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Delay Dynamics in Nonlinear Monopoly with Gradient Adjustment

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Abstract

Two types of boundedly rational monopolies are considered, when they are unable to determine the profit maximizing output levels. In the first case the monopoly knows the price function and in the second case it can access only past output and price values. In applying gradient dynamics the marginal profit is either known or approximated by finite differences based on two past profit data. Stability conditions are derived first with discrete time scales, which are also applied in a special case. Two models of continuous dynamics are then introduced. The first is a natural modification of the discrete model, and the other includes an inertia coefficient with the derivative. In each case a delay differential equation is obtained with two delays. Stability conditions are derived and the stability switching curves are constructed and illustrated.

Keywords Gradient adjustment, Delay differential equation, Boundedly rational monopoly, Discrete-time and continuous-time dynamics, Stability switching curves

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1 Introduction

It is well-known that a monopolist in an elementary textbook of microeconomics is assumed to be *rational* in the sense that it has the perfect information on the market and instantaneous responses to changing circumstances. Accordingly, such a monopolist can choose the levels of price and output that maximizes its overall profit and can adjust its decisions in no time without any difficulties if some exogenous changes occur. It is also well-known that the decision-makers in the real world are *boundedly rational* and thus have to decide under limited information and delayed responses. We can say this behavioral difference in other words. The rational monopolist can jump to the optimal point of output and price in one shot without any adjustments. In consequence, output as well as price will not change over time (i.e., no dynamic consideration is necessary) unless environmental phenomenon changes. The boundedly rational monopolist, on the other hand, can make mistakes due to limited information. It might produce a different amount of output and set a different value of price other than the optimal ones. Noticing the mistake and revising the decision, it experiences time delays in collecting past data of price and output associated with uncertainty, information and implementation delays. Output (and price) will vary in every subsequent time period. The paper's main purpose is to shed light on such an adjustment or dynamic process of output of the boundedly rational monopolist.

In the existing literature, the gradient method is often adopted to describe the adjustment process of the boundedly rational monopolist toward the profit maximizing output. Accordingly, the monopolist increases the output level if its marginal profit is positive, decreases if negative and maintains the same output level if zero. Two types of models are known to introduce the method, discrete-time models and continuous-time models. It is demonstrated that the former could generate chaotic dynamics if the involved nonlinearities are strong enough. Among others, we mention Puu (1995) that follows Baumol and Quandt (1964) constructing a model of monopoly with a linear cost function and a cubic price function with inflection points. Naimzada and Ricchiuti (2008) replace Puu's price function with a cubic function having no inflection points. Askar (2013) assumes a general concave price function. Elsadany and Awad (2015) introduce a log-concave function. In a continuous-time framework, Matsumoto and Szidarovszky (2012, 2014) build a monopoly model, focusing on the effects caused by time delays and show the delay effect can be a source of complex dynamics as well as simple dynamics. In those studies, it is assumed that the form of the demand function could be known or estimated correctly by using the history of output and price.[1] In this study, we introduce two boundedly rational monopolists, one knows the form of the price function and the other knows only a few points on it. Further, neither of them knows the equilibrium output value and search for it based on the gradient method. We analytically and numerically consider how different amounts of information or knowledge on

[1] Even if the price function is known, it might be possible that a monopolist is endowed with limited computational skills to solve the profit maximization problem.

the price function affect their search behavior in discrete-time and continuous-time framework.

The rest of this paper is organized as follows. Section 2 builds a basic model. Section 3 considers the learning process in a discrete-time model. Section 4 constructs a continuous-time model from the discrete-time model via Euler approximation and then considers the same subject in a continuous-time framework. Section 5 investigates a continuous-time model with inertia. Finally, the concluding remarks and future research directions are given in Section 6.

2 Basic Model

Consider a monopoly that produces the quantity of output x. The price and production cost functions are continuously differentiable and are denoted, respectively, as $p(x)$ and $c(x)$. It is also assumed that $p(x)$ has a unique inverse. The profit function is defined to be

$$\pi(x) = p(x)x - c(x).$$

According to the textbook monopoly theory, the interior optimal output x_M satisfies the first-order condition of the profit-maximization,

$$\frac{d\pi(x)}{dx} = 0. \tag{1}$$

The monopoly price p_M is determined by the market-clearing condition for x,

$$p - p(x) = 0. \tag{2}$$

Assuming the strict concavity of the profit function, the rational monopolist determines unique equilibrium values, x_M and $p_M = p(x_M)$, through (1) and (2). It is also implicitly assumed that the rational monopolist has (i) full information on the form of the price function and (ii) enough computability to solve the two conditions, (1) and (2). Hence, the rational monopoly firm can jump to the equilibrium output and price with one-shot (i.e., no dynamics).

In this study, we relax either or both of information requirements (i) and (ii) and consider what happens if the monopolist sets its output at some level other than x_M. First, we introduce two different monopolists according to how much information they have. We call the monopoly the "*knowledgeable monopolist*" or the *k-monopolist* if it has requirement (i) but does not have requirement (ii)[2] and the "*limited monopolist*" or the *ℓ-monopolist* if it has neither of the requirements but possesses the values of x and p in the past two periods of time. It is implicitly assumed that the price is determined so as to clear the market even when x is different from the equilibrium (i.e., $p = p(x)$ for all x). Second, those monopolists with insufficient information adjust their outputs based on the observation of a profit change per a unit change of output. If the change is

[2] Clower (1959) calls a monopolist "knowledgeable monopolist" under the similar circumstance.

positive, negative or unchanged, then the output levels are increased, decreased or maintained at the same levels. Since the k-monopolist knows the analytic form of its profit, its profit change can be obtained by differentiating the profit function. If the change is denoted as $\Delta\pi/\Delta x$, then

$$\frac{\Delta\pi}{\Delta x} \simeq \frac{d\pi(x)}{dx}. \tag{3}$$

On the other hand, the ℓ-monopolist does not know the form of the profit function but can observe its profits at two different periods. The profit change at time t can be described by

$$\frac{\Delta\pi}{\Delta x} = \frac{\pi(x(t-\tau_1)) - \pi(x(t-\tau_2))}{x(t-\tau_1) - x(t-\tau_2)} \tag{4}$$

where $t-\tau_1$ and $t-\tau_2$ are earlier time periods with known profit values, so τ_1 and τ_2 are nonnegative integers and $\tau_1 < \tau_2$.

