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Abstract

Dynamic asymmetric contest games are examined under the assumption that the assessed value of the prize by each agent depends on the total effort of all agents, and each agent has only delayed information about the efforts of the competitors. Assuming gradient dynamics with continuous time scales, first the resulting one-delay model is investigated. Then assuming additional delayed information about the firms' own efforts, a two-delay model is constructed and analyzed. In both cases, first the characteristic equation is derived in the general case, and then two special cases are considered. First symmetric firms are assumed and then general duopolies are examined. Conditions are derived for the local stability of the equilibrium including stability thresholds and stability switching curves.

Keywords: Contest games, Hyperbolic oligopoly, Dynamic systems with time delays, Asymptotic behavior.

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1 Introduction

Contest games are the mathematical models of situations when several agents compete to win a given prize. The value of the prize might be exogenously given as in Perez-Castrillo and Verdies (1992), Szidarovszky and Okuguchi (1997), Cornes and Harley (2005) and Yamazaki (2008). However in many cases the prize should depend on the total effort of all agents, like in research and development, war or armament buildup when more joint effort makes the value of the prize higher. There are cases when more total effort might decrease the value. The studies of Chung (1996), Okuguchi (2005), Corchon (2007), Shaffer (2006) among others considered the possibility of total effort dependent value. Contest games are closely related to other types of games including hyperbolic oligopolies (Bischi et al., 2010), market share attraction games (Hanssens et al.,1990), rent seeking games (Tullock, 1980) among others. In the earlier studies mainly the existence and uniqueness of the pure Nash equilibrium was the main focus. The first attempt to examine dynamic contest games was done in Okuguchi and Szidarovszky (1999) where the local asymptotical stability of the pure Nash equilibrium was studied via linearization. Bischi et al. (2010) gives a detailed analysis of methods of nonlinear dynamics in different oligopoly models including hyperbolic price functions, which are equivalent with contest games. More recently Matsumoto and Szidarovszky (2018) present a comprehensive summary of the most recent results of different versions of nonlinear dynamic oligopolies. These models assume the availability of instantaneous information about the efforts of all agents, however collecting information, data analysis, finding most appropriate decisions and their implementation need time. There is a huge literature on delay differential equations and systems, they are summarized in Matsumoto and Szidarovszky (2018), where delayed nonlinear oligopolies are discusses as well, including the hyperbolic case. In this paper dynamic contest games are investigated. First, delays are assumed in the total effort of the competitors of each firm. The resulting one-delay model is discussed first. Then additional delay is introduced in the firms' own efforts leading to a two-delay model. In both cases the characteristic equations are derived in the general case. Two special cases, symmetric agents and general 2-agent cases will be then analyzed in detail.

The paper is developed as follows. The basic model is introduced in Section 2. One-delay dynamics is examined in Section 3, and the two-delay model is studied in Section 4. Section 5 presents conclusions and offers further research directions.

2 Basic Model

Assume n agents bid for a valuable asset, where y_i denotes the effort of agent i. Asset value assesment R_i by agent i depends on the total effort of all agents. The cost function of agent i is denoted by g_i. For all agents, the following assumptios are made:

Assumption 1. $R_i(\phi) > 0$ and $R_i''(\phi) \leq 0$ for all feasible total effots ϕ of the firms.

Assumption 2. $g_i(0) = 0$, $g_i'(y_i) > 0$ and $g_i''(y_i) > 0$ for all feasible effots y_i of firm i.

Hirai and Szidarovszky (2013) proved that under these conditions there is a unique pure Nash equilibrium. Assume that the total effort of the rest of the agents $\phi_i = \sum_{j \neq i} y_j$ is known only with same delay τ_i for each agent i. The payoff of agent i is

$$\pi_i(t) = R_i\left(y_i(t) + \phi_i(t-\tau_i)\right) \frac{y_i(t)}{y_i(t) + \phi_i(t-\tau_i)} - g_i\left(y_i(t)\right). \tag{1}$$

Differentiating (1) with respect to $y_i(t)$ yields

$$\frac{\partial \pi_i(t)}{\partial y_i(t)} = R_i'\left(y_i(t) + \phi_i(t-\tau_i)\right) \tfrac{y_i(t)}{y_i(t)+\phi_i(t-\tau_i)} + R_i\left(y_i(t) + \phi_i(t-\tau_i)\right) \tfrac{\phi_i(t-\tau_i)}{[y_i(t)+\phi_i(t-\tau_i)]^2} - g_i'\left(y_i(t)\right).$$

Let us denote the left hand side of the last equation by $f\left(y_i(t), \phi_i(t-\tau_i)\right)$.

