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Unawareness of Actions and Closedness under Rational Behavior in Static Games with Unawareness

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Abstract

This note applies "closedness under rational behavior" (CURB) (Basu and Weibull 1991) to "static games with unawareness"(Perea 2018). We characterize a cognitively stable generalized Nash equilibrium (GNE) (Sasaki 2017) in terms of CURB.
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1 Introduction

The cognitively stable generalized Nash equilibrium (GNE) is a solution concept in static games with unawareness (GU) (Sasaki 2017). This note provides a characterization of this solution concept by applying the concept of "closedness under rational behavior" (CURB) developed by Basu and Weibull (1991). A GU represents each player's awareness/unawareness by her type, which is essentially a Cartesian product of nonempty subsets of players' action sets (Perea 2018). CURB is also a property held by such entities. This motivates us to characterize a GNE in terms of CURB. The essential notion is a common CURB set, which is the Cartesian products of subsets of players' action sets that are commonly perceived by every type of player. We show that if a GU has a common CURB set, then there exists a corresponding cognitively stable GNE. We also provide examples of refinement of a GNE and coarsening of a cognitively stable GNE.

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2 Preliminaries

Let us first define a finite standard static game $G = (I, A, u)$. I is a finite set of players. $A = \times_{i\in I} A_i$, where A_i is the nonempty finite set of actions of $i \in I$ and $a_i \in A_i$ is i's technically feasible action. $u = (u_i)_{i\in I}$, where $u_i : A \to \mathbb{R}$ is i's utility function. Denote i's mixed action on A_i by $m_i \in M(A_i)$, where $M(A_i)$ is the set of i's mixed actions, and a mixed action profile on A by $m = (m_i)_{i\in I} \in M(A) = \times_{i\in I} M(A_i)$. We denote i's expected utility for $m \in M(A)$ by $Eu_i(m)$.

First, we define static GU, which is similar to that of Perea (2018). [1] For any standard G, let $V = \times_{i\in I}(2^{A_i} \setminus \{\emptyset\})$ be the set of possible *views* of the G. Like most previous works, this note assumes that the set of players is commonly known and that each player's utility for each action profile do not depend on awareness. Let $v \in V$ and A_i^v be the set of actions of i in $v = \times_{j\in I} A_j^v$. Here, when player i is given v, i is aware of $a \in v$ and unaware of $a \in A \setminus v$. For any $v, v' \in V$, v is contained in v' if A_i^v is a subset of $A_i^{v'}$ for any $i \in I$, i.e., $A_i^v \subseteq A_i^{v'}$. Let $M(A_i^v) = \{m_i \in M(A_i) | \Sigma_{a_i\in A_i^v} m_i(a_i) = 1\}$.

Let $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$ be a static GU. G is called the objective game, and for each $i \in I$:

- T_i is a nonempty finite set of i's types, one of which is her actual type t_i^*.
- $v_i : T_i \to V$ is i's view function.
- $b_i : T_i \to T_{-i}$ is i's belief function, where $T_{-i} = \times_{j\in I\setminus\{i\}} T_j$. Let $b_i(t_i)(j)$ be j's component in $b_i(t_i)$. If $b_i(t_i) = (t_j)_{j\in I\setminus\{i\}}$, for each $j \in I \setminus \{i\}$, $v_j(b_i(t_i)(j)) \subseteq v_i(t_i)$.

The objective game can be interpreted as the "true game" in Γ. i's type t_i describes her view of the game and belief about the opponents' types. At t_i, $v_i(t_i) = v$ means that i is aware of v and unaware of $A \setminus v$; while $b_i(t_i) = (t_j)_{j\in I\setminus\{i\}}$ means that at t_i, i believes that the other types are $(t_j)_{j\in I\setminus\{i\}}$ and that each j's view is $v_j(t_j)$. Here, by compound belief function, each player's type t_i can lead to some player's type t_j. Then, we call that t_j is reachable from t_i, and denote by $t_i \succeq t_j$. In a static GU, each player may be unaware of some types of players, including their own.

