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# The unique limit cycle in post Keynesian systems

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**Abstract**

This paper establishes not only the existence and uniqueness of a limit cycle (representing persistent growth cycles) but also the convergence of every non-equilibrium solution path (describing the state of the economy) to the unique limit cycle in a post Keynesian system.

**Keywords:** Convergence; Growth cycle; Keynesian theory; Unique limit cycle

**JEL classification:** C62; E12; E32; E52

## 1 Introduction

Cyclical economic fluctuations, referred to as business cycles or growth cycles (business cycles around the trend of economic growth), have been one of the major phenomena which call for proper theoretical expositions. Soon after the publication of Keynes' *General Theory*, the foundations of the Keynesian theory of business cycles were laid by, for instance, Kalecki (1935, 1937), Harrod (1936), Samuelson (1939), Kaldor (1940), Metzler (1941), Hicks (1950) and Goodwin (1951).[1] In the Keynesian theory of business cycles, aggregate income is determined by aggregate effective demand composed mainly of consumption and investment (through the principle of effective demand) and fluctuations in the former are brought about by variations in the latter, especially by those in investment (through the multiplier process). For this reason, the Keynesian theory of business cycles focuses mainly on the mechanism of cyclical changes in investment. Indeed, Keynes (1936, chap. 22) attributed the main cause of business cycles to fluctuations in investment on account of variations in the marginal efficiency of capital (or the expected rate of profit on capital). Although Keynes (1936) himself emphasized the effect of expectations (or the long-term expectation) on investment, most Keynesian models of business cycles, including those in the aforementioned literature, slight this effect, postulating that investment is determined by past and current levels of income, profits and other variables.[2]

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[1] Precisely speaking, Kalecki (1935) and Harrod (1936) preceded Keynes (1936), but their theories stand on Keynesian perspectives (cf. Kalecki 1937).

[2] The dominant principles of investment in the Keynesian theory are the followings: the profit principle (cf. Kalecki 1935, 1937; Kalecki 1940), the acceleration one (cf. Harrod 1936; Samuelson 1939; Metzler 1941; Hicks 1950; Goodwin 1951) and the utilization one (cf. Steindl 1952, 1979; Rowthorn 1981; Dutt 1984). Investment is determined by the current aggregate profits or rate of profit in the

To understand the mechanism of business cycles in depth, it may be necessary to explore the role of expectations in investment, following Keynes' (1936) view.

Recently, Murakami (2018, 2020b) examined the effect of expectations on investment and established the existence and uniqueness of a limit cycle, which is regarded as persistent growth cycles, in a post Keynesian system.[3] It was found in Murakami (2018, 2020b) that the existence and uniqueness of a limit cycle is obtained under reasonable assumptions if investment (or the rate of capital formation) is highly elastic to the expected rate of profit and the latter is frequently revised in response to the realized rate of profit. In particular, the uniqueness (as well as existence) of a limit cycle is the distinguished contribution of this paper because it has rarely been explored until the recent studies by Murakami (2018, 2019, 2020a) due to technical difficulty.[4] In Murakami (2018, 2020b), however, the existence and uniqueness of a limit cycle was only confirmed but the convergence of solution paths, which describe the states of the macroeconomic system, to the unique limit cycle was not discussed.

The purpose of this paper is to generalize the conclusion of Murakami's (2018) analysis verifying that, if the revision speed of the expected rate of profit is high enough, every solution path, except for the one starting at the long-run equilibrium, converges to the unique limit cycle, regardless its initial condition. By so doing, we shall demonstrate the inevitability of persistent business cycles in capitalist economies. This paper is organized as follows. In Section 2, we shall formalize a post Keynesian system. In Section 3, we shall analyze the post Keynesian system to establish not only the existence and uniqueness of a limit cycle but also the convergence to the unique limit cycle. In Section 4, we shall conclude this paper. In Appendix, we shall present the mathematical theorem employed in our analysis.

## 2 The post Keynesian system

In this section, we shall formalize a post Keynesian system. In what follows, we shall make use of the following notations; let $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the $n$-dimensional Euclidean space, the subspace of $\mathbb{R}^n$ composed of all nonnegative vectors and the subspace of $\mathbb{R}^n$ composed of all strict positive vectors, respectively; let $\dot{v}$ stand for the time derivative of $v$, i.e., $\dot{v} \equiv dv/dt$.

first, by the current changes in aggregate income in the second and by the current rate of utilization, which is usually identified with the ratio of aggregate income to stock of capital, in the last. Unlike Keynes' (1936, chap. 11) theory of investment, these principles do not put much stress on the role of expectations in decision makings on investment.

[3]Benassy (1984) and Franke (2012, 2014) verified the existence of a periodic orbit in Keynesian models in which investment is affected by expected aggregate demand or the expected rate of profit, but they failed to examine the uniqueness of it.

[4]Until Murakami (2018, 2019, 2020a), the uniqueness of a limit cycle in models of business cycles was investigated by Ichimura (1955), Kosobud and O'Neill (1972), Lorenz (1986, 1993), Galeotti and Gori (1989) and Sasakura (1996) alone, as far as we know.

## 2.1 Aggregate consumption and saving

We shall assume that aggregate saving is determined in the following way:

$$S = s_c \Pi + s_w (Y - \Pi) - A, \tag{1}$$

where $s_c$ and $s_w$ are, respectively, positive and nonnegative constants with $s_w \leq s_c < 1$. In (1), $Y$, $S$, $\Pi$ and $A$ stand for, respectively, aggregate income, aggregate saving, aggregate profits (capitalists' income), aggregate autonomous consumption, which shall be referred to simply as "autonomous demand" in this paper;[5] $s_c$ and $s_w$ represent the marginal propensities to save of capitalists and workers, respectively.[6]

We shall also postulate that aggregate share of capital (or aggregate share of profits) is constant.[7] Aggregate profits can then be written as

$$\Pi = \pi Y, \tag{2}$$

where $\pi$ is a positive constant less than unity. In (2), $\pi$ represents aggregate share of capital. Thus, we can express aggregate saving in the following way:

$$S = s\Pi - A, \tag{3}$$

where $s$ is a positive constant defined by[8]

$$s = \frac{s_c \pi + s_w (1 - \pi)}{\pi} = s_c + s_w \frac{1 - \pi}{\pi}.$$

## 2.2 Aggregate investment

In the light of Keynes' (1936, chap. 11) theory of investment, we shall postulate that the (gross) rate of capital formation (or the ratio of investment to capital stock) is positively influenced by the expected rate of profit on

[5]Our aggregate autonomous demand $A$ can be regarded as aggregate base (or fundamental) consumption. It can also be viewed as autonomous government expenditure or autonomous exports as in the literature on autonomous demand (cf. Trezzini 1995; Serrano 1995; Allain 2015; Lavoie 2016; Dutt 2019).

[6]In this paper, we shall follow the post Keynesian (or Kaleckian) hypothesis that workers and capitalists may differ in spending behavior (cf. Kaldor 1955-1956; Pasinetti 1962).

[7]This postulate is consistent with Kaldor's (1961) stylized fact. According to Karabarbounis and Neiman (2014) and Jones (2016), the U.S. share of capital was almost constant (about 34.2 percent) until around 2000, while it has recently been rising (to 38.7 percent by 2012). In this respect, the postulate does not reflect the recent trend, but it works well as a first approximation to the reality because the recent rise in aggregate share of capital is only slight (at least in most developed countries).

[8]The saving function (3) is similar to Kalecki's (1935), but we allow for the possibility that workers also save. As we shall see, if the marginal propensities to save are the same between capitalists and workers (or if $s_c = s_w$), our post Keynesian system reduces to Murakami's (2018).

capital in the following way:[9]

$$I = f(r^e)K. \tag{4}$$

In (4), $I$, $K$ and $r^e$ stand for aggregate gross investment, aggregate capital stock and the expected rate of profit on capital, respectively; $f$ is the gross capital formation function, whose characteristics shall be discussed below. The investment function formalized reflects Keynes' (1936) theory of investment because it emphasizes the influence of the expected rate of profit (or the marginal efficiency of capital) on investment.[10]

Concerning the capital formation function $f$, we shall make the following reasonable assumption.

**Assumption 1.** *The nonnegative-valued function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable for every $r^e \in \mathbb{R}_+$, and the following condition is satisfied for every $r^e \in \mathbb{R}_+$:*

$$f'(r^e) > 0. \tag{5}$$

Assumption 1 can be seen as reasonable because condition (5) simply implies that the rate of gross capital formation is nonnegative, which is required by its definition, and is a strictly increasing function of the expected rate of profit $r^e$.

