A generalized Nash equilibrium s^* is cognitively stable if for any $i \in I$ and $t_i \in T_i$, $s_i^*(t_i) = s_i^*(t_i^*)$.[6]

Cognitively stable generalized Nash equilibria have the following property.

Remark 1. For any normal-form game with unawareness Γ, let s^* be a cognitively stable generalized Nash equilibrium. Then, the objective outcome $(s_i^*(t_i^*))_{i\in I}$ is a Nash equilibrium on the realizable action set.

proof. Suppose that s^* is a cognitively stable generalized Nash equilibrium, that is, for any $(i, t_i) \in I \times T_i$, $s_i^*(t_i) = s_i^*(t_i^*)$. This is obviously $(s_i^*(t_i^*))_{i\in I} \in A^{v_j(t_j)}$ for any $(j, t_j) \in I \times T_j$. Therefore, $(s_i^*(t_i^*))_{i\in I} \in \times_{j\in I} A_j^{v_j(t_j^*)}$. Assume that $(s_i^*(t_i^*))_{i\in I}$ is not a Nash equilibrium on the realizable action set $\times_{i\in I} A_i^{v_i(t_i^*)}$. In other words, there exists $(i, a_i) \in I \times A_i^{v_i(t_i^*)}$ such that $a_i \notin \mathrm{supp}(s_i^*(t_i^*))$ and $a_i \in \arg\max_{x\in A_i^{v_i(t_i^*)}} Eu_i(x, (s_j^*(t_j^*))_{j\in I})$. However, since s^* is a cognitively stable

[5]Perea (2018) calls an objective game a "base game."

[6]Sasaki (2017) calls cognitive stability "stable belief hierarchy."

generalized Nash equilibrium, this is a contradiction. Therefore, $(s_i^*(t_i^*))_{i \in I}$ is a Nash equilibrium on the realizable action set. ■

2.2 Closedness under Rational Behavior

Some game with unawareness might not have a cognitively stable generalized Nash equilibrium. For example, see the following case[7]:

$v =$

1 / 2	L	R
U	3, 3	0, 5
B	5, 0	1, 1

, $v' =$

1 / 2	L	R
U	3, 3	0, 5

, $v'' =$

1 / 2	L
U	3, 3
B	5, 0

Suppose that $T_1 = \{t_1^*, t_1\}$ and $T_2 = \{t_2^*, t_2\}$ such that:

$v_1(t_1^*) = v'$ and $b_1(t_1^*) = t_2$;
$v_1(t_1) = v''$ and $b_1(t_1) = t_2^*$;
$v_2(t_2^*) = v''$ and $b_2(t_2^*) = t_1$; and
$v_2(t_2) = v'$ and $b_2(t_2) = t_1^*$.

This example is illustrated in Figure 1. In this static game with unawareness, a unique generalized Nash equilibrium $s^* = ([s_1(t_1^*) = U, s_1(t_1) = B], [s_2(t_2^*) = L, s_1(t_1) = R])$ is cognitively *unstable*. Then, each player is already aware of the opponent's actual play. Thus, although both players' beliefs are wrong, since there is no novel action that some player is unaware of, we cannot revise each player's subjective game and the original game with unawareness. Therefore, in this example, a simple discovery process, in which each player only adds actions that some player is unaware of, cannot converge to an update to all players' subjective views and belief hierarchies possessing a cognitively stable generalized Nash equilibrium.

To avoid this case, this section focuses on closedness under rational behavior (Basu and Weibull 1991). Tada (2020) is the first to apply CURB to normal-form games with unawareness.

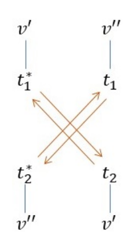


Figure 1: Formulation of example of two players, 1 and 2

[7] We borrow this situation from Schipper (2018, Example 3).

This subsection defines CURB in standard normal-form games. Let $\beta_i(m_{-i}) = \{a_i \in A_i | a_i \in \text{supp}(m_i)$ be such that $m_i \in \arg\max_{x \in M(A_i)} Eu_i(x, m_{-i})\}$ is the set of i's pure-action best responses to her belief about $m_{-i} \in M(A_{-i})$. For any $v \in V$, let $\beta_i(v) = \bigcup_{m_{-i} \in M(\times_{j \in I_{-i}} A_j^v)} \beta_i(m_{-i})$ be the set of i's optimal actions under beliefs in $M(v)$, and let $\beta(v) = \times_{i \in I} \beta_i(v)$. Then, a set $v \in V$ is closed under rational behavior (*CURB*) (in G) if $\beta(v) \subseteq v$. Let us call the v a CURB set.

