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Dynamic Models of Pollution Penalities and Rewards with Time Delays

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Abstract

In cases of non-point pollution sources the regulator can observe the total emission but unable to distinguish between the firms. The regulator then selects an environmental standard. If the total emission level is highter than the standard, then the firms are uniformly punished, and if lower, then uniformly awarded. This environmental regulation is added to n-firm dynamic oligopolies and the asymptotical behavior of the corresponding dynamic systems is examined. Two particular models are considered with linear and hyperbolic price functions. Without delays the equilibrium is always (locally) asymptotically stable. It is shown how the stability can be lost if time delays are introduced in the output quantities of the competitors as well as in the firms' own output levels. Complete stability analysis is presented for the resulting one- and two-delay models including the derivations of stability thresholds, stability switching curves and directions of the stability switches.

Keywords: Non-point source pollution, Ambient charge, Time delays, linear and hyperbolic demand, Stability switching curve

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1 Introduction

Oligopolies are among the most frequently examined models in mathematical economics. The early results up to the mid 70s are summarized in Okuguchi (1976), and their multi-product extensions are discussed in Okuguchi and Szidarovszky (1999). These models and the corresponding dynamic systems are linear, the asymptotical behavior of which is simple, since local asymptotical stability implies global asymptotical stability. From the early 90s an increasing attention was given to nonlinear oligopolies.The asymptotical stability of these models was examined by a variety of concepts including linearization, Lyapunov functions and the critical curve methods among others. Bischi et al. (2010) offers a comprehensive summary of these developments. In all previous models instantaneous information was assumed about the actions of the competitors as well as about the own output selections of the firms. However data collection, determining appropriate actions and their implementations need time, therefore delayed models describe reality more accurately. More recently Matsumoto and Szidarovszky (2018) offer a collection of delayed dynamic oligopolies with brief summary of the used mathematical methodology as well as with discussions on different types of oligopoly models. A large variety of oligopoly models consider environmental issues. The effects of different environmental regulation policies are examined by many authors including Downing and White (1986), Segerson (1988), Jung et al. (1996), Montero (2002), Okuguchi and Szidarovszky (2002, 2007). In the case of non-point pollution sources the government can define a cut-off value for the total emission level of the entire industry but cannot distinguish among the firms. If the total emmission level exceeds the cut-off value, then the firms are punished, otherwise rewarded. In early stages the existence of the Nash equilibrium was the main fucus and how the govenmental policy affects the total polution level of the industry. Depending on the selected model, increased ambient charge in duopolies can lead to higher pollution level (Ganguli and Raju, 2012), and in other cases to lower pollution (Raju and Ganguli, 2013). This result is generalized for n-firm Cournot oligopolies by Matsumoto et al. (2018a). Dynamic models are introduced and their asymptotical behavior examined without and with time delays. The corresponding Bertrand models are considered and investigated in Ishikawa et al. (2019), and in Matsumoto et al. (2018b). Hyperbolic duopolies and triopolies are studied in Matsumoto et al. (2019a) with static and dynamic analysis. Three-stage optimum models are introduced in Matsumoto rt al. (2019b) for Cournot duopolies without product differentiation. This paper extends and further generalizes the earlier methodology and stability results for two particular models. Linear and hyperbolic oligopolies are discussed with ambient charges or rewards, with selected cut-off pollution levels for the entire industry. As the dynamic models are very similar, we will present the complete analysis in detail for the general case including both particular models.

This paper is developes as follows. In Section 2 the mathematical models are introduced. In Sections 3 and 4 one-delay and two-delay models are examined. In both cases two special cases are discussed in detail: symmetric firms and general duopolies. Conclusions and further research directions are outlined in Section 5.

2 The Mathematical Models

An oligopoly of n-firm is considered. Let q_k denote the output of firm k, $Q_k = \sum_{i\neq k} q_i$ the output of the rest of the industry and $Q = q_k + Q_k$ the industry output. Assume that the price function of the product by firm k is given as $P_k(q_k, Q_k)$. Firm k emits pollutions $e_k q_k$ in connection with its production, so the total amount of pollutions is $\sum_{k=1}^n e_k q_k$. The government can measure this total amount and unable to distinguish behavior between the firms. An exogenously determined environmental standard E is selected by the government and a $\theta > 0$ is chosen to determine the emission penalty or award for the firms. If c_k is the production unit cost of firm k, then its payoff is given as

$$\pi_k = [P_k(q_k, Q_k) - c_k]\, q_k - \theta \left(\sum_{i=1}^n e_i q_i - E \right). \tag{1}$$

Assume that at time t, each firm k has only delayed information about the outputs of the competitors, so the payoff of firm k is the following,

$$\pi_k(t) = [P_k(q_k(t), Q_k(t-\tau_k)) - c_k]\, q_k(t) - \theta \left(e_k q_k(t) + \sum_{i\neq k}^n e_i q_i(t-\tau_k) - E \right). \tag{2}$$