3 Discrete-time Dynamics

If discrete time scales are assumed and the earlier time periods are selected as close to t as possible (i.e., $\tau_1 = 1$ and $\tau_2 = 2$), then the k-monopolist adjusts its output in proportion to the marginal profit change,

$$x(t) = x(t-1) + K\frac{d\pi(x(t-1))}{dx(t-1)} \tag{5}$$

where this algorism is often called the gradient adjustment process. The ℓ-monopolist determines its output level, following the formulation,

$$x(t) = x(t-1) + K\frac{\pi(x(t-1)) - \pi(x(t-2))}{x(t-1) - x(t-2)} \tag{6}$$

that follows Puu (1995). In both equations, $K > 0$ is the adjustment coefficient. This section focuses on the discrete dynamics controlled by (5) and (6) and results to be obtained are extending Naimzada and Ricchiuti (2008), Matsumoto and Szidarovszky (2012) and Askar (2013).

The asymptotical behavior of nonlinear model (5) can be obtained by linearization around the steady state x_M. At the steady state $x(t) = x(t-1) = x_M$, we have the linear form of (5),

$$x(t) = x(t-1) + K\pi''(x_M)x(t-1) \tag{7}$$

where $\pi''(x_M)$ is the second derivative evaluated at $x = x_M$. In order to guarantee that the first order condition at x_M provides maximum, we make the following assumption,

Assumption 1. $\pi''(x_M) < 0$.

The eigenvalue of equation (7) is

$$\lambda = 1 - A$$

where

$$A = -K\pi''(x_M) > 0. \tag{8}$$

Since $\lambda < 1$ always, the stability condition for the k-monopolist is $\lambda > -1$ or

$$A < 2.$$

For the ℓ-monopolist,[3] we have the following from (6)

$$x(t) = x(t-1) + K\pi'(z). \tag{9}$$

with z being between $x(t-1)$ and $x(t-2)$. At the steady state, z has to be also x_M, therefore

$$x_M = x_M + K\pi'(x_M),$$

implying that x_M is a stationary point of the profit function. Notice that

$$\frac{\partial x(t)}{\partial x(t-1)} = 1 + K\frac{\pi'(x(t-1))\left[x(t-1) - x(t-2)\right] - \left[\pi(x(t-1)) - \pi(x(t-2))\right]}{\left(x(t-1) - x(t-2)\right)^2}$$

where the numerator can be written as

$$\pi'(x(t-1))\left(x(t-1) - x(t-2)\right) + \left[\pi'(x(t-1))\left(x(t-2) - x(t-1)\right) + \frac{\pi''(z)}{2}\left(x(t-2) - x(t-1)\right)^2\right]$$

where z is between $x(t-1)$ and $x(t-2)$. Therefore

$$\frac{\partial x(t)}{\partial x(t-1)} = 1 + \frac{K\pi''(z)}{2}.$$

At the equilibrium $z = x_M$, therefore this derivative becomes

$$\frac{\partial x(t)}{\partial x(t-1)} = 1 + \frac{K\pi''(x_M)}{2} = 1 - A/2.$$

Similarly, the other derivative at the equilibrium is

$$\frac{\partial x(t)}{\partial x(t-2)} = \frac{K\pi''(x_M)}{2} = -A/2.$$

The linearized equation of (6) becomes

$$x(t) = (1 - A/2)\,x(t-1) - (A/2)\,x(t-2) \tag{10}$$

[3] Notice that the ℓ-monopolist is unable to manipulate the following calculations since it does not know the form of the profit function.

with characteristic equation

$$\lambda^2 - (1 - A/2)\,\lambda + A/2 = 0. \tag{11}$$

The steady state is locally asymptotically stable if

$$\begin{gathered} 1 \pm (1 - A/2) + A/2 > 0 \\ A/2 < 1 \end{gathered} \tag{12}$$

which can be simplified as $0 < A < 2$.

Theorem 1 *The steady state of dynamic equations (5) for the k-monopolist and (6) for ℓ-monopolist is locally asymptotically stable if* $0 < A < 2$ *and locally unstable if* $A > 2$.

Next we adopt a general concave price function to confirm the analytical result by examining numerical examples,

$$p = a - bx^{\alpha},\ \alpha \in \mathbb{Z}^{+} \tag{13}$$

that is used in Askar (2013). With a marginal cost c,[4] the profit function is

$$\pi(x) = (p - c)x = (a - c)x - bx^{1+\alpha},$$

the first derivative is

$$\pi'(x) = a - c - (1 + \alpha)\,x^{\alpha}.$$

Hence the dynamic equation for the k-monopolist is obtained from (5),

$$x(t) = x(t-1) + K\left[a - c - (1 + \alpha)x^{\alpha}(t-1)\right] \tag{14}$$

Solving $\pi'(x) = 0$ presents the equilibrium value, x_M,

$$x_M = \left(\frac{a - c}{(1 + \alpha)b}\right)^{\frac{1}{\alpha}}. \tag{15}$$

The second derivative at the equilibrium point x_M is

$$\pi''(x_M) = -\alpha b(1 + \alpha)\left(\frac{a - c}{(1 + \alpha)b}\right)^{\frac{\alpha - 1}{\alpha}}.$$

Apparently the profit function satisfies the second-order condition for profit maximization due to Assumption 1. The stability condition for the k-monopolist is, according to Theorem 1,

$$A = -K\pi''(x_M) = K\alpha b(1 + \alpha)\left(\frac{a - c}{(1 + \alpha)b}\right)^{\frac{\alpha - 1}{\alpha}} < 2 \tag{16}$$

[4]It has been checked that the convex cost function $c(x) = x^{\beta},\ 2 \le \beta \le \alpha$ does not affect the general properties of the result to be obtained.

that is reduce to

$$bK < 1 \text{ if } \alpha = 1 \text{ or the price function is linear },$$

$$3bKx_M < 1 \text{ if } \alpha = 2 \text{ or the price function is quadratic,}$$

and

$$6bKx_M^2 < 1 \text{ if } \alpha = 3 \text{ or the price function is cubic.}$$

For the ℓ-monopolist, the profit change is given by

$$\pi(x(t-1)) - \pi(x(t-2)) = (a-c)\left[x(t-1) - x(t-2)\right] - b\left[x^{\alpha+1}(t-1) - x^{\alpha+1}(t-2)\right].$$

where the second bracketed term on the right side can be factorized as

$$[(x(t-1) - x(t-2)]\left(\sum_{k=0}^{\alpha} x^{\alpha-k}(t-1)x^k(t-2)\right)$$

Hence the adjustment process is rewritten as

$$x(t) = x(t-1) + K\left\{a - c - bg\left[x(t-1), x(t-2)\right]\right\} \tag{17}$$

where

$$g\left[x(t-1), x(t-2)\right] = \sum_{k=0}^{\alpha} x^{\alpha-k}(t-1)x^k(t-2).$$