3 General Dynamics with One Delay

Based on gradient adjustments the general dynamic equation is

$$\dot{y}_i(t) = K_i f_i\left(y_i(t), \phi_i(t-\tau_i)\right). \tag{2}$$

Notice that (arguments of R_i, R_i' and R_i'' are not shown)

$$\frac{\partial f_i(t)}{\partial y_i(t)} = R_i'' \frac{y_i(t)}{y_i(t) + \phi_i(t-\tau_i)} + 2R_i' \frac{\phi_i(t-\tau_i)}{[y_i(t) + \phi_i(t-\tau_i)]^2} + R_i \cdot \left(-\frac{2\phi_i(t-\tau_i)}{[y_i(t) + \phi_i(t-\tau_i)]^3} \right) - g_i''\left(y_i(t)\right)$$

and

$$\frac{\partial f_i(t)}{\partial \phi_i(t-\tau_i)} = R_i'' \frac{y_i(t)}{y_i(t) + \phi_i(t-\tau_i)} + R_i' \frac{\phi_i(t-\tau_i) - y_i(t)}{[y_i(t) + \phi_i(t-\tau_i)]^2} + R_i \cdot \left(\frac{y_i(t) - \phi_i(t-\tau_i)}{[y_i(t) + \phi_i(t-\tau_i)]^3} \right).$$

Let S_i and T_i denote these derivatives. The common denominator of T_i is $[y_i(t) + \phi_i(t-\tau_i)]^3$ and the numerator is

$$R_i'' y_i(t) \left[y_i(t) + \phi_i(t-\tau_i)\right]^2 + \left[\phi_i(t-\tau_i) - y_i(t)\right] \cdot \left[R_i' \left[y_i(t) + \phi_i(t-\tau_i)\right] - R_i\right].$$

It is easy to see that

$$R_i(0) = R_i\left(y_i(t) + \phi_i(t-\tau_i)\right) - \frac{R_i'(y_i(t)+\phi_i(t-\tau_i))}{1!}\left[y_i(t) + \phi_i(t-\tau_i)\right] + \frac{R_i''(\tilde{\Phi})}{2!}\left[y_i(t) + \phi_i(t-\tau_i)\right]^2 > 0$$

for $\tilde{\Phi} \in (0, y_i(t)+\phi_i(t-\tau_i))$, implying that $R_i'\left[y_i(t) + \phi_i(t-\tau_i)\right] - R_i < 0$, so the numerator of T_i is negative if $\phi_i(t-\tau_i) > y_i(t)$ meaning that there is no dominant agent. This is assumed in the following discussions. Notice that

$$S_i = T_i + u_i$$

with

$$u_i = \frac{R_i'\left[y_i(t) + \phi_i(t-\tau_i)\right] - R_i}{\left[y_i(t) + \phi_i(t-\tau_i)\right]^2} - g_i''\left(y_i(t)\right) < 0.$$

Therefore $S_i < 0$ as well. The linearized equation (2) can be written as

$$\dot{y}_{i\varepsilon}(t) = K_i S_i y_{i\varepsilon}(t) + K_i T_i \phi_{i\varepsilon}(t-\tau_i) \tag{3}$$

where S_i and T_i are on their equilibrium levels and $y_{i\varepsilon}$ and $\phi_{i\varepsilon}$ are their differences from equilibrium levels. To get the characteristic polynomial, assume that

$$y_{i\varepsilon}(t) = e^{\lambda t} u_i.$$

Then from (3)

$$\lambda e^{\lambda t} u_i = K_i S_i e^{\lambda t} u_i + K_i T_i \sum_{j\neq i} e^{\lambda(t-\tau_i)} u_j$$

leading to the characteristic equation

$$\varphi(\lambda) = \det\begin{pmatrix} K_1S_1 - \lambda & K_1T_1e^{-\lambda\tau_1} & \cdots & K_1T_1e^{-\lambda\tau_1} \\ K_2T_2e^{-\lambda\tau_2} & K_2S_2 - \lambda & \cdots & K_2T_2e^{-\lambda\tau_2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ K_nT_ne^{-\lambda\tau_n} & K_nT_ne^{-\lambda\tau_n} & \cdots & K_nS_n - \lambda \end{pmatrix} = 0. \tag{4}$$

This equation is impossible to be solved in general. Therefore, we will consider two special cases: symmetric n agents, and nonsymmetric case of $n = 2$.