The gradient adjustment process of firm k is driven by the delay differential equation

$$\dot{q}_k(t) = K_k \frac{\partial \pi_k(t)}{\partial q_k(t)} \tag{3}$$

where $K_k > 0$ is the speed of adjustment of firm k and the marginal profit is

$$\frac{\partial \pi_k(t)}{\partial q_k(t)} = \frac{\partial P_k}{\partial q_k(t)} q_k(t) + P_k - c_k - \theta e_k.$$

Let

$$g_k(q_k(t), Q_k(t-\tau_k)) = \frac{\partial \pi_k(t)}{\partial q_k(t)}.$$

Then equation (3) can be rewritten as

$$\dot{q}_k(t) = K_k g_k(q_k(t), Q_k(t-\tau_k)). \tag{4}$$

In order to linearize this equation. Let

$$U_k = \frac{\partial g_k}{\partial q_k(t)} \text{ and } V_k = \frac{\partial g_k}{\partial Q_k(t-\tau_k)}$$

and notice that for all $i \neq k$,

$$\frac{\partial g_k}{\partial q_i(t-\tau_k)} = V_k$$

since by differentiation the last term of $g_k(q_k(t), Q_k(t-\tau_k))$ cancels out. Then the linearized equation has the form

$$\dot{q}_{k\varepsilon}(t) = K_k U_k q_{k\varepsilon} + K_k V_k \sum_{i\neq k} q_{i\varepsilon}(t-\tau_k) \tag{5}$$

where $q_{i\varepsilon}$ is the difference of q_i and its equilibrium level.

Before proceeding to the stability analysis of the system, two important cases are introduced.

Case 1.

Assume differentiated products, when the price of the product of firm k is as follows:

$$P_k = \alpha_k - q_k - \gamma_k \sum_{i\neq k} q_i$$

where α_k is the maximum price and γ_k represents the substitutability of the products, $0 \leq \gamma_k \leq 1$. In this case,

$$\pi_k(t) = \left[\alpha_k - q_k(t) - \gamma_k \sum_{i\neq k} q_i(t-\tau_k) - c_k\right] q_k(t) - \theta\left(e_k q_k(t) + \sum_{i\neq k}^{n} e_i q_i(t-\tau_k) - E\right) \tag{6}$$

therefore

$$g_k(q_k(t), Q_k(t-\tau_k)) = \alpha_k - 2q_k(t) - \gamma_k Q_k(t-\tau_k) - c_k - \theta e_k$$

and so

$$U_k = -2 \text{ and } V_h = -\gamma_k.$$

Notice that for all k,

$$U_k < V_k \leq 0.$$

Case 2.

Assume a hyperbolic oligopoly without product differentiation and common price function

$$P = \frac{\alpha}{q_k + Q_k}$$

where α is a positive constant. In this case the profit function is rewritten as

$$\pi_k(t) = \left(\frac{\alpha}{q_k(t) + Q_k(t-\tau_k)} - c_k\right) q_k(t) - \theta\left(e_k q_k(t) + \sum_{i\neq k}^{n} e_i q_i(t-\tau_k) - E\right) \tag{7}$$

consequently,

$$g_k\left(q_k\left(t\right), Q_k(t-\tau_k)\right) = \frac{\alpha Q_k(t-\tau_k)}{\left(q_k(t)+Q_k(t-\tau_k)\right)^2} - c_k - \theta e_k.$$

Then at the equilibrium

$$\begin{aligned} U_k &= -\frac{2\alpha Q_k(t-\tau_k)\left(q_k(t)+Q_k(t-\tau_k)\right)}{\left(q_k(t)+Q_k(t-\tau_k)\right)^4} \\ &= -\frac{2\alpha \bar{Q}_k}{\bar{Q}^3} \end{aligned}$$

where $\bar{Q}_k$ and $\bar{Q}$ are the equilibrium levels of Q_k and Q. Similarly,

$$\begin{aligned} V_k &= \frac{\alpha\left(q_k(t)+Q_k(t-\tau_k)\right)^2 - 2\alpha Q_k(t-\tau_k)\left(q_k(t)+Q_k(t-\tau_k)\right)}{\left(q_k(t)+Q_k(t-\tau_k)\right)^4} \\ &= \frac{\alpha\left(\bar{q}_k - \bar{Q}_k\right)}{\bar{Q}^3} \end{aligned}$$

where $\bar{q}_k$ is the equilibrium level of q_k.

Notice that $U_k < 0,\ U_k < V_k$, and if there is no dominant firm, then $V_k \leq 0$. In the rest of this paper we will assume the absence of a dominant firm. So we will assume that

$$U_k < V_k \leq 0.$$

3 Single Delay Stability

To determine the characteristic equation of model (5) assume exponential solution form, $q_{i\varepsilon}(t) = e^{\lambda t}u_i$, then substituting them into (5) gives

$$\lambda e^{\lambda t}u_k = K_k U_k e^{\lambda t} u_k + K_k V_k e^{\lambda(t-\tau_k)} \sum_{i\neq k} u_i.$$