In a standard normal-form game, as shown by Basu and Weibull (1991), a CURB set has the following property.

Remark 2. For any standard normal-form game G and any CURB set, there exists a Nash equilibrium such that the support is a subset of the CURB set.

Tada (2020) defines a *common CURB set* as follows: For any normal-form game with unawareness Γ and any $v \in V$, v is a common CURB set if v is a non-empty CURB set in the objective game G and $v \subseteq v_i(t_i)$ for any $i \in I$ and $t_i \in T_i$.

He shows the following property.

Lemma 1. Any normal-form game with unawareness possessing a common CURB set has a cognitively stable generalized Nash equilibrium.

proof. Assume that any normal-form game with unawareness has a common CURB set $v \in V$. Then, by Remark 2, there exists a Nash equilibrium in the objective game $G = (I, A, u)$, $m^* \in M(A)$, satisfying $m^* \in M(v)$. Suppose that m^* is not a Nash equilibrium on v. In other words, there exists $(i, m_i) \in I \times M(A_i^v)$ such that $Eu_i(m_i, m_{-i}^*) > Eu_i(m^*)$. However, since v is common CURB set, this is a contradiction. Therefore, m^* is a Nash equilibrium on v. Then, m_i^* is a best response to m_{-i}^* in $v_i(t_i)$ for any $i \in I$ and $t_i \in T_i$. Thus, s^* with $s_i^*(t_i) = m_i^*$ for any $i \in I$ and $t_i \in T_i$ is a cognitively stable generalized Nash equilibrium. ■

As pointed out by Tada (2020), a common CURB set is a refinement of generalized Nash equilibria and a coarsening of cognitively stable generalized Nash equilibria.

2.3 Cognitively Stable Games under CURB

Finally, we define a cognitively stable games under CURB. By the definitions of CURB and common CURB sets, all players are aware of every local action of each player in a common CURB set. Therefore, any game with unawareness possessing a common CURB set is cognitively stable under playing actions in the common CURB set. We call the game a *cognitively stable game under CURB*.

Definition 1. A normal-form game with unawareness Γ is a cognitively stable game under CURB if there exists a non-empty common CURB set in Γ.

The concept of cognitively stable games under CURB is similar to the concept of (rationalizable) self-confirming games (Schipper 2018). In a self-confirming game, each player confirms her belief about a game's structure. Furthermore, in a cognitively stable game under CURB, each player's belief about a game's structure is cognitively stable as long as every player implements her (local) action in the common CURB set.

To show our main results, we introduce the following concept.

Definition 2. For any normal-form game with unawareness Γ, $v \in V$ is a *mutual CURB set* if v is a non-empty CURB set in the objective game G and $v \subseteq v_i(t_i^*)$ for any $i \in I$.

A mutual CURB set has the following property.

Lemma 2. For any normal-form game with unawareness Γ, if $v \in V$ is a mutual CURB set, then v is a CURB set on the realizable action set.

proof. Suppose that v is a mutual CURB set, that is, $v \subseteq v_i(t_i^*)$ for any $i \in I$. In other words, $v \in \bigcap_{i \in I} v_i(t_i^*)$. Here, since $\bigcap_{i \in I} v_i(t_i^*) \subseteq \times_{i \in I} A_i^{v_i(t_i^*)}$, $v \subseteq \times_{i \in I} A_i^{v_i(t_i^*)}$. ■

3 Discovery Process

Schipper (2018) is the first to introduce a *discovery process* in (extensive-form) games with unawareness, which is based on Heifetz, Meier, and Schipper (2013). A discovery process represents an update process by which each player revises her own belief about a game's structure and the opponents' play. Although our definition given here, at first glance, may seem different to that used in Schipper (2018), both definitions are essentially the same.