3.1 Symmetric agents

Assume $\tau_1 = \tau_2 = ... = \tau_n = \tau$ and introduce notation,

$$\boldsymbol{a} = \begin{pmatrix} K_1T_1 \\ K_2T_2 \\ \cdot \\ \cdot \\ \cdot \\ K_nT_n \end{pmatrix} e^{-\lambda\tau}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

and

$$\boldsymbol{D} = diag\left(K_1S_1 - \lambda - K_1T_1e^{-\lambda\tau}, ..., K_nS_n - \lambda - K_nT_ne^{-\lambda\tau}\right)$$

to have

$$\begin{aligned}\varphi(\lambda) &= \det\left(\boldsymbol{D} + \boldsymbol{a}\boldsymbol{b}^T\right) \\ &= \det\left(\boldsymbol{D}\right)\det\left(\boldsymbol{I} + \boldsymbol{D}^{-1}\boldsymbol{a}\boldsymbol{b}^T\right) \\ &= \prod_{i=1}^n\left(K_iS_i - \lambda - K_iT_ie^{-\lambda\tau}\right)\left[1 + \sum_{i=1}^n \frac{K_iT_ie^{-\lambda\tau}}{K_iS_i - \lambda - K_iT_ie^{-\lambda\tau}}\right].\end{aligned} \tag{5}$$

In addition, assume that $K_1 = K_2 = ... = K_n = K$, $S_1 = S_2 = ... = S_n = S$ and $T_1 = T_2 = ... = T_n = T$ which is the case if the equilibrium is symmetric and the agents believe in the same price and cost functions, $g_i(y_i)$ and $R_i(y_i(t) + \phi_i(t - \tau_i))$. We have now two possibilities. Either

$$KS - \lambda - KTe^{-\lambda\tau} = 0 \tag{6}$$

or

$$KS - \lambda - KTe^{-\lambda\tau} + nKTe^{-\lambda\tau} = 0. \tag{7}$$

We first investigate equation (6), since equation (7) is obtained by replacing T with $\bar{T} = T(1 - n)$. We will return to equation (7) later.

Without delay $\tau = 0$, so from (6),

$$\lambda = KS - KT = K(S - T) = Ku < 0$$

implying stability. Stability switch may occur if $\lambda = i\omega$ ($\omega > 0$) which is substituted into equation (6) to get

$$KS - i\omega - KT\left(\cos\omega\tau - i\sin\omega\tau\right) = 0.$$

Separation of the real and imaginary parts gives

$$KT\cos\omega\tau = KS \tag{8}$$

and

$$KT\sin\omega\tau = \omega. \tag{9}$$

By adding the squares of these equations, we have

$$K^2T^2 = K^2S^2 + \omega^2$$

showing that

$$\omega^2 = K^2\left(T^2 - S^2\right). \tag{10}$$

Both T and S are negative and $S < T$ implying that $|S| > |T|$, so there is no positive solution for ω.

In the case of equation (7), T is replaced with $\bar{T} = T(1-n) > 0$. Without delay, from (7), $\lambda = KS + (n-1)KT < 0$. From (10),

$$\omega^2 = K^2\left(\bar{T}^2 - S^2\right) = K^2\left[T^2(1-n)^2 - S^2\right]. \tag{11}$$

So positive solution exists if $|T(n-1)| > |S|$ or $T(n-1) < S$. Otherwise no stability switch occurs. From (11),

$$\omega = K\sqrt{T^2(1-n)^2 - S^2}$$

and since $\bar{T} > 0$ and $S < 0$, from (8) and (9), we see that $\sin\omega\tau > 0$ and $\cos\omega\tau < 0$. The critical values of delay are

$$\tau_m = \frac{1}{\omega}\left[\cos^{-1}\left(\frac{S}{T(1-n)}\right) + 2m\pi\right] \text{ for } m = 0, 1, 2, ...$$