So the characteristic equation can be written as

$$\varphi(\lambda) = \det\begin{pmatrix} A_1-\lambda & B_1e^{-\lambda\tau_1} & \cdots & B_1e^{-\lambda\tau_1} \\ B_2e^{-\lambda\tau_2} & A_2-\lambda & \cdots & B_2e^{-\lambda\tau_2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ B_ne^{-\lambda\tau_n} & B_ne^{-\lambda\tau_n} & \cdots & A_n-\lambda \end{pmatrix} = 0 \tag{8}$$

where the simplifying notation $A_k = K_kU_k$ and $B_k = K_kV_k$ are used. Let

$$\boldsymbol{a} = \begin{pmatrix} B_1e^{-\lambda\tau_1} \\ B_2e^{-\lambda\tau_2} \\ \cdot \\ \cdot \\ \cdot \\ B_ne^{-\lambda\tau_n} \end{pmatrix},\ \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

and

$$\boldsymbol{D} = diag\left(A_1 - \lambda - B_1 e^{-\lambda\tau_1}, \cdots, A_n - \lambda - B_n^{-\lambda\tau_n}\right)$$

to have

$$\begin{aligned}\varphi(\lambda) &= \det\Big(\boldsymbol{D} + \boldsymbol{a}\boldsymbol{b}^T\Big) \\ &= \det(\boldsymbol{D})\det(I + \boldsymbol{D}^{-1}\boldsymbol{a}\boldsymbol{b}^T) \\ &= \det(\boldsymbol{D})\Big[1 + \boldsymbol{b}^T\boldsymbol{D}^{-1}\boldsymbol{a}\Big]\end{aligned}$$

where the identity discussed in Bischi et al. (2010, Appendix E) is used. So

$$\varphi(\lambda) = \prod_{k=1}^{n}\left(A_k - \lambda - B_k e^{-\lambda\tau_k}\right)\left[1 + \sum_{k=1}^{n}\frac{B_k e^{-\lambda\tau_k}}{A_k - \lambda - B_k e^{-\lambda\tau_k}}\right]. \tag{9}$$

We have now two possibilities. Consider first equation

$$A_k - \lambda - B_k e^{-\lambda\tau_k} = 0. \tag{10}$$

Without delay $\tau_k = 0$ and from (10), $\lambda = A_k - B_k < 0$. Stability switch might occur if $\lambda = i\omega$ with some $\omega > 0$. Then (10) implies that

$$A_k - i\omega - B_k\left(\cos\omega\tau_k - i\sin\omega\tau_k\right) = 0.$$

Separation of the real and imaginary parts shows that

$$B_k\cos\omega\tau_k = A_k,$$

$$B_k\sin\omega\tau_k = \omega.$$

Adding the squares of these equations gives that

$$\omega^2 = B_k^2 - A_k^2 < 0$$

since $A_k < B_k \leq 0$. There is no solution for ω.

Consider next equation

$$1 + \sum_{k=1}^{n}\frac{B_k e^{-\lambda\tau_k}}{A_k - \lambda - B_k e^{-\lambda\tau_k}} = 0. \tag{11}$$

Proposition 1 *System (5) is locally asymptotically stable if all roots of equation (11) are negative real numbers or complex with negative real parts.*

In Case 1, system (5) is linear, so the asymptotical stability is global. Equation (11) is very complicated in general, so two special cases are examined.

3.1 Case of symmetric firms

Assume $A_k = A,\ B_k = B,\ \tau_k = \tau$, then (11) is specialized as

$$A - \lambda + (n-1)Be^{-\lambda\tau} = 0. \tag{12}$$

Without delay, $\lambda = A + (n-1)B < 0$. Stability switching might occur with $\lambda = i\omega$ ($\omega > 0$), and substituting it into (12) yields

$$A - i\omega + (n-1)B(\cos\omega\tau - i\sin\omega\tau) = 0,$$

which implies that

$$\begin{aligned} (n-1)B\cos\omega\tau &= -A, \\ (n-1)B\sin\omega\tau &= -\omega. \end{aligned} \tag{13}$$

By adding the squares of these equations, we have

$$\omega^2 = (n-1)^2B^2 - A^2.$$

If $(n-1)B \geq A$, then the right hand side is non-positive with no solution and without stability switch. Otherwise,

$$\omega^* = \sqrt{(n-1)^2B^2 - A^2}. \tag{14}$$

From (13), it is clear that $\cos\omega\tau < 0$ and $\sin\omega\tau > 0$, furthermore the critical values of the delays are

$$\tau^*_m = \frac{1}{\omega^*}\left(\cos^{-1}\left[\frac{-A}{(n-1)B}\right] + 2m\pi\right),\ m \geq 0. \tag{15}$$

The direction of stability switches can be determined by using Hopf bifurcation. Let τ be selected as the bifurcation parameter and consider λ as a function of $\tau : \lambda = \lambda(\tau)$. By implicitly differentiating equation (12) with respect to τ shows that

$$\lambda' = \frac{-(\lambda - A)\lambda}{1 + (\lambda - A)\tau}$$

where equation (12) is used again. At the critical value $\lambda = i\omega$,

$$\lambda' = \frac{\omega^2 + iA\omega}{1 - A\tau + i\omega\tau} \cdot \frac{1 - A\tau - i\omega\tau}{1 - A\tau - i\omega\tau},$$

the real part of which has the same sign as

$$\omega^2(1 - A\tau) + \omega A\omega\tau = \omega^2 > 0.$$

Proposition 2 *In the symmetric case, the equilibrium is locally asymptotically stable if $(n-1)B \geq A$, otherwise if $\tau < \tau^*_0$. At $\tau = \tau^*_0$ stability is lost via Hopf bifurcation and stability cannot be regained with larger values of τ.*