## 2.3 The rate of profit

The Keynesian principle of effective demand implies that aggregate saving is adjusted to aggregate investment in the following way:

$$S = I,$$

which determines aggregate profits as follows:

$$\Pi = \frac{1}{s}[f(r^e)K + A].$$

The (actual or realized) rate of profit can be derived by dividing both sides by aggregate capital stock as follows:

$$r = \frac{\Pi}{K} = \frac{1}{s}\Big[f(r^e) + \frac{1}{k}\Big]. \tag{6}$$

In (6), $r$ and $k$ stand for the rate of profit $\Pi/K$ and the ratio of aggregate stock of capital to autonomous demand $K/A$, respectively. This equation implies that the actual rate of profit $r$ is determined by the expected rate of profit

[9] The marginal efficiency of capital, defined by Keynes (1936), is nothing but the expected rate of profit on capital determined by the long-term expectation.

[10] The investment function (4) is consistent with Robinson's (1962) theory of investment.

$r^e$ through investment.[11]

### 2.4 Revisions of the expected rate of profit

It is seen from (6) that the actual or realized rate of profit $r$ is not necessarily equal to the expected one $r^e$. It is natural to think that the latter is changed based on the former. To describe the revision process, we shall assume that the expected rate of profit $r^e$ is adaptively revised reflecting on the actual or realized rate of profit $r$ in the following fashion:[12]

$$\dot{r}^e = \alpha(r - r^e),$$

or

$$\dot{r}^e = \alpha\Big\{\frac{1}{s}\Big[f(r^e) + \frac{1}{k}\Big] - r^e\Big\}, \tag{7}$$

where $\alpha$ is a positive constant. In (7), $\alpha$ can be regarded as the speed or frequency of revisions of the long-term expectation, based on which the expected rate of profit is determined.

### 2.5 Aggregate capital formation

Aggregate capital formation process can be described as follows:

$$\dot{K} = I - \delta K,$$

or

$$\dot{K} = [f(r^e) - \delta]K, \tag{8}$$

where $\delta$ is a positive constant. In (8), $\delta$ represents the constant rate of capital depreciation.

### 2.6 Changes in autonomous demand

To allow for changes in autonomous demand $A$, we shall assume that it varies at a constant rate as follows:[13]

$$\dot{A} = aA, \tag{9}$$

[11] The ratio of aggregate income to stock of capital $Y/K$ is proportionate to the rate of profit $\Pi/K$ due to (2).

[12] For a justification of adaptive expectation formations in nonlinear systems including ours, see Murakami (2018).

[13] In the literature on autonomous demand (cf. Trezzini 1995; Serrano 1995; Allain 2015; Lavoie 2016; Dutt 2019), its rate of change is assumed to be exogenously given (this is the reason for the name of "autonomous" demand). We can also suppose that autonomous component of aggregate consumption $A$ changes at the same rate as the rate of change in population (cf. Murakami 2018).

where $a$ is a real constant. As we shall see, the rate of change in autonomous demand $a$ can be zero or negative provided that $\delta + a > 0$ (cf. Assumption 2). Note that with the initial value of $A$ given, equation (9) is equivalent to

$$A(t) = A(0)\exp(at) > 0, \tag{10}$$

where $A(0)$ is a positive constant which represents the (positive) initial value of $A$ for $t = 0$ and the inequality holds for all $t \geq 0$.

In the light of (8) and (9), the dynamics of the ratio of capital stock to autonomous demand $k$ can be represented as follows:

$$\dot{k} = \frac{\dot{K}}{A} - \frac{K}{A}\frac{\dot{A}}{A},$$

or

$$\dot{k} = [f(r^e) - (\delta + a)]k. \tag{11}$$

### 2.7 Full system: System (K)

We are now ready to formalize our Keynesian system in the following way:

$$\dot{r}^e = \alpha\Big\{\frac{1}{s}\Big[f(r^e) + \frac{1}{k}\Big] - r^e\Big\}, \tag{7}$$

$$\dot{k} = [f(r^e) - (\delta + a)]k. \tag{11}$$

In what follows, we shall denote the system of (7) and (11) by "System (K)."[14]

## 3 Analysis

We shall now analyze our System (K) to examine the existence and uniqueness of a limit cycle, which can be viewed as persistent growth cycles, and the convergence of every non-equilibrium solution path to the unique limit cycle.

### 3.1 Existence and uniqueness of equilibrium

We shall now define an equilibrium point of System (K). A point $(r^e, k) \in \mathbb{R}^2_{++}$ is said to be an equilibrium point of System (K) if $\dot{r}^e = 0$ and $\dot{k} = 0$ at this point.[15] Then, an equilibrium point of System (K), denoted by $(r^*, k^*)$,

[14] As we have observed, System (K) is a generalized version of Murakami's (2018) post Keynesian system (System (PK)) in that the marginal propensities to save may differ between workers and capitalists.

[15] By this definition, $k = 0$ is ruled out as an equilibrium value of $k$.

can be defined as a solution of the following simultaneous equations:

$$0 = \frac{1}{s}\left[f(r^e) + \frac{1}{k}\right] - r^e,$$
$$0 = f(r^e) - (\delta + a).$$

The unique equilibrium point of System (K), if it exists, is given as follows:

$$(r^*, k^*) = \left(f^{-1}(\delta + a), \frac{1}{sf^{-1}(\delta + a) - (\delta + a)}\right). \tag{12}$$

To ensure the existence and uniqueness of an equilibrium point of System (K), we shall make the following assumption.

**Assumption 2.** *The following conditions are satisfied:*

$$f(0) < \delta + a < \lim_{r^e \to \infty} f(r^e), \tag{13}$$
$$f\left(\frac{\delta + a}{s}\right) < \delta + a. \tag{14}$$

Under Assumption 1, condition (13) ensures the existence and uniqueness of $f^{-1}(\delta + a) > 0$. Also, conditions and (13) and (14) imply that

$$0 < \frac{\delta + a}{s} < f^{-1}(\delta + a),$$

or

$$sf^{-1}(\delta + a) - (\delta + a) > 0. \tag{15}$$

Thus, Assumptions 1 and 2 guarantee the existence of the unique equilibrium point of System (K), defined in (12).

## 3.2 System (K) reformulated

We shall reformulate System (K) as a generalized Liénard system.[16] The reformulation facilitates our analysis of System (K).

For reformulation, we shall introduce the following new variables:

$$x = r^e - r^*, \tag{16}$$

$$y = \ln k^* - \ln k, \tag{17}$$

[16] For a brief exposition of generalized Liénard systems, see Appendix. For other applications of theory of generalized Liénard systems to economic theory, see Murakami (2018, 2019, 2020a).

where $(r^*, k^*)$ is defined by (12) and "ln" represents the natural logarithm (note that $k^*$ is positive under Assumption 2).

Substituting the variables defined in (16) and (17) in System (K), we can reduce System (K) to the following one:

$$\dot{x} = \phi(y) - F(x), \tag{18}$$

$$\dot{y} = -g(x), \tag{19}$$

where

$$g(x) = f(f^{-1}(\delta + a) + x) - (\delta + a), \tag{20}$$

$$F(x) = \alpha\Big[x - \frac{1}{s}g(x)\Big], \tag{21}$$

$$\phi(y) = \alpha\Big[f^{-1}(\delta + a) - \frac{\delta + a}{s}\Big][\exp(y) - 1]. \tag{22}$$

In what follows, we shall denote the system of equations (18) and (19) with (20)-(22) by "System (K*)." It is seen from Appendix that System (K*) can be classified as a generalized Liénard system. Note that the unique equilibrium of System (K*) is $(x^*, y^*) = (0, 0)$, which corresponds to the unique one of System (K), $(r^*, k^*)$.

### 3.3 Existence and uniqueness of a solution path

Before exploring the characteristics of System (K), we shall confirm that for every initial condition $(r^e(0), k(0)) \in \mathbb{R}^2_{++}$, System (K) has a unique solution path $(r^e(t), k(t)) \in \mathbb{R}^2_{++}$ for all $t \geq 0$. For this purpose, it is necessary and sufficient to verify that for every initial condition $(x(0), y(0)) \in D$, System (K*) has a unique solution path $(x(t), y(t)) \in D$ for all $t \geq 0$, where $D$ is defined as follows:

$$D = \{(x, y) \in \mathbb{R}^2 : \ x > -r^*\}. \tag{23}$$

We shall now make two additional assumptions for our analysis. The first assumption is given as follows.