The direction of stability switching can be obtained by considering λ as function of the bifurcation parameter τ, $\lambda = \lambda(\tau)$. Implicitly differentiating equation (7) to get

$$-\dot{\lambda} + (n-1)KTe^{-\lambda\tau}\left(-\dot{\lambda}\tau - \lambda\right) = 0.$$

Therefore from (7),

$$\dot{\lambda} = \frac{(KS-\lambda)\lambda}{1-(KS-\lambda)\tau}.$$

If $\lambda = i\omega$, then

$$\dot{\lambda} = \frac{\omega^2 + i\omega KS}{(1-KS\tau) + i\omega\tau} \cdot \frac{(1-KS\tau) - i\omega\tau}{(1-KS\tau) - i\omega\tau}$$

with real part having same sign as

$$\omega^2(1-KS\tau) + \omega^2 KS\tau = \omega^2 > 0 \tag{12}$$

showing that stability is lost at the smallest critical value,

$$\tau_0 = \frac{1}{K\sqrt{T^2(1-n)^2 - S^2}}\left[\cos^{-1}\left(\frac{S}{T(1-n)}\right)\right] \tag{13}$$

since in this case $\sin\omega\tau$ is positive and $\cos\omega\tau$ is negative, and stability cannot be regained later.

3.2 Asymmetric two agents

Consider the general case of two agents. In this case, (4) is modified as

$$\varphi(\lambda) = \det \begin{pmatrix} K_1S_1 - \lambda & K_1T_1e^{-\lambda\tau_1} \\ K_2T_2e^{-\lambda\tau_2} & K_2S_2 - \lambda \end{pmatrix}.$$

The left hand side is expanded to

$$\lambda^2 - (K_1S_1 + K_2S_2)\,\lambda - K_1K_2T_1T_2e^{-\lambda(\tau_1+\tau_2)} + K_1K_2S_1S_2 = 0. \quad (14)$$

This is a single-delay equation with $\tau = \tau_1 + \tau_2$. Without delay $\tau = 0$ and from (14)

$$\lambda^2 - (K_1S_1 + K_2S_2)\,\lambda + K_1K_2\,(S_1S_2 - T_1T_2) = 0. \quad (15)$$

Notice that $S_1, S_2 < 0$ and $|S_1| > |T_1|\,,\ |S_2| > |T_2|\,,$ implying that $S_1S_2 > T_1T_2$, so both the linear coefficient and the constant term are positive. So without delay the equilibrium is locally asymptotically stable. Stability switch might occurs if $\lambda = i\omega$ with $\omega > 0$, which is then substituted into equation (14) to have

$$\omega^2 - i\omega\,(K_1S_1 + K_2S_2) + K_1K_2S_1S_2 - K_1K_2T_1T_2\,(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separation of the real and imaginary parts shows that

$$K_1K_2T_1T_2\cos\omega\tau = \omega^2 + K_1K_2S_1S_2$$

and

$$K_1K_2T_1T_2\sin\omega\tau = \omega\,(K_1S_1 + K_2S_2)\,.$$

Adding the squares of these equations gives

$$\omega^4 + \left[(K_1S_1 + K_2S_2)^2 + 2K_1K_2S_1S_2\right]\omega^2 + (K_1K_2S_1S_2)^2 - (K_1K_2T_1T_2)^2 = 0. \quad (16)$$

Since $S_1S_2 > T_1T_2$, both the linear coefficient and the constant term are positive implying that no $\omega > 0$ exists. So the equilibrium is asymptotically stable for all $\tau_1 > 0$ and $\tau_2 > 0$.