The corresponding linearized equation is

$$x(t) = \left[\left(1 - bK \left.\frac{\partial g}{\partial x(t-1)}\right|_{x(t-1)=x_M}\right)x(t-1) - bK\left.\frac{\partial g}{\partial x(t-2)}\right|_{x(t-2)=x_M} x(t-2)\right] \tag{18}$$

where

$$\left.\frac{\partial g}{\partial x(t-1)}\right|_{x(t-1)=x_M} = \left.\frac{\partial g}{\partial x(t-2)}\right|_{x(t-2)=x_M} = \frac{(1+\alpha)\alpha}{2}x_M^{\alpha-1}.$$

The linear equation is reduced to

$$x(t) - (1 - bK)\,x(t-1) + bKx(t-2) = 0 \text{ if } \alpha = 1,$$

$$x(t) - (1 - 3bKx_M)\,x(t-1) + 3bKx_M x(t-2) = 0 \text{ if } \alpha = 2$$

and

$$x(t) - \left(1 + 6bKx_M^2\right)x(t-1) + 6bKx_M^2 x(t-2) = 0 \text{ if } \alpha = 3.$$

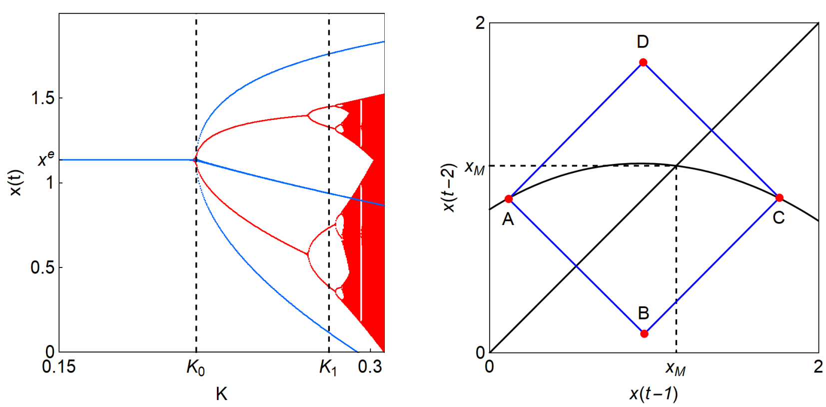
In consequence, the stability condition for the adjustment process for the ℓ-monopolist is

$$bK < 1 \text{ if } \alpha = 1,\ 3bKx_M < 1 \text{ if } \alpha = 2 \text{ and } 6bKx_M^2 < 1 \text{ if } \alpha = 3.$$

Notice that the stability conditions for the k-monopolist and the ℓ-monopolist are identical for $\alpha = 1, 2, 3$. This result is confirmed for any integer α as stated in Theorem 1. In the following numerical examples, since we repeatedly use the parameter values that Naimzada and Ricchiuti (2008) take, we make it as an assumption,

Assumption 2. $a = 4,\ b = 0.6$ and $c = 0.5$.

Figure 1(A) illustrates the two bifurcation diagrams for K with $\alpha = 3$ and Assumption 2. The red one is for the the k-monopolist and the blue one is for the ℓ-monopolist. It is observed that for $K < K_0 = 6bKx_M^2 \simeq 0.216$, both systems are locally asymptotically stable. The stability is lost for $K = K_0$ and bifurcates to a periodic cycle for $K > K_0$ via a Hopf bifurcation. The red diagram has a somewhat familiar shape, and thus the adjustment process for the k-monopolist gives rise to chaotic oscillations via a period-doubling cascade. On the other hand, the blue diagram indicates that for $K > K_0$, the adjustment process for the ℓ-monopolist can generate only a periodic cycle. Further, its lower branch becomes negative for a larger value of K, losing economic meanings. It is worth mentioning that the ℓ-monopolist can arrive at the steady state if it adjusts its output level cautiously with a small value of the adjustment coefficient, although it has limited information only on prices and output levels in the last two periods. In Figure 1(B), the phase diagram for the ℓ-monopolist is plotted for $K_1 = 0.28$ and shows that there is a period-4 cycle. After 5000 iterations of equation (17), the ordinates of points A and C take very close values, 0.937 and 0.938. As a result, it seems that the blue diagram has only three branches for $K > K_0$ in Figure 1(A). However, the middle branch actually consists of two similar curves. The vertical dotted line at $K = 0.28$ crosses the blue diagram four times, although it looks like crossing three times.



(A) Bifurcation diagrams (B) Period-4 cycle

Figure 1. Dynamics generated by (14) and (17)

We now extend our analysis to the case in which only the mth-latest information ($\tau_1 = m$) and the nth-latest information ($\tau_2 = n$) are available for $n > m > 0$ and m and n being coprime intergers. In consequence, the output adjustment process is

$$x(t) = x(t-n) + K\frac{\pi(x(t-m)) - \pi(x(t-n))}{x(t-m) - x(t-n)} \tag{19}$$

where x_M is the steady state. Its linear version is an nth-order difference equation,

$$x(t) = (1 - A/2)\, x(t - m) + (-A/2)\, x(t - n).$$

If $0 < A < 2$, then

$$|1 - A/2| + |-A/2| = 1.$$

Under this special condition, according to Corollary 3.1 of Čermák and Jánský (2015), the steady state of (19) is asymptotically stable if

$$(1 - A/2)^n\, (-A/2)^m < 0 \text{ for any } m \text{ and } n. \tag{20}$$

Since $1 - A/2 > 0$ and $-A/2 < 0$, it is clear that the inequality holds if m is odd. Therefore, when we take $m = 1$, the stability condition for the higher-order difference equation of (19) is given as follows:

Theorem 2 *If $0 < A < 2$, then the steady state of the nth-order difference equation of (19) is locally asymptotically stable for $m = 1$ and any $n \geq 2$,*