**Assumption 3.** *The following condition is satisfied:*

$$f'(f^{-1}(\delta + a)) > s. \tag{24}$$

Assumption 3 means that investment or capital formation is sufficiently elastic to changes in the expected rate of profit at the unique equilibrium. In this respect, it is consistent with Keynes' (1936) view that variations in the expected rate on profit (the marginal efficiency of capital) give rise to violent changes in investment.[17] It also

[17] As Murakami (2018) argued, Assumption 3 does not violate the so-called Keynesian stability condition that the marginal propensity

concerns the (in)stability of the unique equilibrium of System (K) or (K*). The Jacobian matrix of System (K*) evaluated at the unique equilibrium is given as follows:

$$J^* = \begin{pmatrix} -F'(0) & \phi'(0) \\ -g'(0) & 0 \end{pmatrix} = \begin{pmatrix} \alpha[f'(f^{-1}(\delta+a))/s - 1] & \alpha[f^{-1}(\delta+a) - (\delta+a)/s] \\ -f'(f^{-1}(\delta+a)) & 0 \end{pmatrix}.$$

The trace and determinant of $J^*$ are given by

$$\text{tr } J^* = \alpha\Big[\frac{f'(f^{-1}(\delta+a))}{s} - 1\Big] > 0,$$
$$\det J^* = \alpha f'(f^{-1}(\delta+a))\Big[f^{-1}(\delta+a) - \frac{\delta+a}{s}\Big] > 0,$$

where these inequalities hold under Assumptions 2 and 3. The unique equilibrium of System (K*) is then locally asymptotically totally unstable.[18]

The second assumption is concerned with the form of the capital formation function $f$.[19]

**Assumption 4.** *The following condition is satisfied:*

$$\lim_{r^e \to \infty} [sr^e - f(r^e)] = \infty. \tag{25}$$

*There exist exactly two nonnegative constants $\underline{r}^e$ and $\overline{r}^e$ with $\underline{r}^e < \overline{r}^e$ such that the following condition is satisfied:*

$$f'(r^e) = s. \tag{26}$$

*Furthermore, the following condition is satisfied:*

$$f(\overline{r}^e) < s\overline{r}^e. \tag{27}$$

As we shall see, Assumption 4 plays a pivotal role for analysis. In particular, condition (25) is vital for establishing the existence and uniqueness of a solution path and of a limit cycle in System (K) or (K*). This assumption is, on the other hand, reasonable because it holds if $f$ is a logistic function with empirically plausible parameters as shown in Murakami (2019, 2020a).

Under Assumptions 1-4, the capital formation function $f$ can be drawn as in figure 1. It can be seen from this figure that $f$ has a sigmoid shape as in Kaldor's (1940) investment function.

to save is larger than that to invest (with respect to the current income) (cf. Marglin and Bhaduri 1990). This condition does not necessarily guarantee the local asymptotic stability of the long-run equilibrium, if investment is influenced by the expected rate of profit. For debates on the Keynesian stability condition, see, for instance, Hein et al. (2011) or Skott (2012).

[18]We mean by the term "local asymptotic total instability" that the equilibrium point under consideration is either an unstable node or an unstable focus, i.e., that the trace and determinant of the Jacobian matrix evaluated at this equilibrium point are both positive.

[19]It is conceptually identical with Murakami's (2018, p. 298, Assumption 4), whose error was corrected by Murakami (2020b).

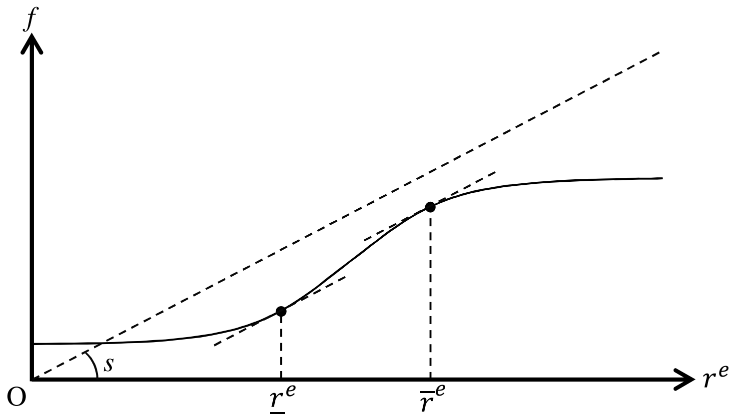


Figure 1: Capital formation function $f$

We shall take a close look at the phase diagram of System (K*) to examine the characteristics of solution paths of System (K). Since the locus $\dot{y} = 0$ is given by $x = 0$ due to (19) and (20), we shall look into that of $\dot{x} = 0$. Since the locus $\dot{x} = 0$ is given by $\phi(y) = F(x)$, we shall now have a close look at $F(x)$. Because of (12), (16) and (21), condition (25) implies[20]

$$\lim_{x \to \infty} F(x) = \infty. \tag{28}$$

Also, it follows from Assumptions 1-3 (and (15)) that

$$F(-r^*) = -\alpha\Big[f^{-1}(\delta + a) - \frac{\delta + a}{s} + \frac{f(0)}{s}\Big] < 0, \tag{29}$$

$$F'(0) = \alpha\Big[1 - \frac{f'(f^{-1}(\delta + a))}{s}\Big] < 0. \tag{30}$$

Due to $F(0) = 0$ (by (21)), it is seen from (30) that, there exists a sufficiently small positive $\varepsilon$ such that $F(-x) > 0$ and $F(x) < 0$ for every $x \in (0, \varepsilon)$. It then follows from the continuity of $F'(x)$ (by Assumption 1) and from (28) and (29) that $F'(x) = 0$ has at least one root both in $(-r^*, 0)$ and in $(0, \infty)$, respectively. Since $F'(x) = 0$ is equivalent to $f'(r^* + x) = s$ for $x \geq -r^*$, it is known from Assumption 4 that $F'(x) = 0$ has exactly two roots and and that the smaller one $\underline{x}' = \underline{r}^e - r^*$ lies in $(-r^*, 0)$, while the bigger one $\underline{x}' = \overline{r}^e - r^*$ in $(0, \infty)$. We can also find from (30) that $F(\underline{x}') > 0$ and $F(\overline{x}') < 0$ are a local maximum and a local minimum, respectively. Moreover, we can see from the continuity of $F$ that $F(x) = 0$ has exactly two roots $\underline{x}_0$ and $\overline{x}_0$, besides $x = 0$, in $(-r^*, \infty)$ with $-r^* < \underline{x}_0 < \underline{x}'$ and $\overline{x}_0 > \overline{x}'$. Thus, the graph of $F(x)$ can be drawn as in figure 2.

[20] As we shall see, condition (28) plays a vital role for our analysis.

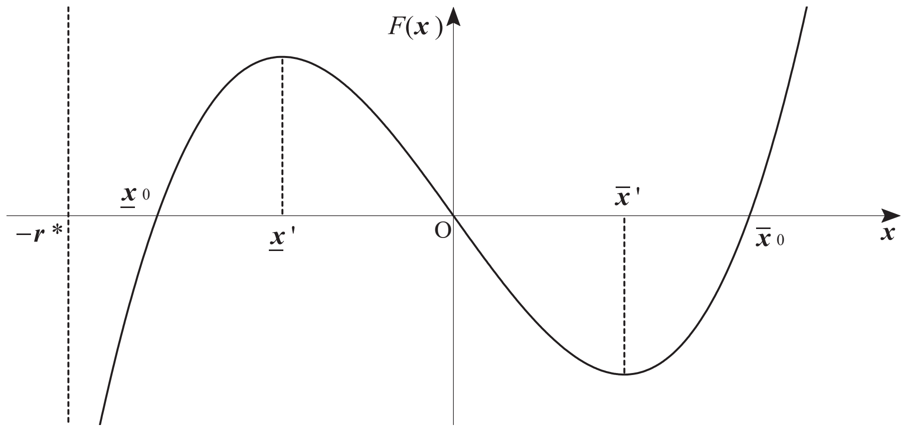


Figure 2: Graph of $F(x)$

Now we shall turn to the locus $\dot{x} = 0$ or $\phi(y) = F(x)$. It follows from (20)-(22) that the locus is given by

$$\phi(y) = \alpha\Big[f^{-1}(\delta + a) - \frac{\delta + a}{s}\Big][\exp(y) - 1] = \alpha\Big[x - \frac{f(f^{-1}(\delta + a) + x) - (\delta + a)}{s}\Big] = F(x),$$

or

$$\exp(y) = 1 + \frac{sF(x)}{\alpha[sf^{-1}(\delta + a) - (\delta + a)]} = \frac{s(r^* + x) - f(r^* + x)}{sf^{-1}(\delta + a) - (\delta + a)}. \tag{31}$$