In the linear case,

$$U_k = -2,\ V_k = -\gamma_k \text{ so } \gamma_k = \gamma.$$

Furthermore with $K_k = K$,

$$A = -2K \text{ and } B = -K\gamma$$

so

$$(n-1)B - A = K\left[-(n-1)\gamma + 2\right]$$

which is non-negative if

$$\gamma \le \frac{2}{n-1}.$$

3.2 General duopolies

In the case of $n = 2$, equation (8) shows that

$$\varphi(\lambda) = (A_1 - \lambda)(A_2 - \lambda) - B_1 B_2 e^{-\lambda(\tau_1+\tau_2)} = 0$$

which is a single-delay equation with $\tau = \tau_1 + \tau_2$:

$$\lambda^2 - (A_1 + A_2)\lambda + A_1 A_2 - B_1 B_2 e^{-\lambda\tau} = 0. \tag{16}$$

Without delay $\tau = 0$, and (16) becomes

$$\lambda^2 - (A_1 + A_2)\lambda + A_1 A_2 - B_1 B_2 = 0.$$

Since $|A_k| > |B_k|$ for $k = 1, 2$, both the linear coefficient and the constant term are positive implying that the roots are negative real values or complex with negative real parts. Stability switch might occur if $\lambda = i\omega$ with $\omega > 0$ when from (16) we have

$$-\omega^2 - i(A_1 + A_2)\omega + A_1 A_2 - B_1 B_2 (\cos\omega\tau - i\sin\omega\tau) = 0$$

implying that

$$B_1 B_2 \cos\omega\tau = -\omega^2 + A_1 A_2$$

$$B_1 B_2 \sin\omega\tau = (A_1 + A_2)\omega.$$

By adding the squares of these equations and arranging the terms, we have

$$\omega^4 + \left(A_1^2 + A_2^2\right)\omega^2 + \left(A_1^2 A_2^2 - B_1^2 B_2^2\right) = 0.$$

Since $|A_k| > |B_k|$ for $k = 1, 2$, all coefficients are positive showing that no stability switch can occur.

Proposition 3 *The equilibrium in duopoly is always locally asymptotically stable with all* $\tau_1, \tau_2 \ge 0$.

4 Two-Delay Stability

Assume next that firm k has a delay τ_1^k in its only output and delay τ_2^k in the outputs of its competitors. Then the dynamic equation (5) modifies as follows:

$$\dot{q}_{k\varepsilon}(t) = A_k q_{k\varepsilon}(t - \tau_1^k) + B_k \sum_{i \neq k} q_{i\varepsilon}(1 - \tau_2^k) \tag{17}$$

Similarly to equations (8) and (9), the characteristic equation can be derived as

$$\varphi(\lambda) = \prod_{k=1}^{n} \left(A_k e^{-\lambda \tau_1^k} - \lambda - B_k e^{-\lambda \tau_2^k} \right) \left[1 + \sum_{k=1}^{n} \frac{B_k e^{-\lambda \tau_2^k}}{A_k e^{-\lambda \tau_1^k} - \lambda - B_k e^{-\lambda \tau_2^k}} \right] = 0. \tag{18}$$

Proposition 4 *The equilibrium of system (17) is locally asymptotically stable if all roots of (18) are negative reals or complex with negative real parts.*

Similarly to the previous model, two special cases will be reexamined.

4.1 Case of symmetric firms

Assume now that $A_k = A, B_k = B$, $\tau_1^k = \tau_1$ and $\tau_2^k = \tau_2$. From (18), we have to consider two cases. First we examine equation

$$Ae^{-\lambda \tau_1} - \lambda - Be^{-\lambda \tau_2} = 0 \tag{19}$$

which can be rewritten as

$$1 + a_1(\lambda) e^{-\lambda \tau_1} + a_2(\lambda) e^{-\lambda \tau_2} = 0$$

with

$$a_1(\lambda) = -\frac{A}{\lambda} \text{ and } a_2(\lambda) = \frac{B}{\lambda}.$$

We will apply the method introduced by Gu et al. (2005) and discussed in details in Matsumoto and Szidarovszky (2018a). Notice that

$$a_1(i\omega) = i\frac{A}{\omega} \text{ and } a_2(i\omega) = -i\frac{B}{\omega},$$

$$|a_1(i\omega)| = -\frac{A}{\omega} \text{ and } |a_2(i\omega)| = -\frac{B}{\omega}$$

and

$$\arg\left[a_1(i\omega)\right] = \frac{3\pi}{2} \text{ and } \arg\left[a_2(i\omega)\right] = \frac{\pi}{2}.$$