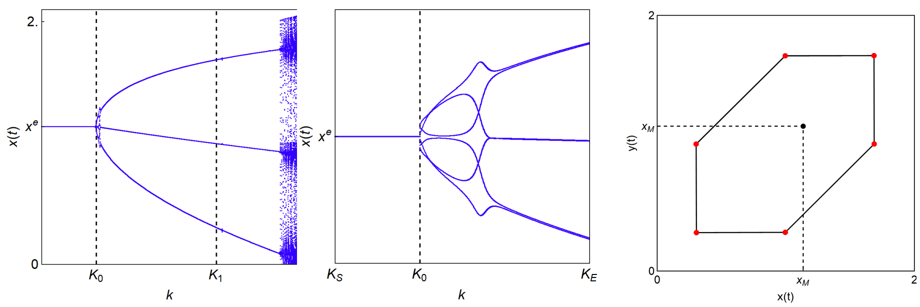
Equation (19) with $m = 1$ and $n = 3$ is cubic,

$$x(t) = x(t - 1) + K \{a - c - bg\, [x(t - 1), x(t - 3)]\} \tag{21}$$

where $g\, [x(t - 1), x(t - 3)]$ has the form,

$$x^3(t - 1) + x^2(t - 1)x(t - 3) + x(t - 1)x^2(t - 3) + x^3(t - 3).$$

Applying the traditional necessary and sufficient stability conditions shown by Farebrother (1973) leads to $0 < A < 2$ that confirms the stability condition of (20). Under Assumption 2, Figure 2(A) presents the bifurcation diagram with respect to $K \in [K_0 - 0.02, 0.29]$, showing the birth of a periodic-cyle after K_0 and complicated dynamics for K close to 0.29. However, the lower part could be negative with no economic meaning. To see the behavior in the neighborhood of K_0, we enlarge the diagram in a small interval $[K_S, K_E]$ with $K_S = K_0 - 0.002$ and $K_E = K_0 + 0.002$. It is observed that equation (21) gives rise to a period-6 cycle just after losing stability. As the value of K increases, the upper two, the middle two and the lower two branches are getting closer, giving the wrong image that they merge to one. After 5000 iterations of equation (19), Figure 2(C) depicts a phase portrait for $K_1 = 0.26$ that is still a period-6 cycle where the ordinates of the higher, middle and lower points are, respectively, the same. Returning to Figure 2(A), the bifurcation diagram indicates that it generates a period-3 cycle after K_0 but in fact, it is a period-6 cycle.



(A) Bifurcation (2) Enlargement (C) Phase portrate

Figure 2. Dynamics of the 3rd-order difference equation

4 Continuous-time Dynamics I

Introducing time delays, we transform the discrete-time models to continuous-time models through Euler approximation (i.e., $\dot{x}(t) \simeq x(t) - x(t-1)$). For the k-monopolist, equation (5) is modified as

$$\dot{x}(t) = K \frac{d\pi(x(t-\tau))}{dx(t-\tau)} \tag{22}$$

and equation (6) for the ℓ-monopolist is converted to

$$\dot{x}(t) = K \frac{\pi(x(t-\tau_1)) - \pi(x(t-\tau_2))}{x(t-\tau_1) - x(t-\tau_2)} \tag{23}$$

where $\tau > 0$, $\tau_1 > 0$ and $\tau_2 > 0$ with $0 \leq \tau_1 < \tau_2$ denote time delays in continuous-time scales. The steady state of these equations are the same as that in the difference equations (5) and (6).

We first consider the k-monopolist case in which a linear version of delay differential equation (22) is

$$\dot{x}(t) = -Ax(t-\tau) \tag{24}$$

where A is defined in equation (8). Substituting an exponential solution $x(t) = e^{\lambda t}u$ into (24) presents a characteristic equation,

$$\lambda + Ae^{-\lambda t} = 0. \tag{25}$$

Since $\lambda = 0$ does not solve equation (25), we suppose a pure imaginary solution, $\lambda = i\omega$, $\omega > 0$ that separates the characteristic equation to the real and imaginary parts,

$$A\cos\omega\tau = 0,$$

$$\omega - A\sin\omega\tau = 0.$$

The first equation implies $\cos\omega\tau = 0$ and the second $\sin\omega\tau = \omega/A = 1$. Hence the critical values of delay are

$$\tau_m = \frac{1}{\omega}\left(\frac{\pi}{2} + 2m\pi\right) \text{ for } m \in \mathbb{Z}^+ \text{ and } \omega = A. \tag{26}$$

It is apparent from (24) that the steady state is locally asymptotically stable for $\tau = 0$. The results are summarized as follows:

Theorem 3 *The steady state of delay difference equation (22) is locally asymptotically stable for $\tau < \tau_0$, loses its stability at $\tau = \tau_0$ and then is replaced with periodic oscillations via a Hopf bifurcation for $\tau > \tau_0$ where*

$$\tau_0 = \frac{\pi}{2A} > 0.$$

We now draw attention to equation (23), the delay output adjustment equation of the ℓ-monopolist. Similar to the discrete case, at the steady state,

$$\frac{\partial \dot{x}(t)}{\partial x(t-\tau_1)} = \frac{K\pi''(x^*)}{2} = -\frac{A}{2}$$

and

$$\frac{\partial \dot{x}(t)}{\partial x(t-\tau_2)} = \frac{K\pi''(x^*)}{2} = -\frac{A}{2}.$$

Hence the linearized model has the form,

$$\dot{x}(t) = -\frac{A}{2}x(t-\tau_1) - \frac{A}{2}x(t-\tau_2). \tag{27}$$

with characteristic equation

$$\lambda + \frac{A}{2}e^{-\lambda\tau_1} + \frac{A}{2}e^{-\lambda\tau_2} = 0. \tag{28}$$

Case 0: $\tau_1 = 0,\ \tau_2 = 0$.

Consider first the no-delay case. The characteristic equation (28) with $\tau_1 = \tau_2 = 0$ is

$$\lambda = -A.$$

The steady state is locally asymptotically stable since $A > 0$.

Case 1: $\tau_1 = 0,\ \tau_2 > 0$

Consider now the special case of $\tau_1 = 0$. Then equation (28) is reduced to a one-delay equation,

$$\lambda + \frac{A}{2} + \frac{A}{2}e^{-\lambda\tau_2} = 0. \tag{29}$$

As is already seen, the steady state is locally asymptotically stable at $\tau_2 = 0$. As the value of τ_2 increases, stability might be lost, when $\lambda = i\omega$ ($\omega > 0$). Assuming positive value of ω does not restrict generality, since if λ is an eigenvalue, then its complex conjugate is also an eigenvalue. Substituting this value of λ into equation (29), we have

$$i\omega + \frac{A}{2} + \frac{A}{2}(\cos\omega\tau_2 - i\sin\omega\tau_2) = 0.$$

The separation of the real and imaginary parts gives

$$A + A\cos\omega\tau_2 = 0,$$

$$2\omega - A\sin\omega\tau_2 = 0.$$

The first equation implies that $\cos\omega\tau_2 = -1$, so $\sin\omega\tau_2 = 0$ which contradicts the second equation. Therefore there is no stability switch.[5]

Theorem 4 *If $\tau_1 = 0$, then the steady state is locally asymptotically stable with all $\tau_2 > 0$.*

Case 2: $\tau_1 > 0,\ \tau_2 > 0$

In the general case of $\tau_1 > 0$ and $\tau_2 > 0$, equation (28) can be rewritten as

$$P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1} + P_2(\lambda)e^{-\lambda\tau_2} = 0 \tag{30}$$

with

$$P_0(\lambda) = \lambda \text{ and } P_1(\lambda) = P_2(\lambda) = A/2.$$

Before looking for stability switching curves, the following conditions should be verified (Gu et al. (2005)):

(i) $\deg[P_0(\lambda)] \geq \max\{\deg[P_1(\lambda)], \deg[P_2(\lambda)]\}$.