Thus, for the locus $\dot{x} = 0$ to be well-defined at least for $x \geq \underline{x}_0$, it is necessary and sufficient that the right-hand side of (31) is positive for every $x \geq \underline{x}_0$.[21] Since it is known from figure 2 that the minimum of $F(x)$ for $x \geq \underline{x}_0$ is given at $x = \overline{x}' = \overline{r}^e - r^*$, the following condition is sufficient for the locus $\dot{x} = 0$ to be well-defined:

$$1 + \frac{sF(\overline{x}')}{\alpha[sf^{-1}(\delta + a) - (\delta + a)]} = \frac{s(\overline{r}^e) - f(\overline{r}^e)}{sf^{-1}(\delta + a) - (\delta + a)} > 0.$$

This is fulfilled by condition (27) in Assumption 4. Also, it is seen from that for every real $y$,

$$\dot{x}|_{x=-r^*} = [\phi(y) - F(x)]|_{x=-f^{-1}(\delta+a)} = \alpha\Big\{\Big[f^{-1}(\delta + a) - \frac{\delta + a}{s}\Big]\exp(y) + \frac{1}{s}f(0)\Big\} > 0, \tag{32}$$

where the inequality follows from $f(0) \geq 0$ and (15) (due to Assumptions 1 and 2). This indicates that the line $x = -r^*$ is wholly located on the region $\dot{x} > 0$ or $\phi(y) > F(x)$ (and that the locus $\dot{x} = 0$ never intersects with the line $x = -r^*$). Therefore, the phase diagram of System (K*) can be drawn as in figure 3.[22]

[21] For our analysis, it is vital for proving the existence and uniqueness of a solution path of System (K) or (K*) that the locus $\dot{x} = 0$ or $\phi(y) = F(x)$ is well-defined for $x \geq \underline{x}_0$ (cf. Appendix).

[22] Note that Assumption 4 (or condition (28)) ensures that, for every (especially, large) $y > 0$, there exists a unique $x$ such that the locus $\dot{x} = 0$ or (31) passes through $(x, y)$; if condition (25) is dropped, there may exist a (sufficiently large) $y > 0$ such that no $x$ satisfies (31) because the left-hand side of (31) tends to $\infty$ as $y \to \infty$. As we shall see, this fact is vital for the existence and uniqueness of a solution path, as well as of a limit cycle, in System (K*).

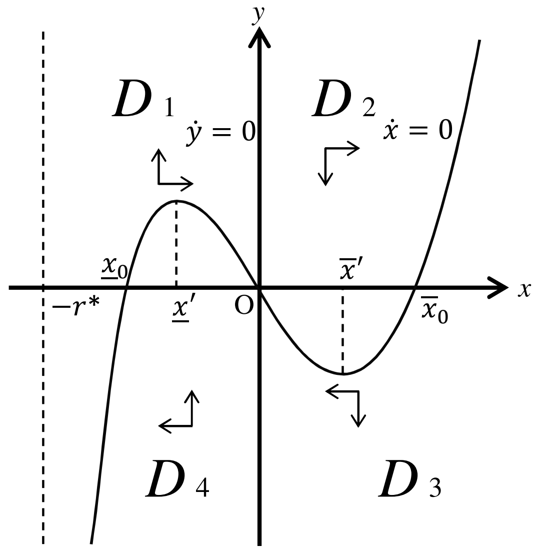


Figure 3: Phase diagram of System (K*)

For our analysis, we shall divide the region $D$, excluding the origin, into the following four parts (cf. figure 3) to examine non-equilibrium solution paths of System (K*), where $D$ is defined by (23):[23]

$$D_1 = \{(x,y) \in D : \ \dot{x} > 0, \ \dot{y} > 0\} = \{(x,y) \in \mathbb{R}^2 : \ -r^* < x < 0, \ \phi(y) > F(x)\}, \tag{33}$$

$$D_2 = \{(x,y) \in D \setminus (0,0) : \ \dot{x} \geq 0, \ \dot{y} \leq 0\} = \{(x,y) \in \mathbb{R}^2 \setminus (0,0) : \ x \geq 0, \ \phi(y) \geq F(x)\}, \tag{34}$$

$$D_3 = \{(x,y) \in D : \ \dot{x} < 0, \ \dot{y} < 0\} = \{(x,y) \in \mathbb{R}^2 : \ x > 0, \ \phi(y) < F(x)\}, \tag{35}$$

$$D_4 = \{(x,y) \in D \setminus (0,0) : \ \dot{x} \leq 0, \ \dot{y} \geq 0\} = \{(x,y) \in \mathbb{R}^2 \setminus (0,0) : \ -r^* < x \leq 0, \ \phi(y) \leq F(x)\}. \tag{36}$$

We shall now verify that, for initial condition $(r^e(0), k(0)) \in \mathbb{R}^2_{++}$, there exists a unique solution path of System (K), $(r^e(t), k(t)) \in \mathbb{R}^2_{++}$, for all $t \geq 0$. Because of (16) and (17) (and of the equivalence of Systems (K) and (K*)), it is (necessary and) sufficient for our purpose to prove the existence and uniqueness of a solution path of System (K*), $(x(t), y(t)) \in D$, for all $t \geq 0$, with every initial condition $(x(0), y(0)) \in D$.

To begin, we shall claim that every solution path of System (K*) with its (arbitrary) initial condition $(x(0), y(0)) \in D$ remains on $D$ all the time (as long as it exists).[24] For the sake of contradiction, we shall assume that a solution path of System (K*) with $(x(0), y(0)) \in D$ eventually does not lie on $D$.[25] By the continuity of solution paths of System (K*) (due to Assumption 1), such a solution path must eventually reach the line $x = -r^*$. But it is known from the continuity of $f$ and (32) that for every real $y$, there exists a (small) positive $\varepsilon$ such that $\dot{x} > 0$ for every $x \in [-r^*, -r^* + 2\varepsilon)$ and that no solution path passing through $(x, y) = (-r^* + \varepsilon, y)$ can cross the line

[23] In this paper, we mean by a "non-equilibrium solution path" a solution path whose initial condition is not an equilibrium point.

[24] At this stage, of course, we may not assume that solution paths of System (K*) exist for all $t \geq 0$.

[25] In this paper, we mean by "eventually" that the phenomenon under consideration occurs for a positive time $t$ (excluding the case of $t = \infty$).

$x = -r^* + \varepsilon > -r^*$ (because $\dot{x} > 0$ for $x = -r^* + \varepsilon$). This contradicts our hypothesis. Our claim can thus be verified.

Next, we shall take an arbitrary point $(x_i, y_i)$ on $D_i$, for $i = 1, 2, 3, 4$, defined by (33)-(36), and consider the solution path of System (K*) with $(x(0), y(0)) = (x_i, y_i)$, denoted by SP-$i$. We shall first claim that SP-1 eventually enters $D_2$ and that SP-3 eventually enters $D_4$. In what follows, we shall only prove the latter; one can prove the former in a similar way.[26] To this end, it is now confirmed that SP-3 never crosses the locus $\dot{x} = 0$ until it leaves $D_3$ for the first time (if it does).[27] It is seen from (35) that $x(t) \in [0, x_3]$ along SP-3 due to $\dot{x} < 0$ on $D_3$ and that there exists an upper limit of $F'(x(t))$, denoted by $\mu$, along SP-3 because of the continuity of $F'(x)$. Letting $u = -\dot{x}$ and noting that $\phi'(y) > 0$ and $\dot{y} < 0$ on $D_3$, it follows from (18) that until SP-3 leaves $D_3$ for the first time,

$$\dot{u} = -\frac{d}{dt}\dot{x} = -[\phi'(y)\dot{y} + F'(x)\dot{x}] \geq -\mu u,$$

or

$$\dot{u} + \mu u \geq 0.$$

Multiplying both sides by $\exp(\mu t) > 0$, we have

$$(\dot{u} + \mu u)\exp(\mu t) \geq 0.$$

Integrating both sides by $t$ in the interval $[0, t]$,[28] we obtain

$$u(t)\exp(\mu t) - u(0) \geq 0,$$

or

$$-\dot{x}(t) = u(t) \geq u(0)\exp(-\mu t) = -\dot{x}(0)\exp(-\mu t).$$

Hence, it is seen that along SP-3,

$$\dot{x}(t) \leq \dot{x}(0)\exp(-\mu t) = [\phi(y_3) - F(x_3)]\exp(-\mu t) < 0,$$

where the inequality holds for all $t$ due to $(x(0), y(0)) = (x_3, y_3) \in D_3$. It is then known from the continuity of solution paths that SP-3 never crosses the locus $\dot{x} = 0$, if it exists and stays on $D_3$, and from (33) and (34) that

[26] For a proof of a similar fact, see Murakami (2018).