The range of ω is determined by conditions,

$$\begin{aligned} |a_1(i\omega)| + |a_2(i\omega)| &\geq 1 \\ -1 \leq |a_1(i\omega)| - |a_2(i\omega)| &\leq 1 \end{aligned} \tag{20}$$

which simplify in our case as

$$-\frac{A}{\omega}-\frac{B}{\omega}\geq 1 \text{ or } \omega\leq -(A+B)$$

and

$$-1\leq -\frac{A}{\omega}+\frac{B}{\omega}\leq 1 \text{ or } \omega\geq B-A.$$

So ω runs through interval $[B-A,-(A+B)]$. Moreover by the law of cosine,

$$\theta_1(\omega)=\cos^{-1}\left[\frac{1+|a_1(i\omega)|^2-|a_2(i\omega)|^2}{2\,|a_1(i\omega)|}\right]=\cos^{-1}\left(\frac{\omega^2+A^2-B^2}{-2A\omega}\right) \tag{21}$$

and

$$\theta_2(\omega)=\cos^{-1}\left[\frac{1+|a_2(i\omega)|^2-|a_1(i\omega)|^2}{2\,|a_2(i\omega)|}\right]=\cos^{-1}\left(\frac{\omega^2+B^2-A^2}{-2B\omega}\right) \tag{22}$$

The stability switching curves are given by pairs $\left(\tau_1^{\pm k},\tau_2^{\mp m}\right)$ with

$$\tau_1^{\pm k}=\frac{1}{\omega}\left[\frac{3\pi}{2}+(2k-1)\pi\pm\theta_1(\omega)\right] \tag{23}$$

and

$$\tau_2^{\mp m}=\frac{1}{\omega}\left[\frac{3\pi}{2}+(2m-1)\pi\mp\theta_2(\omega)\right] \tag{24}$$

The directions of stability switching can be assessed by computing the following expressions:

$$a_1(\omega)e^{-i\omega\tau_1}=i\frac{A}{\omega}\left(\cos\omega\tau_1-i\sin\omega\tau_1\right)$$

and

$$a_2(\omega)e^{-i\omega\tau_2}=-i\frac{B}{\omega}\left(\cos\omega\tau_2-i\sin\omega\tau_2\right)$$

with real and imaginary parts,

$$\begin{aligned}
R_1&=\operatorname{Re}\left(a_1(\omega)e^{-i\omega\tau_1}\right)=\frac{A}{\omega}\sin\omega\tau_1,\\
I_1&=\operatorname{Im}\left(a_1(\omega)e^{-i\omega\tau_1}\right)=\frac{A}{\omega}\cos\omega\tau_1,\\
R_2&=\operatorname{Re}\left(a_2(\omega)e^{-i\omega\tau_2}\right)=-\frac{B}{\omega}\sin\omega\tau_2,\\
I_2&=\operatorname{Im}\left(a_2(\omega)e^{-i\omega\tau_2}\right)=-\frac{B}{\omega}\cos\omega\tau_2
\end{aligned}$$

and finally

$$S_1=R_2I_1-R_1I_2=\frac{AB}{\omega^2}\left(\sin\omega\tau_1\cos\omega\tau_2-\cos\omega\tau_1\sin\omega\tau_2\right)$$

which has the same sign as $\sin\omega(\tau_1 - \tau_2)$.

Consider next equation

$$Ae^{-\lambda\tau_1} - \lambda + (n-1)Be^{-\lambda\tau_2} = 0 \tag{25}$$

which can be rewritten as

$$1 + a_1(\lambda)e^{-\lambda\tau_1} + a_2(\lambda)e^{-\lambda\tau_2} = 0$$

with

$$a_1(\lambda) = -\frac{A}{\lambda} \text{ and } a_2(\lambda) = -\frac{(n-1)B}{\lambda}.$$

Then

$$a_1(i\omega) = i\frac{A}{\omega} \text{ and } a_2(i\omega) = i\frac{(n-1)B}{\omega},$$

$$|a_1(i\omega)| = -\frac{A}{\omega} \text{ and } |a_2(i\omega)| = -\frac{(n-1)B}{\omega}$$

and

$$\arg\left[a_1(i\omega)\right] = \arg\left[a_2(i\omega)\right] = \frac{3\pi}{2}.$$

The range of ω is determined again based on conditions (20), which are the following in this case:

$$-\frac{A}{\omega} - \frac{(n-1)B}{\omega} \geq 1 \text{ or } \omega \leq -\left(A + (n-1)B\right)$$

and

$$-1 \leq -\frac{A}{\omega} + \frac{(n-1)B}{\omega} \leq 1 \text{ or } \omega \geq |A - (n-1)B|$$

so range of ω is in the interval,

$$\left[|A - (n-1)B|, -\left(A + (n-1)B\right)\right].$$

By the low of cosines,

$$\theta_1(\omega) = \cos^{-1}\left(\frac{\omega^2 + A^2 - (n-1)^2B^2}{-2A\omega}\right) \tag{26}$$

and

$$\theta_2(\omega) = \cos^{-1}\left(\frac{\omega^2 + (n-1)^2B^2 - A^2}{-2(n-1)B\omega}\right). \tag{27}$$