Definition 3. $\Gamma' = (G, (T_i')_{i \in I}, (v_i')_{i \in I}, (b_i')_{i \in I})$ is a discovered game with $s = (s_i)_{i \in I}$ in $\Gamma = (G, (T_i)_{i \in I}, (v_i)_{i \in I}, (b_i)_{i \in I})$ if for any $(i, t_i) \in I \times T_i$, there exists $t_i' \in T_i'$ such that:

1. $v_i'(t_i') = \times_{i \in I}[A_j^{v_i(t_i)} \cup \mathrm{supp}(s_j(t_j^*))]$, where t_j^* is j's actual type in Γ; and
2. for any $(j, t_j) \in I_{-i} \times T_j$, if $b_i(t_i)(j) = t_j$ and t_j can be replaced with some $t_j' \in T_j'$ satisfying the first condition in this definition, then $b_i'(t_i')(j) = t_j'$.

Note that some Γ, Γ' may be $T \not\subseteq T'$ and $T' \not\subseteq T$, or $T \cap T' = \emptyset$.

Definition 4. A discovery process $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \ldots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \ldots)$, is defined as follows:

- for any λ, $\Gamma^\lambda = (G, (T_i^\lambda)_{i \in I}, (v_i^\lambda)_{i \in I}, (b_i^\lambda)_{i \in I})$;
- when $\lambda = 0$, $s^0 = \phi$, while for any $\lambda \geq 1$, s^λ is a played generalized strategy profile in Γ^λ; and

- for any $\lambda \geq 2$, Γ^{λ} is a discovered game with $s^{\lambda-1}$ in $\Gamma^{\lambda-1}$.

Let us call Γ^1 the initial game (in P).

By Definition 3, Definition 4 implicitly assumes perfect recall. If we exclude the assumption, some player may forget some action at λ even if she is aware of the action at $\lambda - 1$.

Simply, this study assumes that every player can play and observe a mixed action.[8]

4 Discovery Process with Markov Best Responses

Schipper's (2018) discovery process assumes that every player implements her rationalizable strategy in each discovered game. Playing a rationalizable strategy would be appropriate in a game in which every player faces the situation for the first time. However, rationalizability notions have an issue in discovery processes. Consider the following example:

$v =$

3 / 4	L	R
U	3, 3	0, 5
B	5, 0	1, 1

, $v' =$

3 / 4	L
U	3, 3
B	5, 0

, $v'' =$

3 / 4	L	R
U	3, 3	0, 5

In initial game $\Gamma^{\lambda=1}$, suppose that $T_3^{\lambda=1} = \{t_3^{(\lambda=1)*}, t_3^{(\lambda=1)}\}$ and $T_4^{\lambda=1} = \{t_4^{(\lambda=1)*}, t_4^{(\lambda=1)}\}$ such that:

$v_3(t_3^{(\lambda=1)*}) = v'$ and $b_3(t_3^{(\lambda=1)*}) = t_4^{(\lambda=1)}$;
$v_3(t_3^{(\lambda=1)}) = v''$ and $b_3(t_3^{(\lambda=1)}) = t_4^{(\lambda=1)*}$;
$v_4(t_4^{(\lambda=1)*}) = v''$ and $b_4(t_4^{(\lambda=1)*}) = t_3^{(\lambda=1)}$; and
$v_4(t_4^{(\lambda=1)}) = v'$ and $b_4(t_4^{(\lambda=1)}) = t_3^{(\lambda=1)*}$.

Figure 2 (a) is a formulation of the example at $\lambda = 1$. Then, a rationalizable strategy play is $s^{(\lambda=1)*} = ([s_3(t_3^{(\lambda=1)*}) = B, s_3(t_3^{(\lambda=1)}) = U], [s_4(t_4^{(\lambda=1)*}) = R, s_4(t_4^{(\lambda=1)}) = L])$, player 3 discovers 4's action R and player 4 discovers 3's action B. Thus, in the discovered game $\Gamma^{\lambda=2}$, $T_3^{\lambda=2} = \{t_3^{(\lambda=2)*}\}$ and $T_4^{\lambda=2} = \{t_4^{(\lambda=2)*}\}$ such that:

$v_3(t_3^{(\lambda=2)*}) = v$ and $b_3(t_3^{(\lambda=2)*}) = t_4^{(\lambda=2)*}$;
$v_4(t_4^{(\lambda=2)*}) = v$ and $b_4(t_4^{(\lambda=2)*}) = t_3^{(\lambda=2)*}$.