[27] We do not exclude the possibility that SP-3 never leaves $D_3$.

[28] Note that

$$\frac{d}{dt}(u\exp(\mu t)) = (\dot{u} + \mu u)\exp(\mu t).$$

SP-3 never enters $D_1$ or $D_2$ before entering $D_4$.

We shall then claim that SP-3 eventually enters $D_4$, crossing the locus $\dot{y} = 0$ or the $y$-axis. It is now confirmed that if $x_3 \geq \overline{x}'$, SP-3 eventually crosses the line $x = \overline{x}'$, as long as it exists. For this purpose, it is assumed for the sake of contradiction that SP-3 remains the region $x \geq \overline{x}'$, as long as it exists. We can find from (20) and (35) that along SP-3, as long as it exists and remains on the region $x \geq \overline{x}'$ in $D_3$, we have $x(t) \in [\overline{x}', x_3]$ and

$$\dot{y} = -g(x) \leq -g(\overline{x}') < 0,$$

or

$$y(t) \leq y(0) - g(\overline{x}')t = y_3 - g(\overline{x}')t.$$

Since there exists a real $\overline{y}'$ such that $\phi(\overline{y}') < F(\overline{x}')$ (because $\phi^{-1}(F(\overline{x}'))$ is well-defined by Assumption 4), it is seen that $y(t) \leq \overline{y}'$ for $t \geq \tilde{t}_1 \equiv \max[0, (y_3 - \overline{y}')/g(\overline{x}')]$ along SP-3 as long as it exists and remains on the region $x \geq \overline{x}'$. If $x_3 \geq \overline{x}'$, we can then find from (35) that for $t \geq \tilde{t}_1$ along SP-3, as long as it exists and remains on the region $x \geq \overline{x}'$,

$$\dot{x} = \phi(y) - F(x) \leq \phi(\overline{y}') - F(\overline{x}') < 0,$$

or

$$x(t) \leq x(0) + [\phi(\overline{y}') - F(\overline{x}')]t = x_3 + [\phi(\overline{y}') - F(\overline{x}')]t$$

because $F(x) \geq F(\overline{x}')$ for $x \geq \overline{x}'$ (cf. figure 2). It then follows that if $x_3 \geq \overline{x}'$, SP-3 enters the region $x < \overline{x}'$, crossing the line $x = \overline{x}'$, by the time $t = \tilde{t}_2 \equiv \tilde{t}_1 + (x_3 - \overline{x}')/[F(\overline{x}') - \phi(\overline{y}')]$ (as long as it exists until this time). It is shown below that if $x_3 < \overline{x}'$, SP-3 eventually enters $D_4$, crossing the $y$-axis, as long as it exists. To this end, it is assumed for the sake of contradiction that SP-3 with $x_3 < \overline{x}'$ remains on $D_3$, as long as it exists. We can then see from (35) that if $x_3 < \overline{x}'$, along SP-3 as long as it exists, $0 \leq x(t) \leq x_3 < \overline{x}'$ and

$$\dot{x} = \phi(y) - F(x) \leq \phi(y_3) - F(x_3) < 0,$$

or

$$x(t) = x_3 + [\phi(y_3) - F(x_3)]t.$$

because $F(x) \geq F(x_3)$ for $0 \leq x \leq x_3 < \overline{x}'$. Hence, we have $x(t) \leq 0$ for $t \geq \tilde{t}_3 = x_3/[F(x_3) - \phi(y_3)]$ along SP-3, as long as it exists and remains on $D_3$, and this is a contradiction. It then follows that if $x_3 < \overline{x}'$, SP-3 enters $D_4$, crossing the $y$-axis by the time $t = \tilde{t}_3$ (as long as it exists by this time). Thus, we have shown that SP-3 enters $D_4$, crossing the $y$-axis for a positive time $t = T_1 \leq \tilde{t}_2 + \tilde{t}_3$ (for the first time), as long as it exists for $t \in [0, T_1]$. It remains to prove that SP-3 exists (at least) until it enters $D_4$ for $t = T_1$ (for the first time). It is known from (20) and (35) that for $t \in [0, T_1]$ along SP-3 (on $D_3$) as long as it exists, $(x(t), y(t)) \in D$, especially, $x(t) \in [0, x_3]$ and

$$-g(x_3) \leq \dot{y} = -g(x) \leq 0,$$

or

$$y_3 - g(x_3)T_1 \leq y(t) \leq y_3.$$

Then, it can be verified from the argument on continuation of solution paths (cf. Coddington and Levinson 1955, chap. 1) that SP-3 exists for $t \in [0, T_1]$. Also, it can be shown that SP-3 uniquely exists at least for $t \in [0, T_1]$ because System (K*) satisfies the Lipschitz condition (cf. Coddington and Levinson 1955, chap. 1) on the following (nonempty) compact rectangular (convex) region $\tilde{D}_1$:[29]

$$\tilde{D}_1 = \{(x, y) \in D : \ 0 \leq x \leq x_3, \ y_3 - g(x_3)T_1 \leq y \leq y_3\}.$$

Therefore, we can verify that SP-3 eventually enters $D_4$ (and uniquely exists until entering $D_4$ for the first time) and (in a similar manner) that SP-1 eventually enters $D_2$ (and uniquely exists until entering $D_2$ for the first time). Note that it is possible to prove in a similar manner that for every real $y$, a solution path of System (K*) with $(x(0), y(0)) = (-r^*, y)$ eventually enters $D_2$ (and uniquely exists until entering $D_2$ for the first time) because System (K*) is defined for $x = -r^*$ and such a solution path cannot leave the region $x \geq -r^*$ due to (32).

We shall second claim that SP-2 eventually enters $D_3$ and that SP-4 eventually enters $D_1$. We shall only prove the former; one can prove the latter in a similar manner. It is now confirmed that SP-2 never crosses the locus $\dot{y} = 0$ or the $y$-axis until it leaves $D_2$ for the first time. It is seen from (19) that if SP-2 reaches the $y$-axis on $D_2$ at $(x, y) = (0, \tilde{y})$, we have $\tilde{y} > 0$ (because $(x, y) = (0, \tilde{y})$ is known (from figure 3) not to lie on $D_2$ if $\tilde{y} \leq 0$) and

$$\dot{x}|_{(x,y)=(0,\tilde{y})} = \phi(\tilde{y}) - F(0) = \phi(\tilde{y}) > 0,$$

where the inequality follows from (22). It follows from the continuity of solution paths that SP-2 never crosses the $y$-axis until it leaves $D_2$ for the first time and never enters $D_1$ or $D_4$ before entering $D_3$.

[29] Since $g$, $F$ and $\phi$ are all continuously differentiable on $D$ by Assumption 1, it can be shown by the mean-value theorem that there exists a positive $M$ such that for every $(x', y')$, $(x'', y'') \in \tilde{D}_1$,

$$|[\phi(y'') - F(x'')] - [\phi(y') - F(x')]| + |g(x'') - g(x')| \leq M(|x'' - x'| + |y'' - y'|).$$

This implies that System (K*) satisfies the Lipschitz condition on $\tilde{D}_1$. See Murakami (2014), for details of the proof that the Lipschitz condition is satisfied on a compact rectangular subset of $\mathbb{R}^n$ if all the functions are continuously differentiable on this set.