4 Two-delay model

Assume agent i has delay τ_i^1 on its own effective effort and delay τ_i^2 on the effective efforts of the competitors. Then equation (3) has the modified form,

$$\dot{y}_{i\varepsilon}(t) = K_iS_iy_{i\varepsilon}(t - \tau_i^1) + K_iT_i\phi_{i\varepsilon}(t - \tau_i^2) \quad (17)$$

The exponential solution forms $y_{k\varepsilon}(t) = e^{\lambda t} u_k$ give

$$\lambda e^{\lambda t} u_i = K_i S_i e^{\lambda(t-\tau_i^1)} u_i + K_i T_i \sum_{j \neq i} e^{\lambda(t-\tau_i^2)} u_j$$

leading to the characteristic equation,

$$\varphi(\lambda) = \det \begin{pmatrix} K_1 S_1 e^{-\lambda\tau_1^1} - \lambda & K_1 T_1 e^{-\lambda\tau_1^2} & \cdots & K_1 T_1 e^{-\lambda\tau_1^2} \\ K_2 T_2 e^{-\lambda\tau_2^2} & K_2 S_2 e^{-\lambda\tau_2^1} - \lambda \cdots & & K_2 T_2 e^{-\lambda\tau_2^2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ K_n T_n e^{-\lambda\tau_n^2} & K_n T_n e^{-\lambda\tau_n^2} & \cdots & K_n S_n e^{-\lambda\tau_n^1} - \lambda \end{pmatrix} = 0. \tag{18}$$

This equation is more complex than in the case of a single delay, so we discuss two special cases.

4.1 Symmetric agents

Assume $K_i = K,\ S_i = S,\ T_i = T,\ \tau_k^1 = \tau^1$ and $\tau_k^2 = \tau^2$ for $k = 1, 2, ..., n$. Similarly to the derivation of equations (6) and (7), we can see that there are two possibilities. Either

$$KSe^{-\lambda\tau^1} - \lambda - KTe^{-\lambda\tau^2} = 0 \tag{19}$$

or

$$KSe^{-\lambda\tau^1} - \lambda + (n-1)KTe^{-\lambda\tau^2} = 0. \tag{20}$$

For these equations we follow the procedure offered by Gu et al. (2005). Consider first equation (19), which can be rewritten as

$$1 + a_1(\lambda)e^{-\lambda\tau^1} + a_2(\lambda)e^{-\lambda\tau^2} = 0$$

with

$$a_1(\lambda) = -\frac{KS}{\lambda} \text{ and } a_2(\lambda) = \frac{KT}{\lambda}.$$

So

$$a_1(i\omega) = i\frac{KS}{\omega} \text{ and } a_2(i\omega) = -i\frac{KT}{\omega},$$

$$|a_1(i\omega)| = -\frac{KS}{\omega} \text{ and } |a_2(i\omega)| = -\frac{KT}{\omega},$$

and

$$\arg(a_1(i\omega)) = \frac{3\pi}{2} \text{ and } \arg(a_2(i\omega)) = \frac{\pi}{2}.$$

Then the range of ω defined by relations

$$|a_1(i\omega)| + |a_2(i\omega)| \geq 1$$

and

$$-1 \leq |a_1(i\omega)| - |a_2(i\omega)| \leq 1$$

implies in our case that

$$K(T-S) \leq \omega \leq -K(S+T) \tag{21}$$

since $S < T$. Define

$$\theta_1 = \cos^{-1}\left(\frac{1+|a_1(i\omega)|^2 - |a_2(i\omega)|^2}{2\,|a_1(i\omega)|}\right) = \cos^{-1}\left(\frac{\omega^2 + K^2S^2 - K^2T^2}{-2KS\omega}\right) \tag{22}$$

and

$$\theta_2 = \cos^{-1}\left(\frac{1+|a_2(i\omega)|^2 - |a_1(i\omega)|^2}{2\,|a_2(i\omega)|}\right) = \cos^{-1}\left(\frac{\omega^2 + K^2T^2 - K^2S^2}{-2KT\omega}\right) \tag{23}$$

and the points on the stability switching curves are given as

$$\tau_n^{1\pm}(\omega) = \frac{1}{\omega}\left(\frac{3\pi}{2} + (2n-1)\pi \pm \theta_1\right) \tag{24}$$

and

$$\tau_m^{2\mp}(\omega) = \frac{1}{\omega}\left(\frac{\pi}{2} + (2m-1)\pi \mp \theta_2\right). \tag{25}$$

The stability switching curves are formed by the set of points $(\tau_n^{1\pm}(\omega), \tau_m^{2\mp}(\omega))$ when ω runs through interval defined in (21), which are illustrated in Figure 1 with $m = 0, 1, 2, 3$ and $n = 1, 2, 3$. The red segments are the points $(\tau_n^{1+}(\omega), \tau_m^{2-}(\omega))$ and the blue segments represent the points $(\tau_n^{1-}(\omega), \tau_m^{2+}(\omega))$.