[8]In proofs of each result in Section 5, since we assume that not that every player implements a generalized Nash equilibrium, as in Tada (2018), but rather that every player implements any generalized strategy profile, our main results do not depend on the assumption. In fact, a pure action is a mixed action assigning probability 1 to the action. The proofs of our main results are provided in Section 5.

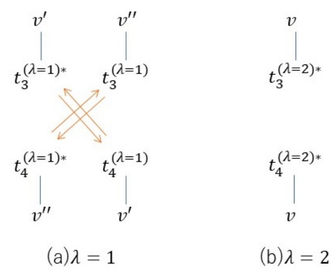


Figure 2: Formulation of example of two players, 3 and 4

At $\lambda = 2$, the formulation is depicted in Figure 2 (b). Here, each player discovers novel action profiles: 3 discovers (U, R) and (U, L); while 4 discovers (B, L) and (B, R). Then, rationalizability assumes that 3 is aware of 4's payoff $u_4(U, R) = 5$ and 4 is aware of 3's payoff $u_3(B, L) = 5$. However, even if each player were unaware of each profile and the profiles were not realized in the initial game $\Gamma^{\lambda=1}$, it would not explain why each player is aware of every opponent's utility in the next game $\Gamma^{\lambda=2}$. This seems to be a strict assumption.

By contrast, this study assumes that in each discovered game, every player implements a best response to the opponents' play in the immediately preceding discovered game. Then, since each player has to know only the opponents' immediately preceding play but not the opponents' novel payoff, we can avoid the abovementioned issue. Let us call the best response a *Markov best response.*

Definition 5. For any discovered game Γ^λ from $\Gamma^{\lambda-1}$, s_i^λ is i's Markov best response if for any $t_i \in T_i$,

$$s_i^\lambda(t_i) \in \arg \max_{x \in M(A_i^{v_i(t_i)})} Eu_i(x, (s_j^{\lambda-1}(t_j^{(\lambda-1)*})_{j \in I_{-i}})).$$

Note that for any $j \in I_{-i}$, $t_j^{(\lambda-1)*} \in T_j$ is j's actual type in $\Gamma^{\lambda-1}$. Let us call the generalized strategy profile $s^\lambda = (s_i^\lambda)_{i \in I}$ a *Markov best response profile.*

Each player's Markov best response is *not* a best response in belief hierarchies in each discovered game. Therefore, her Markov best response might not be rationalizable in her belief hierarchy.

Next, we define a discovery process with Markov best response as follows.

Definition 6. Any discovery process $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \ldots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \ldots)$, is a discovery process with Markov best responses if for any $\lambda \geq 2$, s^λ is Markov best response profile at λ.

5 Results

This section shows a convergence of a discovery process with Markov best responses. By Definition 4, every player adds realized novel actions to every view.

By Definition 5, a discovery process with Markov best responses is essentially a sequence of best responses *on the realizable action set*. Therefore, we first show convergence to a CURB set.

Theorem 1. For any normal-form game with unawareness, every discovery process with Markov best responses converges to some discovered game. In that game, support of an objective outcome induced from the Markov best response profile is a subset of a CURB set on the realizable action set.