For any $i \in I$, let $s_i : T_i \to M(A_i)$. Then, given t_i, $s_i(t_i) \in M(A_i^{v_i(t_i)})$ is i's local action at t_i. Denote i's generalized strategy by $s_i = (s_i(t_i))_{t_i\in T_i}$, and a generalized strategy profile by $s = (s_i)_{i\in I}$. In s, each player i's actual play is $m_i \in M(A_i)$ with $m_i = s_i(t_i^*)$, and then the profile is called the objective outcome induced from s. Here, s^* is a GNE (Halpern and Rêgo 2014) if for any $i \in I$ and $t_i \in T_i$,

$$s_i^*(t_i) \in \arg \max_{x\in M(A_i^{v_i(t_i)})} Eu_i(x, (s_j^*(b_i(t_i)(j)))_{j\in I\setminus\{i\}}).$$

[1] Note that there are two differences to his models. First, we assume that "actual types" of agents are given. Second, each player's belief function does not depend on their beliefs about the choices of others.

A GNE s^* is cognitively stable (Sasaki, 2017) if for any $i \in I$ and $t_i \in T_i$, $s_i^*(t_i) = s_i^*(t_i^*)$.

Focusing on CURB, (Basu and Weibull 1991). A CURB set, which is a refinement of rationalizability (Bernheim 1984; Pearce 1984), is the set of best responses to each Nash equilibrium in a standard model. In a standard $G = (I, A, u)$, let $\mathbb{A} = \{A' \subseteq A | A' = \times_{i \in I} A_i'$ and $\forall i \in I \quad \emptyset \neq A_i' \subseteq A_i\}$ be the family of Cartesian products of nonempty subsets of players' action sets. For any $i \in I$ and nonempty $A_i' \subseteq A_i$, $M(A_i') = \{m_i \in M(A_i) | \sigma_{a_i \in A_i'} m_i(a_i') = 1\}$. and for any $A' \in \mathbb{A}$, $M(A') = \times_{i \in I} M(A_i')$. Let $\beta_i(m_{-i}) = \{a_i \in A_i | a_i \in \text{supp}(m_i)$ such that $m_i \in \arg\max_{x \in M(A_i)} Eu_i(x, m_{-i})\}$, be the set of i's pure-action best responses to her belief $m_{-i} \in M(A_{-i})$. For any $A' \in \mathbb{A}$, let $\beta_i(A') = \bigcup_{m_{-i} \in M(A'_{-i})} \beta_i(m_{-i})$ be the set of i's optimal actions under beliefs in $M(A')$ and let $\beta(A') = \times_{i \in I} \beta_i(A')$.

Remark 1. For any G, $V = \mathbb{A}$.

Remark 1 suggests that we can analyze any static GU in terms of CURB. [2] A set $A' \in \mathbb{A}$ is CURB if $\beta(A') \subseteq A'$. Then, let us call A' a CURB set. A CURB set is known for the following property (Basu and Weibull 1991).

Remark 2. For any G and any CURB set in G, there exists a Nash equilibrium whose support is a subset of the CURB set.

3 Applying CURB to static GUs

Applying CURB to static GUs. We define a *common CURB set*.

Definition 1. $v \in V$ is called a common CURB set of the static GU if v is CURB in G and $v \subseteq v_i(t_i)$ for any $i \in I$ and $t_i \in T_i$.

A common CURB set has the following property.

Proposition 1. Any static GU possessing a nonempty common CURB set has a cognitively stable GNE.

proof. Suppose that there exists a common CURB set $v \in V$. Let $m^* \in M(v)$ be a Nash equilibrium in v. Then, by Remark 2, there exists a Nash equilibrium in the objective game $G = (I, A, u)$, $m^* \in M(A)$, satisfying $m^* \in M(v)$. Suppose that m^* is not a Nash equilibrium on v. That is, there exists $(i, m_i) \in I \times M(A_i^v)$ such that $Eu_i(m_i, m_{-i}^*) > Eu_i(m^*)$. However, since v is a common CURB set, it contradicts. Therefore, m^* is a Nash equilibrium on v. Then, m_i^* is the best response to m_{-i}^* in $v_i(t_i)$ for any $i \in I$ and $t_i \in T_i$. Thus, s^* with $s_i^*(t_i) = m_i^*$ for any $i \in I$ and $t_i \in T_i$, is a cognitively stable GNE. ■

[2] Simply, this note does not focus on *minimal CURB sets* (Basu and Weibull 1991).