We shall proceed to prove that SP-2 eventually enters $D_3$, crossing the locus $\dot{x} = 0$. It is assumed for the sake of contradiction that SP-2 remains on $D_2$ as long as it exists. Let $(x, y) = (\tilde{x}, y_2)$ be the unique intersection of the line $y = y_2$ and the locus $\dot{x} = 0$ on the region $x \geq \overline{x}'$.[30] If $x_2 = 0$ (i.e., if $\dot{y} = 0$ for $t = 0$), we can find from (34) that $\dot{x} > 0$ for $t = 0$ along SP-2 (because $y_2 > 0$) and that there exists a (small) positive $\tilde{t}_4$ such that $x(t) > 0$ for $t = \tilde{t}_4$. It then follows that if $x_2 = 0$, SP-2 (immediately) enters the region $x > 0$ (in $D_2$). If $x_2 > 0$, we can see from (20) and (34) that along SP-2 as long as it exists, $x(t) \in [x_2, \tilde{x}]$ and

$$\dot{y} = -g(x) \leq -g(x_2) < 0,$$

or

$$y(t) \leq y(0) - g(x_2)t = y_2 - g(x_2)t.$$

Hence, we have $y(t) \leq \overline{y}' < \phi^{-1}(F(\overline{x}'))$ for $t \geq \tilde{t}_5 \equiv (y_2 - \overline{y}')/g(x_2) > 0$ along SP-2 as long as it exists and remains on $D_2$, where $\overline{y}'$ has been defined above (and confirmed to exist). This is a contradiction because $y \geq \phi^{-1}(F(\overline{x}'))$ if $(x, y) \in D_2$ (cf. figure 3). It is then seen that if $x_2 > 0$, SP-2 enters $D_3$ by the time $t = \tilde{t}_5$ (as long as it exists by this time). Thus, we have proved that SP-2 enters $D_3$, crossing the locus $\dot{x} = 0$, for a positive time $t = T_2 \leq \tilde{t}_4 + \tilde{t}_5$ (for the first time), as long as it exists at least for $t \in [0, T_2]$. Since $(x(t), y(t)) \in D$ (because $x(t) \in [x_2, \tilde{x}]$ and $y(t) \in [\phi^{-1}(F(\overline{x}')), y_2]$) for $t \in [0, T_2]$ along SP-2 as long as it exists, it can be proved from the argument on continuation of solution paths (cf. Coddington and Levinson 1955, chap. 1) that SP-2 exists for $t \in [0, T_2]$. Also, it can be confirmed that SP-2 uniquely exists at least for $t \in [0, T_2]$ because System (K*) satisfies the Lipschitz condition on the following (nonempty) compact rectangular (convex) region $\tilde{D}_2$, defined by

$$\tilde{D}_2 = \{(x, y) \in D : \ x_2 \leq x \leq \tilde{x}, \ \phi^{-1}(F(\overline{x}')) \leq y \leq y_2\}.$$

Therefore, we can verify that SP-2 eventually enters $D_3$ (and uniquely exists until entering $D_3$ for the first time) and (in a similar manner) that SP-4 eventually enters $D_1$ (and uniquely exists until entering $D_1$ for the first time). Now we have established the fact that all solution paths of System (K*) starting on $D_1$, $D_2$, $D_3$ and $D_4$ for $t = 0$ eventually enter $D_2$, $D_3$, $D_4$ and $D_1$, respectively.

Finally, we shall make sure that for every initial condition $(x(0), y(0)) \in D$, a unique solution path of System (K*) exists (or is defined) for all $t \geq 0$.[31] To this end, we shall claim that there exists a (nonempty) positively invariant

[30] We can ensure by Assumption 4 that such a positive $\tilde{x}$ uniquely exists.

[31] By a method similar to ours, Hirsch and Smale (1974, p. 220, Proposition 2) proved the existence and uniqueness of a solution path with respect to every initial condition for Van der Pol's system, which is a special case of generalized Liénard systems and given as follows:

$$\dot{x} = y - (x^3 - x)$$
$$\dot{y} = -x.$$

They also gave a proof of the convergence of non-equilibrium solution paths to the unique limit cycle in this system (cf. Hirsch and compact region with respect to System (K*)[32] and then that every solution path starting on $D$ eventually enters the positively invariant region. As we have confirmed, a solution path of System (K*) with $(x(0), y(0)) = (-r^*, 0)$ eventually enters $D_2$, and it can be similarly shown to eventually enter $D_2$, $D_3$ and $D_4$ in order and then eventually return to $D_1$. It is also seen that a solution path with $(x(0), y(0)) = (-r^*, 0)$ uniquely exists until it returns to $D_1$ for the first time. It then follows that the solution path starting at P$(-r^*, 0)$, crosses the $y$-axis and the $\dot{x} = 0$ for the first time at Q$(0, y_q)$ and R$(x_r, y_r)$, respectively, and it crosses the $y$-axis and the $\dot{x} = 0$ for the second time at S$(0, y_s)$ and T$(x_t, y_t)$, respectively (cf. figure 4).[33] Taking the point U$(-r^*, y_t)$, we have a (nonempty) compact region enclosed by the arc PQRST and the segments TU and UP and denote it by $D_0$. By the uniqueness of the solution path starting at P (until it reaches T), no solution path of System (K*) starting from the inside (or outside) of $D_0$ can cross the arc PQRST; no solution path starting from the inside of $D_0$ can cross the segment TU or UP because $\dot{y} \geq 0$ along TU and $\dot{x} > 0$ on UP. It then follows that $D_0$ is a positively invariant region with respect to System (K*). It can thus be verified from the argument on continuation of solution paths that every solution path with $(x(0), y(0)) \in D_0$ uniquely exists on $D_0$ for all $t \geq 0$.[34] Since every solution path starting on $D$ is already known to remain on $D$ as long as it exists, it can also be proved that for every $(x(0), y(0)) \in D_0 \cap D$, a unique solution path exists on $D_0 \cap D$ for all $t \geq 0$. Moreover, it can be shown from the above argument that every solution path starting on the region $D$ not contained in $D_0$ eventually enters the region $y < y_t$ (below the segment UT) in $D_1$ (because of the uniqueness of the solution path, it never meets or crosses the arc PQRST) and then $D_0 \cap D$ before entering $D_2$ again (and that it uniquely exists until it enters $D_0 \cap D$). Therefore, we have established the fact that for every $(x(0), y(0)) \in D$, a unique solution path of System (K*) exists on $D$ for all $t \geq 0$, which implies the following proposition.

**Proposition 1.** *Let Assumptions 1-4 hold. Then, for every initial condition* $(r^e(0), k(0)) \in \mathbb{R}^2_{++}$, *there exists a unique solution path of System (K),* $(r^e(t), k(t)) \in \mathbb{R}^2_{++}$, *for all* $t \geq 0$.

*Proof.* It is straightforward to draw the conclusion from the above argument (as well as from (16) and (17)). □

Smale 1974, p. 218, Theorem). Their method of proof was, however, different from ours.

[32] A closed (usually compact) region is $R$ said to be positively invariant with respect to the dynamical system under consideration if every positive semi-trajectory (i.e., every solution path for $t \geq 0$) of this system which starts at an arbitrary point in $R$ will remain in $R$ for ever after (i.e., for all $t \geq 0$).

[33] Points Q, R, S and T are, of course, uniquely defined. Note that Assumption 4 plays a vital role for R to be properly defined.

[34] The uniqueness of a solution path on $D_0$ follows because it can be shown by the continuity of $g$, $F$ and $\phi$ and the mean-value theorem that System (K*) satisfies the Lipschitz condition on the following (nonempty) compact rectangular set, which includes $D_0$:

$$D_0^* = \{(x, y) \in \mathbb{R}^2 : \ -r^* \leq x \leq x_r, \ y_t \leq y \leq y_q\}.$$

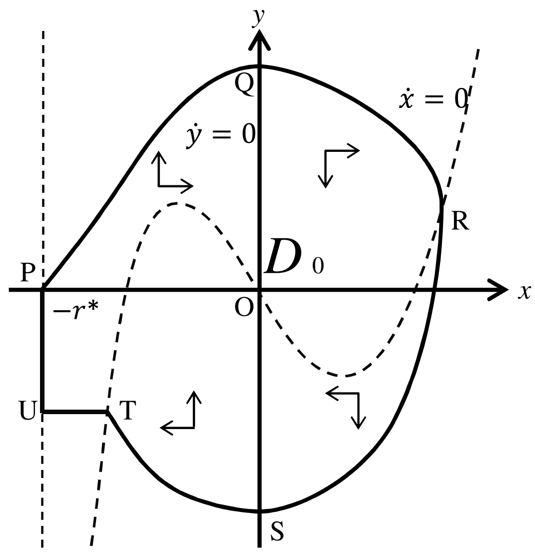


Figure 4: Positively invariant region $D_0$

### 3.4 Convergence to a periodic orbit

We shall now examine the existence of a periodic orbit, which may be regarded as persistent business cycles, in System (K) or (K*). For this purpose, we shall make use of the Poincaré-Bendixson theorem (cf. Coddington and Levinson 1955, chap. 16) to ensure the existence of a periodic orbit. In so doing, we shall also confirm the convergence of every non-equilibrium solution path to a periodic orbit.[35] Since it has already been confirmed in the last subsection that the unique equilibrium (0,0) is locally asymptotically totally unstable and that every solution path of System (K*) starting on $D$ eventually enters the positively invariant compact set $D_0$, it is straightforward to present the following theorem.