proof. Since we consider a discovery process with Markov best responses, it is necessary only to focus on the realizable action set. For any objective outcome in the initial game m, let $\beta^{\lambda}(m)$ be an objective outcome induced from a Markov best response profile and it is defined as follows: $\beta^0(m) = \text{supp}(m)$, $\beta^1(m) = \beta \circ \beta^0(m)$, $\beta^2(m) = \beta \circ \beta^1(m)$, $\ldots$, $\beta^{\lambda}(m) = \beta \circ \beta^{\lambda-1}(m)$, $\ldots$. Suppose that for any CURB set on the realizable action set $v \subseteq \times_{i\in I} A_i^{v_i(t_i^*)}$ and natural number λ, $\beta^{\lambda}(m) \nsubseteq v$. As pointed out by Basu and Weibull (1991), since the set of rationalizable strategy profile on $\times_{i\in I} A_i^{v_i(t_i^*)}$, $R \subseteq \times_{i\in I} A_i^{v_i(t_i^*)}$, is CURB,[9] $\beta^{\lambda}(m) \nsubseteq R$ for any λ. This is obviously a contradiction. Therefore, there exists a CURB set v on $\times_{i\in I} A_i^{v_i(t_i^*)}$ and natural number n such that $\beta^n(m) \subseteq v$. Suppose that there exist $\lambda \geq n$ satisfying $\beta^{\lambda}(m) \nsubseteq v$. In other words, $\beta \circ \cdots \circ \beta \circ \beta^n(m) \nsubseteq v$. However, since v is CURB, that is, $\beta(v') \subseteq v$ for any $v' \subseteq v$ with $\emptyset \neq A_i^{v'} \subseteq A_i^{v}$, this is a contradiction. Therefore, $\beta^{\lambda}(m) \subseteq v$ for any $\lambda \geq n$. Since $\beta^{\lambda}(m)$ supports an objective outcome induced from a Markov best response profile at λ and v is a CURB set on the realizable action set, the support is a subset of a CURB set on the realizable action set. ■

It is known that many intuitively appealing adjustment processes eventually settle down in a minimal curb set, cf. Hurkens (1995) and Young (1998). Theorem 1 adds to the previous literature, highlighting the importance of CURB set. However, the process therein converges to a general CURB set, and not necesssarily a "minimal" CURB set like Hurkens (1995) and Young (1998).

By Definition 3, for any discovery process with Markov best responses P, since an objective outcome induced from each Markov best response profile in P is added in $v_i(t_i)$ for any $(i, t_i) \in I \times T_i$, the support must be commonly known through P. Note that a CURB set on the realizable action set that is commonly known might *not* be a common CURB set, because a CURB set on the realizable action set might not be a CURB set in an objective game.[10]

The following theorem suggests a condition for which a CURB set on the realizable action set is a common CURB set in a discovery process with Markov best responses.

[9] Specifically, R is a (maximum) tight CURB set. An action profile set $v \in V$ is a tight CURB set if $\beta(v) = v$.

[10] In this case, the CURB set on the realizable action set might not be CURB in each player's view.

Theorem 2. Suppose a normal-form game with unawareness has a mutual CURB set. Then, there exists a discovery process with Markov best responses converging to some cognitively stable game under CURB.

proof. Suppose a normal-form game with unawareness has a mutual CURB set. Assume that every discovery process with Markov best responses does not converge to any cognitively stable game under CURB, i.e., every discovery process with Markov best responses does not converge to any discovered game possessing a common CURB set. That is, any action profile in the mutual CURB set is not realized in any discovery process with Markov best responses. However, by Lemma 2, since every mutual CURB set is a CURB set on the realizable action set, it contradicts. Therefore, since every action profile on the mutual CURB set is realizable, in some discovery process with Markov best responses the set is a common CURB set. ■

By Lemma 1, we can deduce the following corollary.

Corollary 1. Suppose a normal-form game with unawareness has a mutual CURB set. Then, there exists a discovery process with Markov best responses converging to some discovered game possessing a cognitively stable generalized Nash equilibrium.

Example 1. Consider the following game played by Alice (A) and Bob (B):

$v^0 =$

A / B	b_1	b_2	b_3
a_1	3, 3	0, 5	0, -1
a_2	5, 0	1, 1	1, 0
a_3	-1, 0	0, 1	2, 2

, $v^1 =$

A / B	b_2	b_3
a_1	0, 5	0, -1
a_3	0, 1	2, 2

, $v^2 =$

A / B	b_1	b_3
a_1	3, 3	0, -1
a_2	5, 0	1, 0

At $\lambda = 1$, suppose that $T_A^{\lambda=1} = \{t_A^{(\lambda=1)*}, t_A^{(\lambda=1)}\}$ and $T_B^{\lambda=1} = \{t_B^{(\lambda=1)*}, t_B^{(\lambda=1)}\}$ such that:

$v_A(t_A^{(\lambda=1)*}) = v^1$ and $b_A(t_A^{(\lambda=1)*}) = t_B^{(\lambda=1)}$;
$v_A(t_A^{(\lambda=1)}) = v^2$ and $b_A(t_A^{(\lambda=1)}) = t_B^{(\lambda=1)*}$;
$v_B(t_B^{(\lambda=1)*}) = v^2$ and $b_B(t_B^{(\lambda=1)*}) = t_A^{(\lambda=1)}$; and
$v_B(t_B^{(\lambda=1)}) = v^1$ and $b_B(t_B^{(\lambda=1)}) = t_A^{(\lambda=1)*}$.