Proposition 1 holds even if the set of actions of some i at actual type t_i^* satisfies $A_i \setminus A_i^{v_i(t_i^*)} \neq \emptyset$. [3] In the course of the above proof, we have also shown that if each local action is a Nash equilibrium in the common CURB set, then the generalized strategy profile constitutes a cognitively stable GNE and that the objective outcome induced by the GNE is a Nash equilibrium in the objective game.

4 An Example of Refinement and Coarsening

As pointed out by Basu and Weibull (1991), in a standard static game, a CURB set has two notions; one is a refinement of a Nash equilibrium; the other is a coarsening of a Nash equilibrium. A common CURB set also has two notions, a refinement of a GNE and a coarsening of a cognitively stable GNE. The following examples show their concepts.

Example 1. Consider the following game played by players 1 and 2:

$v =$

1 / 2	L	R
U	1, 1	0, 0
M	1, 0	0, 1
B	0, 0	1, 1

, $v' =$

1 / 2	L	R
U	1, 1	0, 0
B	0, 0	1, 1

Suppose that $T_1 = \{t_1^*, t_1\}$, and $T_2 = \{t_2^*\}$, such that

$v_1(t_1^*) = v$ and $b_1(t_1^*) = t_2^*$;
$v_1(t_1) = v'$ and $b_1(t_1) = t_2^*$; and
$v_2(t_2^*) = v'$ and $b_2(t_2^*) = t_1$.

In this GU, the objective game has two CURB sets $A_1^C = \{B\} \times \{R\}$ and $A_2^C = \{U, M, B\} \times \{L, R\}$. Here, a common CURB set is only A_1^C, in which there exists a Nash equilibrium (B, R) that constitutes a cognitively stable GNE $s^* = ([s_1(t_1^*) = B, s_1(t_1) = B], [s_2(t_2^*) = R])$.

This example suggests that a refinement of the GNEs. □

Example 2. Consider the following game played by players 3 and 4:

$\hat{v} =$

3 / 4	L	R
U	1, 1	1, 0
B	1, 0	0, 0

, $\hat{v}' =$

3 / 4	L
U	1, 1
B	1, 0

Suppose that $T_3 = \{t_3^*, t_3\}$, and $T_2 = \{t_3^*, t_3\}$, such that

[3] This result is similar to Sasaki's (2017) Proposition 2. He shows that if every view has a Nash equilibrium in the objective game and each player's local action consists of the Nash equilibrium action, then a cognitively stable GNE constituted by the local actions induces the objective outcome which is a Nash equilibrium in the objective game. In contrast, we show that a cognitively stable GNE constituted by a common CURB set induces a Nash equilibrium.

$v_3(t_3^*) = \hat{v}$ and $b_3(t_3^*) = t_4^*$;
$v_3(t_3) = \hat{v}'$ and $b_3(t_3) = t_4$;
$v_4(t_4^*) = \hat{v}$ and $b_4(t_4^*) = t_3$; and
$v_4(t_4) = \hat{v}'$ and $b_4(t_4) = t_3$.

This GU has a unique common CURB set $A_3^C = \{U, B\} \times \{L\}$, where there exist two Nash equilibria (U, L) and (B, L). Then, the GNEs are:

$s_1^* = ([s_3(t_3^*) = U, s_3(t_3) = U], [s_4(t_4^*) = L, s_4(t_4) = L])$;
$s_2^* = ([s_3(t_3^*) = U, s_3(t_3) = B], [s_4(t_4^*) = L, s_4(t_4) = L])$;
$s_3^* = ([s_3(t_3^*) = B, s_3(t_3) = U], [s_4(t_4^*) = L, s_4(t_4) = L])$; and
$s_4^* = ([s_3(t_3^*) = B, s_3(t_3) = B], [s_4(t_4^*) = L, s_4(t_4) = L])$.

Cognitively stable GNEs are s_1^* and s_4^*; while cognitively unstable GNEs are s_2^* and s_3^*. However, in each GNE, every local action of any player is A_3^C.

That is, this example suggests a coarsening of cognitively stable GNEs. □

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