(ii) $P_0(0) + P_1(0) + P_2(0) \neq 0$.

(iii) The polynomials $P_0(\lambda),\ P_1(\lambda)$ and $P_2(\lambda)$ do not have common roots.

(iv) $\lim_{\lambda\to\infty}\left(\left|\frac{P_1(\lambda)}{P_0(\lambda)}\right| + \left|\frac{P_2(\lambda)}{P_0(\lambda)}\right|\right) < 1$.

[5] Mathematically, we have the same result if $\tau_1 > 0$ and $\tau_2 = 0$. However this symmetric case is assumed away by assumption $\tau_1 < \tau_2$.

Equation (30) satisfies these conditions. Since the followings hold,

$$\deg\left[P_0(\lambda)\right] = 1 \text{ and } \deg\left[P_1(\lambda)\right] = \deg\left[P_2(\lambda)\right] = 0,$$

condition (i) is satisfied. Condition (ii) is satisfied as $P_0(0) + P_1(0) + P_2(0) = A \neq 0$. Condition (iii) is apparently satisfied as $P_0(\lambda),\ P_1(\lambda)$ and $P_2(\lambda)$ have no common roots. Condition (iv) also holds, since

$$\left(\left| \frac{P_1(\lambda)}{P_0(\lambda)} \right| + \left| \frac{P_2(\lambda)}{P_0(\lambda)} \right| \right) = \frac{A}{|\lambda|} \to 0 \text{ as } \lambda \to \infty.$$

Dividing equation (30) by λ, we get

$$1 + a_1(\lambda)e^{-\lambda\tau_1} + a_2(\lambda)e^{-\lambda\tau_2} = 0, \tag{31}$$

where new functions are

$$a_1(\lambda) = \frac{A}{2\lambda} \text{ and } a_2(\lambda) = \frac{A}{2\lambda}.$$

We examine the stability switches of the non-trivial solution of dynamic equation (27) as the delays (τ_1, τ_2) vary. The modified characteristic equation (31) must have a pair of pure conjugate imaginary roots and stability switch occurs for the corresponding critical delays. So let $\lambda = i\omega,\ \omega > 0$ and substitute it into equation (31),

$$1 + a_1(i\omega)e^{-i\omega\tau_1} + a_2(i\omega)e^{-i\omega\tau_2} = 0 \tag{32}$$

where

$$a_1(i\omega) = a_2(i\omega) = -i\frac{A}{2\omega}.$$

We now solve equation (32). To this purpose, we treat the three terms in the left hand side of equation (32) as three vectors in the complex plane with the magnitudes, 1, $|a_1(i\omega)|$ and $|a_2(i\omega)|$ where the absolute values are

$$|a_1(i\omega)| = |a_2(i\omega)| = \frac{A}{2\omega}.$$

The right hand side of equation (32) is zero, implying that if we put these vectors head to tail, then they form a triangle as illustrated in Figure 3. Similar triangle can be formed under the real axis. Since the sum of lengths of any two line segments is not shorter than that of the remaining line segment in a triangle, these absolute values satisfy the following inequality conditions

$$1 \leq |a_1(i\omega)| + |a_2(i\omega)|, \tag{33}$$

and

$$-1 \leq |a_1(i\omega)| - |a_2(i\omega)| \leq 1 \tag{34}$$

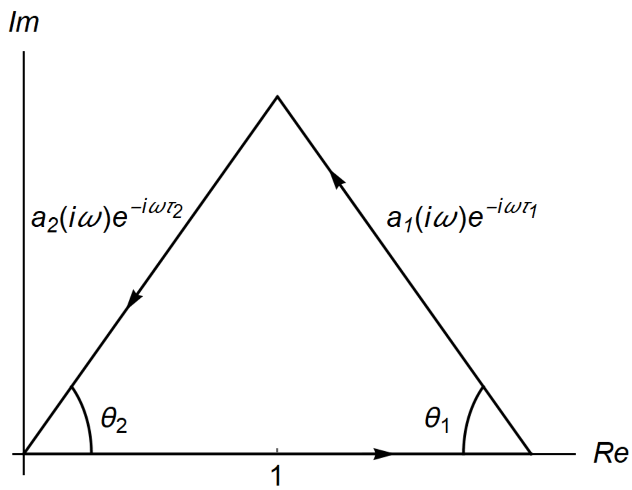


Figure 3. Triangle representation of equation (32)

Relation (34) is clearly satisfied, and (33) requires that

$$\frac{A}{\omega} \geq 1 \text{ or } 0 < \omega \leq A. \tag{35}$$

By using the cosine rule,

$$\begin{aligned} \theta_1 = \theta_2 \quad &= \cos^{-1}\left(\frac{1 + |a_1(i\omega)|^2 - |a_2(i\omega)|^2}{2\,|a_1(i\omega)|}\right) \\ &= \cos^{-1}\left(\frac{\omega}{A}\right). \end{aligned} \tag{36}$$

Notice also that

$$\theta_1 = \theta_2 \in \left(0, \frac{\pi}{2}\right)$$

and

$$\arg[a_1(i\omega)] = \arg\left[a_2(i\omega)\right] = \arg\left[-i\frac{A}{2\omega}\right] = \frac{3\pi}{2}.$$

Hence the angle balance relations at points $(0,0)$ and $(1,0)$ imply that the stability switching curves are given as

$$\tau^{\pm}_{1,m}(\omega) = \frac{1}{\omega}\left[\frac{3\pi}{2} + (2m-1)\,\pi \pm \theta_1(\omega)\right] \tag{37}$$

and

$$\tau^{\mp}_{2,n}(\omega,) = \frac{1}{\omega}\left[\frac{3\pi}{2} + (2n-1)\,\pi \mp \theta_2(\omega)\right] \tag{38}$$

where both τ_1 and τ_2 are positive with all nonnegative integer values of m and n. The results obtained are summarized in the following:

Theorem 5 *The stability switching curve consists of the following segments,*

$$SW^1_{m,n}(\omega) = \left\{ \left(\tau^+_{1,m}(\omega), \tau^-_{2,n}(\omega,)\right) \;\middle|\; \omega \in [0, A],\ (m,n) \in \mathbb{Z} \right\},$$
$$SW^2_{m,n}(\omega) = \left\{ \left(\tau^-_{1,m}(\omega), \tau^+_{2,n}(\omega,)\right) \;\middle|\; \omega \in [0, A],\ (m,n) \in \mathbb{Z} \right\}.$$

Using numerical example with Assumption 2, we visualize the analytical results. With the concave price function (13) with $\alpha = 3$, the dynamic equation (30) of the k-monopolist is modified as

$$\dot{x}(t) = K\left[a - c - 4bx^3(t-\tau)\right] \tag{39}$$

and its linear version is

$$\dot{x}(t) = -Ax(t-\tau).$$

Suppose that the linear equation has an exponential solution $e^{\lambda t}$ and $\lambda = i\omega$ with $\omega > 0$. Then the linear equation is separated to the real and imaginary parts,

$$A\cos\omega\tau = 0,$$

$$\omega - A\sin\omega\tau = 0.$$

The first equation implies $\cos\omega\tau = 0$ and the second $\sin\omega\tau = \omega/A = 1$. Hence the critical values of delay are

$$\tau_m(\omega) = \frac{1}{2\omega}\left(\frac{\pi}{2} + 2m\pi\right) \text{ for } m \in \mathbb{Z}^+ \text{ and } \omega = A.$$

The output dynamic equation for the ℓ-monopolist is

$$\dot{x}(t) = K\left\{a - c - bg\left[x(t-\tau_1), x(t-\tau_2)\right]\right\} \tag{40}$$

where $g\left[x(t-\tau_1), x(t-\tau_2)\right]$ has the form,

$$x^3(t-\tau_1) + x^2(t-\tau_1)x(t-\tau_2) + x(t-\tau_1)x^2(t-\tau_2) + x^3(t-\tau_2).$$

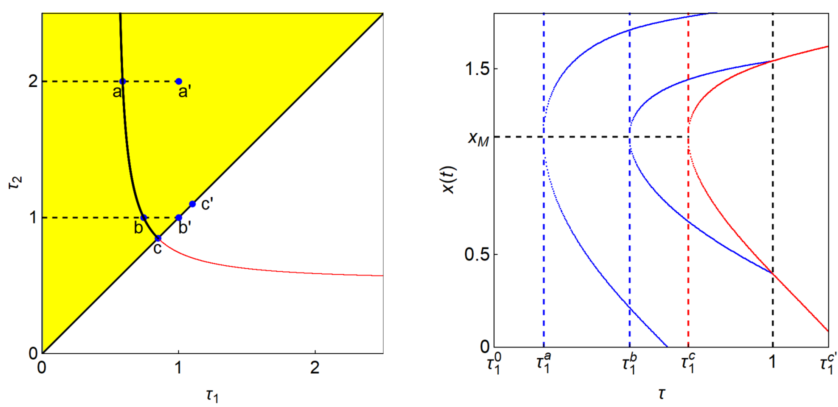
Notice that x_M is the steady state of equation (40). Some stability switching curves are illustrated in Figure 4(A) for $K = 0.2$. The feasible region satisfying $\tau_2 \geq \tau_1$ is in yellow and divided into two subregions by the black segment $SW^1_{0,0}(\omega)$. The steady state is stable to the left and unstable to the right. The red segment in the white region is the loci of $SW^2_{0,0}(\omega)$. Any other segments are located outside of Figure 4(A). Three bifurcation diagrams are illustrated in Figure 4(B) in which $x(t)$ is plotted against the parameter τ_1. For the left-most blue diagram, τ_2 is fixed at 2 and τ_1 is increased from 0 to $\tau_1^{a'} = 1$. In Figure 4(A), the horizontal line at $\tau_2 = 2$ crosses the black stability curve at point a where its abscissa is $\tau_1^a \simeq 0.589$.[6] After τ_1^a, the steady state is no longer stable and replaced with a limit cycle whose amplitude gets larger as τ_1 is increased. As a natural consequence, the lower branch of the cycle becomes negative and

[6] From (37) and (38), solving $2 = \tau^+_{2,0}(\omega)$ gices $\omega_a \simeq 1.214$ that is substituted into $\tau^-_{1,0}(\omega_a) \simeq 0.589$.

loses its economic meaning. For the middle bifurcation diagram, the fixed value of τ_2 is decreased to 1 from 2 and τ_1 is increased to $\tau_1^{b'} = 1$. A threshold value is changed to $\tau_1^b \simeq 0.743$. The dynamic equation (39) gives rise to the red bifurcation diagram for $\tau \in [0, \tau_1^{a'}]$ with $\tau_1^{a'} = 1.2$ and a threshold value $\tau_1^c \simeq 0.848$. By the formulations of the profit change per a unit output change in (22) and (23), we have

$$\lim_{\tau_1 \to \tau_2} \frac{\pi(x(t-\tau_1)) - \pi(x(t-\tau_2))}{x(t-\tau_1) - x(t-\tau_2)} = \frac{d\pi(x(t-\tau))}{dx(t-\tau)} \text{ for } \tau = \tau_1 = \tau_2.$$

It is also confirmed that equation (40) is reduced to equation (39) when $\tau_1 = \tau_2 = \tau$. Hence the k-monopolist's behavior can be described by increasing the value of τ_1 along the diagonal in Figure 4(A). The middle blue diagram arrives at the lower and upper branches of the red diagram at $\tau_1 = 1 = \tau_2$ at which the dynamic equation of the ℓ-monopolist is identical with the dynamic equation of k-monopolist.



(A) Feasible stable region (B) Bifurcation diagrams

Figure 4. Dynamics generated by (39) and (40)

5 Continuous-time Dynamics II

In this section, for analytical simplicity (at the expense of generality), we impose Assumption 2 and assume a concave price function from the beginning,

$$p = a - bx^3.$$

The output adjustment equation for the k-monopolist in discrete-time scales with the delay length τ is

$$x(t) = x(t-\tau) + K\left[a - c - 4bx^3(t-\tau)\right]. \tag{41}$$

There are many ways to transform a discrete-time model to a continuous-time model. Euler approximation is one of them. In this section, applying Berezowski transformation,[7] we convert equation (41) to the following continuous-time equation,

$$\delta \dot{x}(t) + x(t) = x(t-\tau) + K\left[a - c - 4bx^3(t-\tau)\right] \tag{42}$$

where δ is the inertia coefficient. It is clearly seen that for $\delta = 0$, equation (42) reduces identically to the original discrete-time equation, (41). The steady state x_M in (41) is also a steady state of the new equation.