The formulation at $\lambda = 1$ is Figure 3 (a). Then, the realizable action set is as follows:

$\hat{v} =$

A / B	b_1	b_3
a_1	3, 3	0, -1
a_3	-1, 0	2, 2

In $\hat{v}$, there are three CURB sets: $X_1 = \{a_1\} \times \{b_1\}$, $X_2 = \{a_2\} \times \{b_2\}$ and $X_3 = \{a_2, a_3\} \times \{b_2, b_3\}$. Moreover, X_2 is a mutual CURB set.

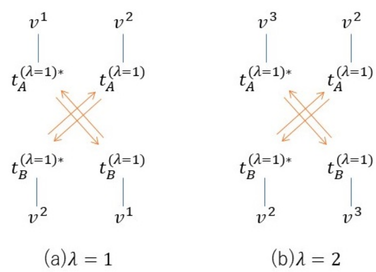


Figure 3: Formulation of example of two players, Alice and Bob

At $\lambda = 1$, suppose that both players play a generalized Nash equilibrium:

$$s_1^{(\lambda=1)*} = ([s_A(t_A^{(\lambda=1)*}) = a_1, s_A(t_A^{(\lambda=1)}) = a_2], [s_B(t_B^{(\lambda=1)*}) = b_1, s_B(t_B^{(\lambda=1)}) = b_2]).$$

The generalized Nash equilibrium is obviously cognitively unstable and the objective outcome is $(s_A(t_A^{(\lambda=1)*}), s_B(t_B^{(\lambda=1)*})) = (a_1, b_1)$. Then, since Alice is unaware of b_1, she is surprised by it and revises her beliefs as follows:

$v^3 =$

A / B	b_1	b_2	b_3
a_1	3, 3	0, 5	0, -1
a_3	-1, 0	0, 1	2, 2

Then, at $\lambda = 2$, suppose that $T_A^{\lambda=2} = \{t_A^{(\lambda=2)*}, t_A^{(\lambda=2)}\}$ and $T_B^{\lambda=2} = \{t_B^{(\lambda=2)*}, t_B^{(\lambda=2)}\}$ such that:

$v_A(t_A^{(\lambda=2)*}) = v^3$ and $b_A(t_A^{(\lambda=2)*}) = t_B^{(\lambda=2)}$;
$v_A(t_A^{(\lambda=2)}) = v^2$ and $b_A(t_A^{(\lambda=2)}) = t_B^{(\lambda=2)*}$;
$v_B(t_B^{(\lambda=2)*}) = v^2$ and $b_B(t_B^{(\lambda=2)*}) = t_A^{(\lambda=2)}$; and
$v_B(t_B^{(\lambda=2)}) = v^3$ and $b_B(t_B^{(\lambda=2)}) = t_A^{(\lambda=2)*}$.

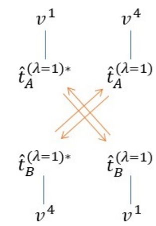
The formulation playing $s_1^{(\lambda=1)*}$ is depicted as Figure 3 (b). At $\lambda = 2$, when they play a Markov best response, the generalized strategy profile is:

$$s_1^{(\lambda=2)*} = ([s_A(t_A^{(\lambda=2)*}) = a_1, s_A(t_A^{(\lambda=2)}) = a_1], [s_B(t_B^{(\lambda=2)*}) = b_1, s_B(t_B^{(\lambda=2)}) = b_1]).$$

Although there is no generalized Nash equilibrium in the discovered game, the objective outcome $(s_A(t_A^{(\lambda=2)*}), s_B(t_B^{(\lambda=2)*})) = (a_1, b_1)$ satisfies $(s_A(t_A^{(\lambda=2)*}), s_B(t_B^{(\lambda=2)*})) \in X_1$ and X_1 is a CURB set in the realizable action set in the discovered game. This case suggests Theorem 1.