**Theorem 1.** *Let Assumptions 1-4 hold. Then, for every initial condition* $(r^e(0), k(0)) \in \mathbb{R}^2_{++}$ *with* $(r^e(0), k(0)) \neq (r^*, k^*)$*, the unique solution path of System (K),* $(r^e(t), k(t)) \in \mathbb{R}^2_{++}$*, either is a periodic orbit on* $\mathbb{R}^2_{++}$ *or converges to a periodic orbit on* $\mathbb{R}^2_{++}$ *as* $t \to \infty$*.*

*Proof.* Because of (16) and (17), it suffices for the proof of this proposition to verify that every non-equilibrium solution path of System (K*) starting on $D$ either is or converges to a periodic orbit on $D$.

It has been confirmed in the last subsection (especially in Proposition 1) that every solution path of System (K*) with $(x(0), y(0)) \in D$ uniquely exists on $D$ for all $t \geq 0$ and eventually enters the positively invariant set compact $D_0$ (cf. figure 4). Also, the unique equilibrium of System (K*) has been shown to be locally asymptotically unstable (by Assumption 3). We can then enclose the unique equilibrium by a sufficiently small rectangle such that every non-equilibrium solution path starting on its interior eventually leaves the interior and never enters the interior

[35]In a related post Keynesian system, Murakami (2018, p. 301, Proposition 1; 2020b) failed to ensure the convergence to a periodic orbit.

again ever after. We can construct a (nonempty) positively invariant compact set, denoted by $D_0^*$, eliminating the interior of the small rectangle from $D_0$. It follows that every non-equilibrium solution path starting on $D$ eventually enters the positively invariant compact set free of the unique equilibrium $D_0^*$ and remains on $D_0^*$ ever after. We can thus apply the Poincaré-Bendixson theorem (cf. Coddington and Levinson 1955, chap. 16, especially pp. 391-392, Theorem 2.1) to conclude that every non-equilibrium solution path starting on $D$, uniquely defined for all $t \geq 0$, either is or converges to a periodic orbit on $D_0^*$ as $t \to \infty$. Also, each of such periodic orbits is entirely located on $D_0^* \cap D$, because every solution path starting on $D$ remains on $D$ for all $t \geq 0$. It then follows that every non-equilibrium solution path starting on $D$ either is or converges to a periodic orbit on $D$. □

Theorem 1 confirms not only the existence of a periodic orbit, which may be viewed as persistent growth cycles, but also the convergence of every non-equilibrium solution path to a periodic orbit. It is then implied from this theorem that unless the macroeconomic system happens to be in the (long-run) equilibrium state, it is surely subjected to persistent cyclical fluctuations along (the way of convergence to) a periodic orbit. We can also draw from this theorem the economic implication that the macroeconomic system (with Keynesian features) necessarily (precisely speaking, almost surely) undergoes persistent business cycles around the trend path determined by the rate of change in autonomous demand $a$. In this respect, it may be stated that business cycles or growth cycles are an inevitable phenomenon in capitalist economies, which are characterized by investment behavior highly responsive to prospected profits or to the long-term expectation (as expressed by Assumption 3). Note that along each of the growth cycles, aggregate income also undergoes cyclical fluctuations around its trend level $r^*k^*A(t)/\pi$, proportional to autonomous demand $A$, because it is determined by $Y(t) = r(t)k(t)A(t)/\pi$ (cf. (2), (6) and (10)).

### 3.5 Convergence to the unique limit cycle

We shall proceed to establish not only the uniqueness of a periodic orbit (or a limit cycle in this case) and but also the convergence of every non-equilibrium solution to the unique periodic orbit in System (K) with the revision speed of expectations $\alpha$ sufficiently large.[36] To this end, we shall utilize the theorem of Xiao and Zhang (2003), which is reproduced as Theorem 3 in Appendix.

It is easily seen that Assumptions 5 and 6, required in Theorem 3, are fulfilled in System (K*), if $\underline{x} = -r^*$, $\overline{x} = \infty$, $\underline{y} = -\infty$ and $\overline{y} = \infty$ (and if $\underline{x}_0$ and $\overline{x}_0$ are defined as in figure 2). Because of Theorem 1, it suffices for our purpose to make sure that the remaining hypothesis for Theorem 3, Assumption 7, is satisfied if $\alpha$ is sufficiently large.

Now we shall take a look at Assumption 7. Since we have $\underline{x}' \in [\underline{x}_0, 0]$ and $\overline{x}' \in [0, \overline{x}_0]$ (cf. figure 2), condition

[36] The argument that follows draws on Murakami (2018). Note that Murakami (2018) only proved the uniqueness of a limit cycle but not the convergence of every non-equilibrium solution path to it.

(49) in Assumption 7 holds if the following condition is fulfilled:[37]

$$\begin{cases} G(\overline{x}') + \Phi(\phi^{-1}(F(\overline{x}'))) \geq G(\underline{x}_0), \text{ if } G(\underline{x}_0) \geq G(\overline{x}_0), \\ G(\underline{x}') + \Phi(\phi^{-1}(F(\underline{x}'))) \geq G(\overline{x}_0), \text{ if } G(\overline{x}_0) > G(\underline{x}_0), \end{cases}$$

or

$$\begin{cases} \Phi(\phi^{-1}(F(\overline{x}'))) \geq G(\underline{x}_0) - G(\overline{x}'), \text{ if } G(\underline{x}_0) \geq G(\overline{x}_0), \\ \Phi(\phi^{-1}(F(\underline{x}'))) \geq G(\overline{x}_0) - G(\underline{x}'), \text{ if } G(\overline{x}_0) > G(\underline{x}_0). \end{cases} \tag{37}$$

It is straightforward to obtain

$$G(x) = \int_0^x g(s)ds = \int_0^x f(f^{-1}(\delta + a) + \tau)d\tau - (\delta + a)x,$$

which implies that

$$G(\underline{x}_0) - G(\overline{x}') = \int_{\overline{x}'}^{\underline{x}_0} f(f^{-1}(\delta + a) + \tau)d\tau - (\delta + a)(\underline{x}_0 - \overline{x}'), \tag{38}$$

$$G(\overline{x}_0) - G(\underline{x}') = \int_{\underline{x}'}^{\overline{x}_0} f(f^{-1}(\delta + a) + \tau)d\tau - (\delta + a)(\overline{x}_0 - \underline{x}'). \tag{39}$$

It is seen from (21) and (22) that

$$\phi^{-1}(F(x)) = \ln\Big(\frac{s(r^* + x) - f(r^* + x)}{sf^{-1}(\delta + a) - (\delta + a)}\Big),$$

$$\Phi(y) = \int_0^y \phi(\tau)d\tau = \alpha\Big[f^{-1}(\delta + a) - \frac{\delta + a}{s}\Big][\exp(y) - y - 1],$$

It follows from (12) that

$$\begin{aligned} \Phi(\phi^{-1}(F(x))) &= \alpha\Big[f^{-1}(\delta + a) - \frac{\delta + a}{s}\Big]\Big[\frac{sx - f(r^* + x) + \delta + a}{sf^{-1}(\delta + a) - (\delta + a)} - \ln\Big(\frac{s(r^* + x) - f(r^* + x)}{sf^{-1}(\delta + a) - (\delta + a)}\Big)\Big] \\ &= \alpha\Big[f^{-1}(\delta + a) - \frac{\delta + a}{s}\Big][z(x) - \ln(1 + z(x))], \end{aligned}$$

where

$$z(x) = \frac{s(r^* + x) - f(r^* + x)}{sf^{-1}(\delta + a) - (\delta + a)} - 1 = \frac{sx - f(r^* + x) + f(r^*)}{sf^{-1}(\delta + a) - (\delta + a)}. \tag{40}$$

[37] Due to $\phi'(y) > 0$ (by (22)), the function $\phi^{-1}(F(x))$ can be defined at least for $x \in [\underline{x}_0, \overline{x}_0]$ because $\phi(y) = F(x)$ is well-defined for $x \in [\underline{x}_0, \overline{x}_0]$ (by Assumption 4).