The stability switching curves are given by pairs $\left(\tau_1^{\pm k}, \tau_2^{\mp m}\right)$ with

$$\bar{\tau}_1^{\pm k} = \frac{1}{\omega}\left[\frac{3\pi}{2} + (2k-1)\pi \pm \theta_1(\omega)\right] \tag{28}$$

and

$$\bar{\tau}_2^{\mp m} = \frac{1}{\omega}\left[\frac{\pi}{2} + (2m-1)\pi \mp \theta_2(\omega)\right]. \tag{29}$$

The directions of stability switches can be determined similarly to the previous case. Notice that

$$a_1(\omega)e^{-i\omega\tau_1} = i\frac{A}{\omega}\left(\cos\omega\tau_1 - i\sin\omega\tau_1\right)$$

and

$$a_2(\omega)e^{-i\omega\tau_2} = i\frac{(n-1)B}{\omega}\left(\cos\omega\tau_2 - i\sin\omega\tau_2\right)$$

with real and imaginary parts,

$$\begin{aligned}
R_1 &= \mathrm{Re}\left(a_1(\omega)e^{-i\omega\tau_1}\right) = \frac{A}{\omega}\sin\omega\tau_1, \\
I_1 &= \mathrm{Im}\left(a_1(\omega)e^{-i\omega\tau_1}\right) = \frac{A}{\omega}\cos\omega\tau_1, \\
R_2 &= \mathrm{Re}\left(a_2(\omega)e^{-i\omega\tau_2}\right) = \frac{(n-1)B}{\omega}\sin\omega\tau_2, \\
I_2 &= \mathrm{Im}\left(a_2(\omega)e^{-i\omega\tau_2}\right) = \frac{(n-1)B}{\omega}\cos\omega\tau_2
\end{aligned}$$

and therefore

$$S_1 = R_2 I_1 - R_1 I_2 = \frac{(n-1)AB}{\omega^2}\left(\sin\omega\tau_2\cos\omega\tau_1 - \cos\omega\tau_2\sin\omega\tau_1\right)$$

which has the same sign as $\sin\omega(\tau_2 - \tau_1)$.

Proposition 5 *The stability switching curves are formed by points* $\left(\tau_1^{\pm k}, \tau_2^{\mp m}\right)$ *when* ω *runs through interval* $[B-A, -(A+B)]$ *and points* $\left(\tau_1^{\pm k}, \tau_2^{\mp m}\right)$ *when* ω *runs through* $[|A-(n-1)B|, -(A+(n-1)B)]$

Proposition 6 *Let* (τ_1, τ_2) *be a point on the stability switching curve and assume that the curve is crossed at this point from right to left when we are looking forward increasing values of* ω *on the curve. If* S_1 *(or* S_2*) is positive, then at least one pair of eigenvalues changes the sign of the real part from negative to positive. If* S_1 *(or* S_2*) is negative, then the sign change is in the opposite direction.*

4.2 General duopolies

In the case of $n = 2$ from (18), we have

$$\begin{aligned}
\varphi(\lambda) &= \left(A_1 e^{-\lambda\tau_1^1} - \lambda - B_1 e^{-\lambda\tau_2^1}\right)\left(A_2 e^{-\lambda\tau_1^2} - \lambda - B_2 e^{-\lambda\tau_2^2}\right) \\
&\quad + B_1 e^{-\lambda\tau_2^1}\left(A_2 e^{-\lambda\tau_1^2} - \lambda - B_2 e^{-\lambda\tau_2^2}\right) + B_2 e^{-\lambda\tau_2^2}\left(A_1 e^{-\lambda\tau_1^1} - \lambda - B_1 e^{-\lambda\tau_2^1}\right) \\
&= 0
\end{aligned}$$

which can be simplified as

$$\left(A_1 e^{-\lambda\tau_1^1} - \lambda\right)\left(A_2 e^{-\lambda\tau_1^2} - \lambda\right) - B_1 B_2 e^{-\lambda(\tau_2^1+\tau_2^2)} = 0. \tag{30}$$

This equation is analytically intractable, since it has four delays, $\tau_1^1,\ \tau_1^2,\ \tau_1^1 + \tau_1^2,\ \tau_2^1 + \tau_2^2$. Therefore we make the following simplifying assumption,

$$\tau_1^1 = \tau_1^2 = \tau_2^1 = \tau_2^2 = \tau$$

when (30) becomes

$$\lambda^2 - (A_1 + A_2)\, e^{-\lambda\tau} + (A_1 A_2 - B_1 B_2)\, e^{-2\lambda\tau} = 0.$$

By multiplying both sides by $e^{\lambda\tau}$ we have

$$\lambda^2 e^{\lambda\tau} - (A_1 + A_2)\,\lambda + (A_1 A_2 - B_1 B_2)\, e^{-\lambda\tau} = 0. \tag{31}$$