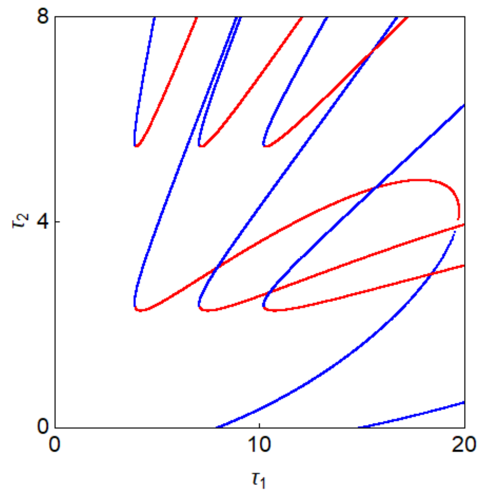

Figure 1. Stability switching curves with $K = 0.8,\ S = -3/2$ and $T = -1$

For determing the directions of stability switching we first compute quantities,

$$X_1 = a_1(i\omega)e^{-i\omega\tau^1} = i\frac{KS}{\omega}\left(\cos\omega\tau^1 - i\sin\omega\tau^1\right)$$

and

$$X_2 = a_2(i\omega)e^{-i\omega\tau^2} = -i\frac{KT}{\omega}\left(\cos\omega\tau^2 - i\sin\omega\tau^2\right)$$

from which

$$R_1 = \mathrm{Re}\,(X_1) = \frac{KS}{\omega}\sin\omega\tau^1,$$

$$R_2 = \mathrm{Re}\,(X_2) = -\frac{KT}{\omega}\sin\omega\tau^2,$$

$$I_1 = \mathrm{Im}\,(X_1) = \frac{KS}{\omega}\cos\omega\tau^1,$$

$$I_2 = \mathrm{Im}\,(X_2) = -\frac{KT}{\omega}\cos\omega\tau^2$$

and compute

$$\begin{aligned} R_2I_1 - R_1I_2 &= -\frac{K^2ST}{\omega^2}\left(\sin\omega\tau^2\cos\omega\tau^1 - \cos\omega\tau^2\sin\omega\tau^1\right) \\ &= -\frac{K^2ST}{\omega^2}\sin\left(\omega\left(\tau^2 - \tau^1\right)\right), \end{aligned} \tag{26}$$

the sign of which is the same as that of $\sin\left(\omega\left(\tau^1-\tau^2\right)\right)$. From Gu et al. (2005, Proposition 6.1) or Matsumoto and Szidarovszky (2018, Theorem A.2), we can apply the following result. Let ω be a point in interval (21) and a point $(\tau_n^{1\pm}(\omega), \tau_m^{2\pm}(\omega))$ on the stability switching curve and assume we are moving along the curve in increasing direction of ω. Then as (τ^1, τ^2) moves from the region on the right to the region on the left of the corresponding curve, then a pair of eigenvalues cross the imaginary axis to the right if $R_2 I_1 - R_1 I_2 > 0$, otherwise the crossing is in the opposite direction.

The case of equation (20) is very similar, T has to be replaced by $-(n-1)T$ in (22) and (23), furthermore $\arg\left(a_1(i\omega)\right) = 3\pi/2$ but $\arg\left(a_2(i\omega)\right)$ becomes $3\pi/2$. In addition, $|a_2(i\omega)|$ becomes $-(n-1)TK/\omega$ and (21) changes to

$$K\left|(n-1)T - S\right| \leq \omega \leq -K\left[S + (n-1)T\right].$$

4.2 General case of $n = 2$

In case of $n = 2$, (18) implies that

$$\varphi(\lambda) = \left(K_1 S_1 e^{-\lambda\tau_1^1} - \lambda\right)\left(K_2 S_2 e^{-\lambda\tau_2^1} - \lambda\right) - K_1 K_2 T_1 T_2 e^{-\lambda\left(\tau_1^2+\tau_2^2\right)} = 0. \tag{27}$$