To proceed to the stability consideration, we linearize equation (42) in the neighborhood of x_M to obtain

$$\delta \dot{x}(t) + x(t) - (1-A)\, x(t-\tau) = 0 \tag{43}$$

where $A = -K\pi''(x_M)$ is given by

$$A = 12bKx_M^2 > 0.$$

The corresponding characteristic equation for $x(t) = e^{\lambda t}u$ is

$$\delta\lambda + 1 - (1-A)\, e^{-\lambda\tau} = 0.$$

When λ is pure imaginary (i.e., $\lambda = i\omega,\ \omega > 0$), the characteristic equation can be separated into the real and imaginary parts,

$$\begin{aligned} 1 - (1-A)\cos\omega\tau &= 0, \\ \delta\omega + (1-A)\sin\omega\tau &= 0 \end{aligned} \tag{44}$$

Moving the constants of both equations in (44) to the right side, squaring them, adding them and then solving the resultant equation for ω give a positive solution,

$$\omega_+ = \frac{\sqrt{(A-2)A}}{\delta} > 0 \text{ for } A > 2.$$

Hence if $A \leq 2$, then no stability switch occurs, otherwise, the stability might switch for the critical delay value,

$$\tau_m = \frac{1}{\omega_+}\left[\cos^{-1}\left(\frac{1}{1-A}\right) + 2m\pi\right] \text{ for } m = 0, 1, 2, ... \tag{45}$$

The stability of x_M in (42) is summarized as follows:

Theorem 6 *The steady state of equation (42) with $A > 2$ is locally stable for $\tau < \tau_0$, loses stability for $\tau = \tau_0$ and bifurcates to a oscillatory cycle via a Hopf bifurcation for $\tau > \tau_0$ where from (45) with $m = 0$,*

$$\tau_0 = \frac{\delta}{\sqrt{(A-2)A}} \cos^{-1}\left(\frac{1}{1-A}\right).$$

[7] See Berezowski (2001) for more details.

With additional specification of $K = 0.3$ and $\delta = 0.2$, the bifurcation diagram given in Figure 5 confirms the analytical results in Theorem 6. Further, it indicates the birth of chaotic oscillations for larger values of τ through an àla period-doubling cascade.

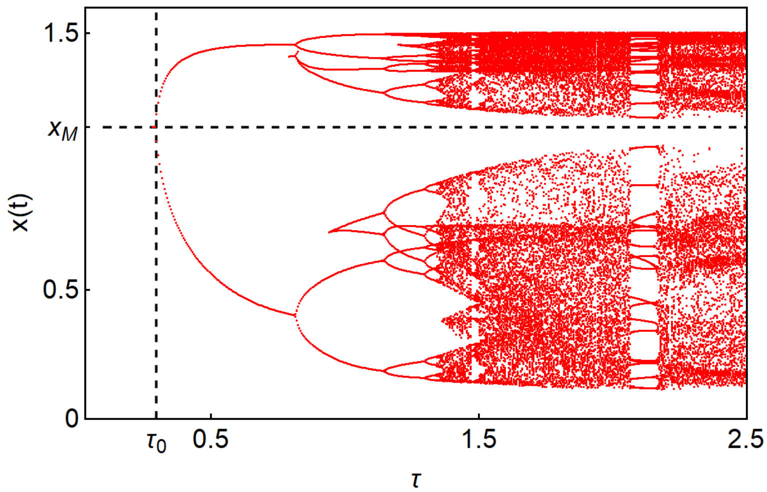


Figure 5. Bifurcation diagram for τ generated by equation (30)

Berezowski transformation of the discrete-time equation for the ℓ-monopolist can be formulated from (17). First, we rewrite it as

$$x(t) = x(t - \tau_1) + K \{a - c - bg[x(t - \tau_1), x(t - \tau_2)]\} \tag{46}$$

where, for $\alpha = 3$, $g[x(t - \tau_1), x(t - \tau_2)]$ has the form of

$$x^3(t - \tau_1) + x^2(t - \tau_1)x(t - \tau_2) + x(t - \tau_1)x^2(t - \tau_2) + x^3(t - \tau_2).$$

Replacing the left-hand side of (46) by $\delta\dot{x}(t) + x(t)$, we have a continuous-time output adjustment equation in Berezowski's sense,

$$\delta\dot{x}(t) = -x(t) + x(t - \tau_1) + K \{a - c - bg[x(t - \tau_1), x(t - \tau_2)]\} . \tag{47}$$

This is a nonlinear delay differential equation, which is reduced to the difference equation (46) if $\delta = 0$. Its linearized version is

$$\delta\dot{x}(t) = -x(t) + \left(1 - \frac{A}{2}\right) x(t - \tau_1) - \frac{A}{2} x(t - \tau_2) \tag{48}$$

and its characteristic equation for $x(t) = e^{\lambda t}u$ is

$$2\delta\lambda + 2 - (2 - A)e^{-\lambda\tau_1} + Ae^{-\lambda\tau_2} = 0.$$

Dividing each term by $2(1+\delta\lambda)$, we get again equation (32) with

$$a_1(\lambda) = \frac{A-2}{2(1+\delta\lambda)} \text{ and } a_2(\lambda) = \frac{A}{2(1+\delta\lambda)}.$$

Hence

$$a_1(i\omega) = \frac{A-2}{2(1+i\omega\delta)} = \frac{A-2}{2\left(1+(\delta\omega)^2\right)} - i\frac{\delta\omega\,(A-2)}{2\left(1+(\delta\omega)^2\right)}$$

and

$$a_2(i\omega) = \frac{A}{2(1+i\omega\delta)} = \frac{A}{2\left(1+(\delta\omega)^2\right)} - i\frac{\delta\omega A}{2\left(1+(\delta\omega)^2\right)}$$

implying that

$$|a_1(i\omega)|^2 = \frac{(A-2)^2}{4\left[1+(\delta\omega)^2\right]}$$

and

$$|a_2(i\omega)|^2 = \frac{A^2}{4\left[1+(\delta\omega)^2\right]}.$$

The triangle conditions (33) and (34) have the forms

$$\frac{|A-2|}{2\sqrt{1+(\delta\omega)^2}} + \frac{A}{2\sqrt{1+(\delta\omega)^2}} \geq 1 \tag{49}$$

and

$$-1 \leq \frac{|A-2|-A}{2\sqrt{1+(\delta\omega)^2}} \leq 1 \tag{50}$$

Now we have to consider two cases:

(i) $0 < A \leq 2$

In this case, the left-hand side of (49) gives

$$|a_1(i\omega)| + |a_2(i\omega)| = \frac{1}{\sqrt{1+(\delta\omega)^2}} < 1.$$

The last inequality violates the direction of the inequality of (49), so there is no stability switch.