Without delay a quadratic equation is obtained by λ

$$\lambda^2 - (A_1 + A_2)\,\lambda + (A_1 A_2 - B_1 B_2) = 0.$$

The linear coefficient and constant term are positive, the roots are negative real values, since the discriminant is positive. Stability switch might occur if $\lambda = i\omega$ with $\omega > 0$, then from (31),

$$-\omega^2 (\cos\omega\tau + i\sin\omega\tau) - i\omega\,(A_1 + A_2) + (A_1 A_2 - B_1 B_2)(\cos\omega\tau - i\sin\omega\tau) = 0.$$

By separating the real and imaginary parts, we have

$$\begin{aligned}&\left[-\omega^2 + (A_1 A_2 - B_1 B_2)\right]\cos\omega\tau = 0\\ &\left[-\omega^2 - (A_1 A_2 - B_1 B_2)\right]\sin\omega\tau = \omega\,(A_1 + A_2)\end{aligned} \tag{32}$$

We have to consider now two possibilities from the first equation of (32).

(i) $\cos\omega\tau \neq 0$,

then,

$$\omega^2 = A_1 A_2 - B_1 B_2$$

and from the second equation of (32),

$$-2\,(A_1 A_2 - B_1 B_2)\sin\omega\tau = \omega\,(A_1 + A_2)\,. \tag{33}$$

However

$$\begin{aligned}&\omega^2 (A_1 + A_2)^2 - 4\,(A_1 A_2 - B_1 B_2)^2\\ =\ &(A_1 A_2 - B_1 B_2)\left(A_1^2 + A_2^2 + 2A_1 A_2\right) - 4\,(A_1 A_2 - B_1 B_2)^2\\ =\ &(A_1 A_2 - B_1 B_2)\left[(A_1 - A_2)^2 + 4B_1 B_2\right]\end{aligned}$$

where the first term is positive and the second is being nonnegative. If it is positive, (33) has no solution. If it is zero, then $\sin\omega\tau = +1$ implying that $\cos\omega\tau = 0$, which is contradiction. The other possibility is

(ii) $\cos\omega\tau = 0$,

then, from the second equation of (32), $\sin\omega\tau = 1$ and

$$\omega^2 + (A_1 + A_2)\,\omega + (A_1A_2 - B_1B_2) = 0. \quad (34)$$

The discriminant is

$$\Delta = (A_1 - A_2)^2 + 4B_1B_2 \geq 0,$$

furthermore,

$$(A_1 + A_2)^2 - \Delta = 4\,(A_1A_2 - B_1B_2) > 0,$$

so both roots are positive,

$$\omega_\pm = \frac{-\,(A_1 + A_2) \pm \sqrt{\Delta}}{2} > 0. \quad (35)$$

Notice that $\Delta = 0$ if $A_1 = A_2$ and one of B_1 and B_2 equals zero. Then $\omega_\pm = -A_1 = -A_2$. From the first equation of (32), the critical value of the delay are

$$\tau_m^\pm = \frac{1}{\omega_\pm}\left(\frac{\pi}{2} + 2m\pi\right) \text{ for } m = 0, 1, 2, ... \quad (36)$$

The direction of the stability switches are assessed by considering λ as the function of the bifurcation parameter, $\lambda = \lambda(\tau)$ and implicitly differentiating equation (31):

$$2\lambda\lambda' e^{\lambda\tau} + \lambda^2 e^{\lambda\tau}\left(\lambda'\tau + \lambda\right) - \lambda'\,(A_1 + A_2) + (A_1A_2 - B_1B_2)\,e^{-\lambda\tau}\left(-\lambda'\tau - \lambda\right) = 0.$$

The multiplier of λ' and the constant term are

$$2\lambda e^{\lambda\tau} + \lambda^2\tau e^{\lambda\tau} - A_1 - A_2 - (A_1A_2 - B_1B_2)\,e^{-\lambda\tau}\tau$$

and

$$\lambda^3 e^{\lambda\tau} - (A_1A_2 - B_1B_2)\,e^{-\lambda\tau}\lambda,$$

respectively, so

$$\left(\lambda'\right)^{-1} = \frac{-2\lambda e^{\lambda\tau} + A_1 + A_2}{\lambda^3 e^{\lambda\tau} - (A_1A_2 - B_1B_2)\,\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}. \quad (37)$$

Notice that at the critical values of $\tau_m^\pm$, $\cos\omega\tau = 0$ and $\sin\omega\tau = 1$ implying that

$$e^{i\omega\tau} = \cos\omega\tau + i\sin\omega\tau = i$$

and

$$e^{-i\omega\tau} = \cos\omega\tau - i\sin\omega\tau = -i.$$

At $\lambda = i\omega$, we are interested in the real part of (37), where the second term is pure complex. Therefore

$$\begin{aligned}\operatorname{Re}\left[\left(\lambda'\right)^{-1}\right] &= \operatorname{Re}\left[\frac{-2\lambda e^{\lambda\tau} + A_1 + A_2}{\lambda^3 e^{\lambda\tau} - (A_1A_2 - B_1B_2)\,\lambda e^{-\lambda\tau}}\right] \\ &= \frac{2\omega + (A_1 + A_2)}{\omega\left[\omega^2 - (A_1A_2 - B_1B_2)\right]}.\end{aligned}$$

Substituting (35) with numerator $\pm\sqrt{\Delta}$ into the denominator presents

$$\frac{\omega}{4}\left[2(A_1-A_2)^2+8B_1B_2\mp 2(A_1+A_2)\sqrt{\Delta}\right]=\frac{\omega}{2}\sqrt{\Delta}\left(\sqrt{\Delta}\mp(A_1+A_2)\right)$$

We know however that $|A_1+A_2|>\sqrt{\Delta}$, so this expression is positive at ω_+ and negative at ω_-. Hence $\mathrm{Re}\left[\left(\lambda'\right)^{-1}\right]$ is always positive.