Then, we have

$$\Phi(\phi^{-1}(F(\overline{x}'))) = \alpha\Big[f^{-1}(\delta+a) - \frac{\delta+a}{s}\Big][z(\overline{x}') - \ln(1+z(\overline{x}'))] > 0, \tag{41}$$

$$\Phi(\phi^{-1}(F(\underline{x}'))) = \alpha\Big[f^{-1}(\delta+a) - \frac{\delta+a}{s}\Big][z(\underline{x}') - \ln(1+z(\underline{x}'))] > 0, \tag{42}$$

where the inequalities hold because it is seen from (21), (31) and (40) that $z(x) \neq 0$ and $z(x) > -1$ for $x = \underline{x}'$ and for $x = \overline{x}'$ (by Assumption 4). Since $\underline{x}_0$ and $\overline{x}_0$ are the roots of $F(x) = 0$ or $sx = f(r^* + x)$ and $\underline{x}'$ and $\overline{x}'$ are the roots of $F'(x) = 0$ or $f'(r^* + x) = s$, all of them are determined independently from the value of $\alpha$. Thus, we can find from (38), (39), (41) and (42) that the left-hand side of (37) is positive and proportional to $\alpha$ while the right-hand is fixed and independently from $\alpha$ and that if $\alpha$ is large enough, condition (37) is fulfilled (in either case).

Therefore, we can present the main theorem, which sharpens Theorem 1 for the case of $\alpha$ being large enough.[38]

**Theorem 2.** *Let Assumptions 1-4 hold. Assume that $\alpha$ is sufficiently large. Then, for every initial condition $(r^e(0), k(0)) \in \mathbb{R}^2_{++}$ with $(r^e(0), k(0)) \neq (r^*, k^*)$, the unique solution path of System (K), $(r^e(t), k(t)) \in \mathbb{R}^2_{++}$, either is the unique (and periodically stable) limit cycle on $\mathbb{R}^2_{++}$ or converges to the unique limit cycle on $\mathbb{R}^2_{++}$ as $t \to \infty$.*

*Proof.* If $\alpha$ is sufficiently large, the periodic orbit (or limit cycle) is confirmed by the above argument to be unique in System (K*), and it follows from Theorem 1 that every non-equilibrium solution path of System (K) either is or converges to the unique limit cycle, which corresponds to the one in System (K*). It is obvious from Theorem 1 that the unique limit cycle of System (K) lies entirely on $\mathbb{R}^2_{++}$. □

Theorem 2 establishes not only the existence and uniqueness of a limit cycle but also the convergence of every non-equilibrium solution path to the unique limit cycle in the case of the speed of revisions of expectations $\alpha$ being large enough. The main economic implication from this theorem is that if expectations on the future rate of profit are frequently revised (or if $\alpha$ is large enough), the macroeconomic system with Keynesian features asymptotically approaches the unique business cycle around its trend level determined by autonomous demand, regardless its initial condition (except for the case in which it is the unique equilibrium). We may then state that the ultimate (or final) state of capitalist economies, where prospected profits (or the long-term expectation) are frequently revised in response to realized ones, is persistent cyclical fluctuations with constant period and amplitude along the unique growth cycle. Also, we can confirm that Theorem 2 strengthens Keynes' (1936) view that violent fluctuations in the marginal efficiency of capital (or the expected rate of profit) are the essential cause of business cycles.[39]

[38] The conclusion of Theorem 2 is stronger than Murakami's (2018) related proposition (p. 302, Proposition 2) because the convergence of every non-equilibrium solution path to the uniqueness is obtained in the former but not in the latter.

[39] Theorem 2 sharpens Keynes' (1936) view on business cycles more than Murakami's (2018, p. 302, Proposition 2) related proposition because the former guarantees the inevitability of the unique growth cycle irrespective of initial conditions.

# 4 Conclusion

In this paper, we have verified that every non-equilibrium solution path converges to the unique limit cycle (if the revision speed of the expected rate of profit is high enough) in a post Keynesian system which emphasizes the role of the expected rate of profit. By so doing, we have demonstrated that persistent cyclical fluctuations are inevitable if the expected rate of profit (or the marginal efficiency of capital) is frequently revised and given a strong theoretical support to Keynes' (1936) view on business cycles. We hope that our present analysis is helpful for understanding the mechanism of business cycles.


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# Appendix: Generalized Liénard systems

We shall introduce the theorem by Xiao and Zhang (2003) on the uniqueness of a (stable) limit cycle in generalized Liénard systems.

We shall consider the following generalized Liénard system:

$$\dot{x} = \phi(y) - F(x), \tag{43}$$

$$\dot{y} = -g(x). \tag{44}$$

In what follows, the system of equations (43) and (44) is denoted by "System (L)."

Following Xiao and Zhang (2003), we shall impose the following assumptions concerning System (L).

**Assumption 5.** *The real-valued functions $g(x)$ and $F(x)$ are, respectively, continuous and continuously differentiable on $(\underline{x}, \overline{x})$, and the real valued function $\phi(y)$ is continuously differentiable on $(\underline{y}, \overline{y})$ with $-\infty \leq \underline{x} < 0 < \overline{x} \leq \infty$ and $-\infty \leq \underline{y} < 0 < \overline{y} \leq \infty$. Furthermore, the following conditions are satisfied:*

$$xg(x) > 0 \text{ for } x \neq 0, \tag{45}$$

$$\phi(0) = 0,\ \phi'(y) > 0 \text{ for } y \in (\underline{y}, \overline{y}). \tag{46}$$

**Assumption 6.** *There exist* $\underline{x}_0$ *and* $\overline{x}_0$ *with* $\underline{x} < \underline{x}_0 < 0 < \overline{x}_0 < \overline{x}$ *such that the following conditions are satisfied:*

$$F(\underline{x}_0) = F(0) = F(\overline{x}_0) = 0, \tag{47}$$

$$\begin{cases} xF(x) \leq 0 \text{ for } x \in (\underline{x}_0, \overline{x}_0), \\ xF(x) > 0,\ F'(x) \geq 0 \text{ for } x \in (\underline{x}, \underline{x}_0) \text{ or } x \in (\overline{x}_0, \overline{x}). \end{cases} \tag{48}$$

*Furthermore,* $F(x)$ *is not identically equal to 0 for* $x$ *sufficiently close to 0.*

**Assumption 7.** *The curve of* $\phi(y) = F(x)$ *is well-defined for* $x \in [\underline{x}_0, \overline{x}_0]$.[40] *Furthermore, the following condition is satisfied:*

$$\begin{cases} \sup_{x\in[0,\overline{x}_0]}(G(x) + \Phi(\phi^{-1}(F(x)))) \geq G(\underline{x}_0),\ \text{if } G(\underline{x}_0) \geq G(\overline{x}_0), \\ \sup_{x\in[\underline{x}_0,0]}(G(x) + \Phi(\phi^{-1}(F(x)))) \geq G(\overline{x}_0),\ \text{if } G(\overline{x}_0) > G(\underline{x}_0), \end{cases} \tag{49}$$

*where*

$$G(x) = \int_0^x g(\tau)d\tau,$$
$$\Phi(y) = \int_0^y \phi(\tau)d\tau.$$

As regards the uniqueness of a limit cycle in System (L), the following theorem was established by Xiao and Zhang (2003).

**Theorem 3.** *Let Assumptions 5-7 hold. Then, System (L) has at most one limit cycle, and it is (periodically) stable if it exists.*

*Proof.* See Xiao and Zhang (2003, p. 1187, Theorem 2.2). □

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[40] Xiao and Zhang (2003) assumed that the curve of $\phi(y) = F(x)$ is well-defined for $x \in (\underline{x}, \overline{x})$, but our assumption suffices for the proof of their theorem (cf. Xiao and Zhang 2003, pp. 1187-1190).

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