Theorem 7 *If $0 \leq A \leq 2$, then the steady state is locally asymptotically stable with all $\tau_1 \geq 0$ and $\tau_2 \geq 0$.*

(ii) $A > 2$

This condition is equivalent to $A - 2 > 0$, then the left-hand side of (49) gives

$$|a_1(i\omega)| + |a_2(i\omega)| = \frac{A-1}{\sqrt{1+(\delta\omega)^2}} \geq 1$$

or

$$\omega^2 \leq \frac{A(A-2)}{\delta^2}$$

showing that this condition holds if

$$0 < \omega \leq \omega_1 = \frac{1}{\delta}\sqrt{A(A-2)}. \tag{51}$$

Similarly, condition (50) has the form

$$-1 \leq \frac{-1}{\sqrt{1+(\delta\omega)^2}} \leq 1$$

which always holds.

Theorem 8 *If $A > 2$, then stability switch might occur with all ω values satisfying relation (51).*

Similarly to the previously discussed model, based on Figure 3, the application of the law of cosine presents,

$$\theta_1(\omega) = \cos^{-1}\left(\frac{1+(\delta\omega)^2-(A-1)}{|A-2|\sqrt{1+(\delta\omega)^2}}\right) \tag{52}$$

and

$$\theta_2(\omega) = \cos^{-1}\left(\frac{1+(\delta\omega)^2+(A-1)}{|A|\sqrt{1+(\delta\omega)^2}}\right). \tag{53}$$

Notice that $\theta_1 \in [0, \pi]$ and $\theta_2 \in [0, \pi/2]$, furthermore, the arguments of $a_1(i\omega)$ and $a_2(i\omega)$ are

$$\arg[a_1(i\omega)] = -\tan^{-1}(\delta\omega) + 2\pi$$

and

$$\arg[a_2(i\omega)] = -\tan^{-1}(\delta\omega) + 2\pi.$$

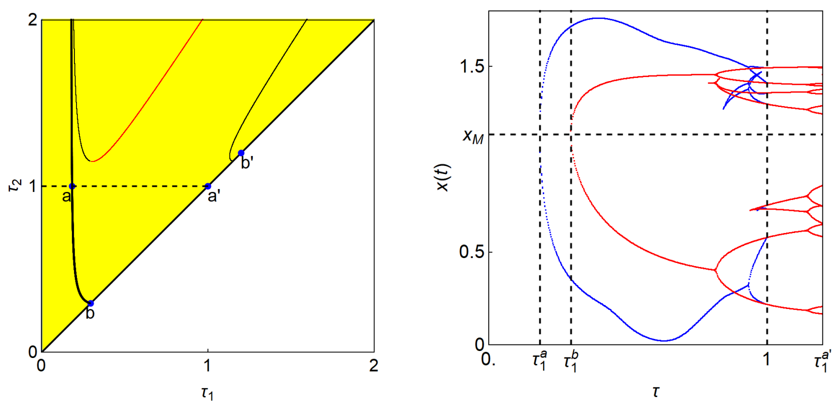
At points $(0, 0)$ and $(1, 0)$ in Figure 3, the angle balance relations imply the followings,

$$\tau^{\pm}_{1,m}(\omega) = \frac{1}{\omega}\left(\arg[a_1(i\omega)] + (2m-1)\pi \pm \theta_1(\omega)\right) \tag{54}$$

and

$$\tau^{\mp}_{2,n}(\omega) = \frac{1}{\omega}\left(\arg[a_2(i\omega)] + (2n-1)\pi \mp \theta_2(\omega)\right) \tag{55}$$

where m and n are integers such that both τ_1 and τ_2 are nonnegative. Similarly to the previous case, there are infinitely many stability switching curves. Some curves are illustrated in Figure 6(A), where the locus of $(\tau^{\pm}_{1,m}(\omega), \tau^{\mp}_{2,n}(\omega))$ in the feasible region are shown for $\omega \in [0, \omega_1]$ and $m, n = 0, 1$. The loci of $(\tau^{-}_{1,0}(\omega), \tau^{+}_{2,0}(\omega))$ is the bold black curve that divides the yellow region into the stability and instability regions. Two numerical results are depicted in Figure 6(B). Taking $\tau_2 = 1$, we simulate equation (47) for $\tau_1 \in [\tau_1^a, \tau_1^{a'}]$ with $\tau_1^a \simeq 0.183$ and $\tau_1^{a'} = 1$ to obtain the blue bifurcation diagram. It is seen that a multiple-periodic cycle is possible for τ_1 closer to $\tau_1^{a'}$. The red bifurcation diagram reproduces a part of the bifurcation diagram of Figure 6 for $\tau_1 \in [\tau_1^b, \tau_1^{b'}]$ with $\tau_1^b \simeq 0.295$ and $\tau_1^{b'} = 1.2$.



(A) Feasible region

(B) Bifurcation diagram

Figure 6. Dynamics generated by (42) amd (46)

6 Concluding Remarks

In this paper gradient adjustment process was introduced in a limited monopoly, when it does not know an analytic form of its profit function but can observe the actual profit at any time. The marginal profit was approximated with a simple finite difference formula based on two past profit observations leading to dynamic models with two time delays. For the sake of comparison, a knowledgeable monopoly knowing the form is also introduced. In the discrete-time adjustment process, two main results are obtained: (1) The stability condition is the same for both monopolists; (2) After losing stability, the knowledgeable monopolist experiences various dynamics ranging from periodic oscillations to chaos as the adjustment coefficient increases, whereas the limited monopolist produces only periodic cycles.

Two different delay continuous dynamics were then constructed based on the discrete-time model and analyzed to derive the stability switching curves. In the first continuous-time model with Euler approximation, two main results are summarized as follows: (3) The stability threshold value of τ_1 for the limited monopolist is less than that of the knowledgeable monopolist; (4) After losing stability, both monopolists present only periodic oscillations. In the second continuous-time model with Beresowski transformation, two main results are the followings: (5) The stability threshold value of τ_1 for the limited monopolist is less than that of the knowledgeable monopolist due to the negative sloping of the stability switching curve. This is the the same as result (3) above; (6) After losing stability, the limited monopolist can produce at most a multi-periodic cycle, where as the knowledgeable monopolist goes into chaotic oscillations, passing through an àla period doubling cascade.

In approximating the marginal profit a very simple differentiation formula was used. However, more sophisticated formulas could provide better approximations with increased numbers of delays. It is an interesting task to see how the more sophisticated differentiation formulas alter the stability conditions.

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