This equation has four delayed terms with $\tau_1^1,\ \tau_2^1,\ \tau_1^1+\tau_2^1$ and $\tau_1^2+\tau_2^2$ which is analytically untractable. Theorefore we make the simplifying assumption that

$$\tau_1^1 = \tau_2^1 = \tau_1^2 = \tau_2^2 = \tau.$$

Then

$$\varphi(\lambda) = \lambda^2 - \left(K_1 S_1 + K_2 S_2\right)\lambda e^{-\lambda\tau} + K_1 K_2\left(S_1 S_2 - T_1 T_2\right)e^{-2\lambda\tau} = 0.$$

Multiplying both sides by $e^{\lambda\tau}$ to get

$$\lambda^2 e^{\lambda\tau} - \left(K_1 S_1 + K_2 S_2\right)\lambda + K_1 K_2\left(S_1 S_2 - T_1 T_2\right)e^{-\lambda\tau} = 0. \tag{28}$$

Without delay

$$\lambda^2 - \left(K_1 S_1 + K_2 S_2\right)\lambda + K_1 K_2\left(S_1 S_2 - T_1 T_2\right) = 0.$$

Since $S_i < T_i \leq 0,\ S_1 S_2 > T_1 T_2$ so all coefficients are positive, equilibrium is stable. Stability switching occurs when $\lambda = i\omega$, then (28) becomes

$$-\omega^2\left(\cos\omega\tau + i\sin\omega\tau\right) - i\omega\left(K_1 S_1 + K_2 S_2\right) + K_1 K_2\left(S_1 S_2 - T_1 T_2\right)\left(\cos\omega\tau - i\sin\omega\tau\right) = 0.$$

Separating the real and imigainary parts,

$$\left[-\omega^2 + K_1K_2\left(S_1S_2 - T_1T_2\right)\right]\cos\omega\tau = 0 \tag{29}$$

and

$$\left[-\omega^2 - K_1K_2\left(S_1S_2 - T_1T_2\right)\right]\sin\omega\tau = \omega\left(K_1S_1 + K_2S_2\right). \tag{30}$$

Consider first equation (29), where we have two possibilities, $-\omega^2 + K_1K_2\left(S_1S_2 - T_1T_2\right) = 0$ and $\cos\omega\tau = 0$.

Case (i)

$-\omega^2 + K_1K_2\left(S_1S_2 - T_1T_2\right) = 0$ so

$$\omega = \sqrt{K_1K_2\left(S_1S_2 - T_1T_2\right)}$$

where the expression under the square root is positive. Then from (30)

$$-2\omega^2\sin\omega\tau = \omega\left(K_1S_1 + K_2S_2\right).$$

Since $\omega \neq 0$,

$$-2\omega\sin\omega\tau = K_1S_1 + K_2S_2.$$

We can easilty show that there is no solution, since

$$\left|-2\omega\right| < \left|K_1S_1 + K_2S_2\right|,$$

which follows from the following:

$$\begin{aligned}
&4\omega^2 - \left(K_1S_1 + K_2S_2\right)^2\\
&= 4K_1K_2\left(S_1S_2 - T_1T_2\right) - K_1^2S_1^2 - 2K_1K_2S_1S_2 - K_2^2S_2^2\\
&= -4K_1K_2T_1T_2 - \left(K_1S_1 - K_2S_2\right)^2 < 0.
\end{aligned}$$

Case (ii)

$\cos\omega\tau = 0$, and from (30), $\sin\omega\tau > 0$, necessarily $\sin\omega\tau = 1$. Then from (30),

$$\omega^2 + \left(K_1S_1 + K_2S_2\right)\omega + K_1K_2\left(S_1S_2 - T_1T_2\right) = 0. \tag{31}$$

The discriminant is

$$\Delta = \left(K_1S_1 - K_2S_2\right)^2 + 4K_1K_2T_1T_2 > 0. \tag{32}$$

Furthermore,

$$\left(K_1S_1 + K_2S_2\right)^2 - \Delta = 4K_1K_2\left(S_1S_2 - T_1T_2\right) > 0. \tag{33}$$