Next, suppose that both players implement a different generalized Nash equilibrium at $\lambda = 1$ as follows:

$$s_2^{(\lambda=1)*} = ([s_A(t_A^{(\lambda=1)*}) = a_3, s_A(t_A^{(\lambda=1)}) = a_2], [s_B(t_B^{(\lambda=1)*}) = b_3, s_B(t_B^{(\lambda=1)}) = b_3]).$$



(b')$\lambda = 2$

Figure 4: Formulation of example of two players, Alice and Bob (continued.)

The generalized Nash equilibrium is cognitively unstable and the objective outcome is $(s_A(t_A^{(\lambda=1)*}), s_B(t_B^{(\lambda=1)*})) = (a_3, b_3)$. Then, since Bob is unaware of a_3, he is surprised at it and revises his beliefstw as follows:

$v^4 =$

A / B	b_1	b_3
a_1	1, 1	0, 0
a_2	0, 0	1, 1
a_3	-1, 0	2, 2

Then, at $\lambda = 2$, suppose that $\hat{T}_A^{\lambda=2} = \{\hat{t}_A^{(\lambda=2)*}, \hat{t}_A^{(\lambda=2)}\}$ and $\hat{T}_B^{\lambda=2} = \{\hat{t}_B^{(\lambda=2)*}, \hat{t}_B^{(\lambda=2)}\}$ such that:

$v_A(\hat{t}_A^{(\lambda=2)*}) = v^1$ and $b_A(\hat{t}_A^{(\lambda=2)*}) = \hat{t}_B^{(\lambda=2)}$;
$v_A(\hat{t}_A^{(\lambda=2)}) = v^4$ and $b_A(\hat{t}_A^{(\lambda=2)}) = \hat{t}_B^{(\lambda=2)*}$;
$v_B(\hat{t}_B^{(\lambda=2)*}) = v^4$ and $b_B(\hat{t}_B^{(\lambda=2)*}) = \hat{t}_A^{(\lambda=2)}$; and
$v_B(\hat{t}_B^{(\lambda=2)}) = v^1$ and $b_B(\hat{t}_B^{(\lambda=2)}) = \hat{t}_A^{(\lambda=2)*}$.

Figure 4 (b') is a formulation at $\lambda = 2$ playing $s_2^{(\lambda=2)}$ in a previous situation. At $\lambda = 2$, when they play a Markov best response, the generalized strategy profile is:

$$s_2^{(\lambda=2)*} = ([s_A(\hat{t}_A^{(\lambda=2)*}) = a_3, s_A(\hat{t}_A^{(\lambda=2)}) = a_3], [s_B(\hat{t}_B^{(\lambda=2)*}) = b_3, s_B(\hat{t}_B^{(\lambda=2)}) = b_3]).$$

This is a cognitively stable generalized Nash equilibrium, the objective outcome $(s_A(\hat{t}_A^{(\lambda=2)*}), s_B(\hat{t}_B^{(\lambda=2)*})) = (a_3, b_3)$ satisfies $(s_A(\hat{t}_A^{(\lambda=2)*}), s_B(\hat{t}_B^{(\lambda=2)*})) \in X_2$ and X_2 is a common CURB set in the discovered game. In other words, the discovered game is a cognitively stable game under CURB. This case suggests Theorem 2. $\square$

In Example 1, although the initial game in Figure 3 (a) has mutual CURB set X_2, some discovery process with Markov best responses may converge to a discovered game, as in Figure 3 (b), which is not a common CURB set.

However, as shown by the proof of Theorem 2, a mutual CURB set in the initial game is a common CURB set in cognitively stable game under CURB to which some discovery process with Markov best responses converge. Therefore, we can deduce the following corollaries.

Corollary 2. Suppose that every CURB set on the realizable action set is a mutual CURB set in a normal-form game with unawareness. Then, every discovery process with Markov best responses converges to a cognitively stable game under CURB.

Corollary 3. Suppose that every CURB set on the realizable action set is a mutual CURB set in a normal-form game with unawareness. Then, every discovery process with Markov best responses converges to a discovered game possessing a cognitively stable generalized Nash equilibrium.

Note that the process we consider in the present paper starts from an arbitrary generalized strategy profile. Our convergence result holds even if the starting point is not necessarily a generalized Nash equilibrium.[11]

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