Proposition 7 *The equilibrium is locally asymptotically stable for $\tau>\tau_0^+$, stability is lost at $\tau=\tau_0^+$ via Hopf bifurcation, and stability cannot be regained with larger values of τ.*

5 Conclusions

Environmental regulations were added to the classical n-firm Cournot model. In cases of non-point source pollution the regulator can measure only the total emission level without knowing the individual emissions of the firms. Therefore in the regulation the firms are uniformly punished if the total emission is higher then a regulator selected standard, and awarded otherwise. In the dynamic extensions we considered three cases. First, no delays were introduced about information on the outputs of all firms, second, delayed data were assumed about the output levels of the competitors and third, additional delays were added in the firms' own output levels. The stability analysis was conducted under general conditions which are satisfied in cases of linear and hyperbolic price functions. Models without delays, with single-delay and with two delays were analyzed in detail. It was demonstrated how the stability of the no-delay models can be lost by introducing delays. The stability thresholds, stability switching curves and directions of stability switches were analytically derived.

It will be an interesting project to extend the results of this paper to more general cases including nonlinear Cournot models, multi-product, labor managed oligopolies and rent-seeking games among others. This will be the subject of our continued research project.

References

Bischi, G-I, Chiarella, C., Kopel, M., and Szidarovszky, F., *Nonlinear oligopolies: stability and bifurcations*, Berlin/Heidelberg, Springer, 2010.

Downing, P. and White, L., Innovation in pollution control, *Journal of Environmental Economics and Management*, 13, 18-29, 1986.

Ganguli, S., and Raju, S., Perverse environmental effects of ambient charges in a Bertrand duopoly, *Journal of Environmental Economics and Policy*, 1, 289-296, 2012.

Ishikawa, T., Matsumoto, A., and Szidarovszky, F., Regulation of non-point source pollution under n-firm Bertrand competition, *Environmental Economics and Policy Studies*, doi.org/10.1007/s100-18-019-00243-9, 2019.

Jung, C., Krutilla, K., and Boyd, R., Incentives for advanced pollution abetment technology at the industry level: an evaluation of policy alternative, *Journal of Environmental Economics and Management*, 30, 95-111, 1996.

Matsumoto, A., Nakayama, K., and Szidarovszky, F., The ambient charge in hyperbolic duopoly and triopoly: static and dynamic analysis, in Nakayama, K., Miyata, Y. (ed) *Theoretical and Empirical Analysis in Environmental Economics*, 3-24, Springer-Verlag, 2019a.

Matsumoto, A., Nakayama, K., Okamura, M., and Szidarovszky, F., Environmental regulation for non-point source pollution in a Cournot three-stage game, *mimeo*, 2019b.

Matsumoto, A., and Szidarovszky, F., *Dynamic oligopolies with time delays*, Tokyo, Springer-Verlag, 2018.

Matsumoto, A., Szidarovszky, F., and Yabuta, M., Environmental effects of ambient charges in Cournot oligopoly, *Journal of Environmental Economics and Policy,* 7, 41-56, 2018a.

Matsumoto, A., Nakayama, K., and Szidarovszky, K., Environmental policy for non-point source pollutions in a Bertrand duopoly, *Theoretical Economic Letters*, 8, 1058-1069, 2018b

Montero, J-P., Permits, standards, and technology innovation, *Journal of Environmental Economics and Management*, 44, 23-44, 2002. (2002)

Okuguchi, K., *Expectations and stability in oligopoly models*, Berlin, Springer, 1976.

Okuguchi, K., and Szidarovszky, F., Existence of Nash-Cournot equilibrium in oligopoly game with pollution treatment cost-sharing, *Pure Mathematics and Applications*, 13, 455-462, 2002.

Okuguchi, K., and Szidarovszky, F., Environmental R&D in Cournot oligopoly with emission or performance standards, *Pure Mathematics and Applications*, 18, 111-119, 2007.

Okuguchi, K., and Szidarovszky, F. *The theory of oligopoly with multi-product firms* (2nd ed.), Berlin, Springer, 1999.

Raju, S., and Ganguli, S., Strategic firm interaction, returns to scales, environmental regulation and ambient charges in a Cournot duopoly, *Technology and Investment*, 4, 113-122, 2013.

Segerson, K., Uncertainty and incentives in nonpoint pollution control, *Journal of Environmental Economics and Management*, 15, 87-98, 1988.