Therefore there are two positive solutions,

$$\omega_{\pm} = \frac{-\left(K_1S_1 + K_2S_2\right) \pm \sqrt{\Delta}}{2} \text{ with } \omega_+ > \omega_-. \tag{34}$$

The critical values for τ are

$$\tau_m^{\pm} = \frac{1}{\omega_{\pm}}\left(\frac{\pi}{2} + 2m\pi\right) \text{ for } m = 0, 1, 2, ... \tag{35}$$

The direction of the stability switches can be obtained by considering λ as function of τ, $\lambda = \lambda(\tau)$, and impilicitly differentiating equation (28) with respect to τ:

$$2\lambda\lambda' e^{\lambda\tau} + \lambda^2\left(\lambda'\tau + \lambda\right)e^{\lambda\tau} - \left(K_1S_1 + K_2S_2\right)\lambda' - K_1K_2\left(S_1S_2 - T_1T_2\right)e^{-\lambda\tau}\left(\lambda'\tau + \lambda\right) = 0.$$

The multiplier of λ' equals

$$-\left(K_1S_1 + K_2S_2\right) + (2\lambda + \lambda^2\tau)e^{\lambda t} - K_1K_2\left(S_1S_2 - T_1T_2\right)\tau e^{-\lambda\tau}$$

and the constant term is

$$\left[\lambda^2 e^{\lambda\tau} - K_1K_2\left(S_1S_2 - T_1T_2\right)e^{-\lambda\tau}\right]\lambda.$$

So

$$\frac{1}{\lambda'} = \frac{K_1S_1 + K_2S_2 - 2\lambda e^{\lambda\tau}}{\left[\lambda^2 e^{\lambda\tau} - K_1K_2\left(S_1S_2 - T_1T_2\right)e^{-\lambda\tau}\right]\lambda} - \frac{\tau}{\lambda}. \tag{36}$$

At $\lambda = i\omega$,

$$\lambda\tau = i\omega\tau = i\left(\frac{\pi}{2} + 2m\pi\right)$$

so $e^{\lambda\tau} = i$ and $e^{-\lambda\tau} = -i$. As the real part is concerned, only the first term counts, the numerator of which becomes

$$K_1S_1 + K_2S_2 - 2i\omega(i) = K_1S_1 + K_2S_2 + 2\omega \tag{37}$$

the denominator is

$$-i\omega^3 i - K_1K_2\left(S_1S_2 - T_1T_2\right)(-i)i\omega = \omega\left[\omega^2 - K_1K_2\left(S_1S_2 - T_1T_2\right)\right]. \tag{38}$$

Both are real values. Notice first that

$$K_1S_1 + K_2S_2 + 2\omega_{\pm} = \pm\sqrt{\Delta}$$

which is positive if $\omega = \omega_+$ and negative if $\omega = \omega_-$.

Similarly, (37) has the same sign as

$$\omega^2 - K_1 K_2 (S_1 S_2 - T_1 T_2) = \frac{\sqrt{\Delta}}{2} \left(\mp (K_1 S_1 + K_2 S_2) + \sqrt{\Delta} \right)$$

where ω^2 from (34) is substituted to get the final form. Using (33), we see that this is positive at ω_+ and negative at ω_-. Hence $\mathrm{Re}(\lambda') > 0$ at all critical values. Hence the equilibrium is locally asymptotically stable for $\tau < \tau_0^+$ (the smalles critical value), at $\tau = \tau_0^+$ stability is lost via Hopf bifurcation, and stability cannot be regained later.

5 Conclusions

In this paper dynamic contest games were introduced and examined when the assessment of the value of the prize by each agent depends on the total effort of all agents. First delays are assumed in the total effort of the competitors of each agent, then a one-delay model is obtained. By introducing additional delays in the firms' own efforts a two-delay model is obtained in the cases of symmetric firms and the 2-agent cases. The stability threshold for the one-delay case and the stability switching curves for the two-delay case are analytically determined. For analytic simplicity simplified models were mathematically investigated. In our further studies we will consider ways to relax these conditions to reach more realistic models and stability results.

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