

Dynamics in Delay Monopoly*

Akio Matsumoto[†]
Chuo University

Ferenc Szidarovszky[‡]
University of Arizona

Abstract

The purpose of this paper is to study the dynamics of a delay differential monopoly with the gradient method. The paper is divided into two parts. First we assume constant speed of adjustment and investigate stability switches of a stationary state by analyzing the corresponding characteristic equation in cases of single fixed time delay, two fixed time delays and a continuously distributed time delay. Second, we adopt output-dependent speed of adjustment and demonstrate the birth of Hopf bifurcation with respect to the time delay when a stationary state is locally unstable.

Keywords: delay monopoly, single and multiple fixed time delay, continuously distributed time delay, Hopf bifurcation

JEL Classification: D42

*The paper was prepared when the first author visited the Department of Systems and Industrial Engineering of the University of Arizona. He appreciated its hospitality over his stay. The usual disclaimer applies.

[†]Department of Economics, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan. akiom@tamacc.chuo-u.ac.jp

[‡]Department of Systems and Industrial Engineering, Tucson, Arizona, 85721-0020, USA. szidar@sie.arizona.edu

1 Introduction

In recent year an increasing attention has been given to the periodic and aperiodic behavior of systems that can be described by delay differential equation (see for example, Bélair and Mackey, 1989 and Invernizzi and Medio, 1991). With the infinite dimensionality created by delays, even a single first-order equation is transformed into an equation with a sufficient number of degrees of freedom to permit the occurrence of chaotic phenomena (e.g., see an der Heiden and Mackey (1982)). This finding indicates that delay models of a dynamic economy may explain various complex dynamic behavior of the economic variables. It also indicates that in modelling firm's dynamic behavior, it is necessary and essential to consider time delays inherent in phenomena like information and implementation delays. In spite of this, it is usually assumed that the economic agents have instantaneous information about their own behavior and also on the competition's behavior mainly due to mathematical convenience.

In constructing dynamic economic models, the most common processes are based on either the gradients of the profit functions or the best replies of the agents. Time delays can be modeled in two different ways: fixed time delay and continuously distributed time delays. Chiarella and Khomin (1996) and Chiarella and Szidarovszky (2001) examine delay differential oligopolies with best replies by using continuously distributed time delays. Howroyd and Russel (1984) detect the stability conditions of delay output adjustment processes in a general N -firm oligopoly with fixed time delay. In the case of the gradient method, we mention the works of Bischi and Naimuzada (1999) and Bischi and Lamantia (2002) for discrete oligopoly dynamics without time delays and the works of Puu (2003) and Ahmad et al. (2000) for discrete monopoly dynamics with time delays. Recently, Matsumoto et al. (2009) introduce the gradient method into delay differential Cournot models and investigate their dynamics under both fixed and continuously distributed time delays. Although it has been shown by Ahmad et al. (2000) that delay increases stability and by Puu (2000) that chaos can emerge in delay discrete monopoly, dynamics of a delay differential monopoly with the gradient method has not yet been revealed in the existing literature. The main purpose of this study is to demonstrate the possibility of the birth of limit cycles in such a delay differential monopoly with the gradient method.

The paper is organized as follows. Section 2 constructs a basic monopoly model with linear price and cost functions and then introduces time delay into the basic model provided that the speed of adjustment is constant. Section 3 discusses a nonlinear extension of the basic model by adopting the growth rate adjustment process. Section 4 concludes the paper.

2 Delay Linear Monopoly

Dynamics in a monopoly model with delays is considered. The model is simple and standard but suffices to bring out cyclic fluctuations in output production.

Let q be the output produced by a monopoly firm with a unit production cost, c . The price function is assumed to be linear

$$f(q) = a - bq, \quad a > 0 \text{ and } b > 0.$$

Then the profit of the monopoly firm is given by

$$\pi = (a - bq)q - cq.$$

The optimal output that maximizes the profit is obtained from the first-order condition for an interior solution,

$$q^M = \frac{a - c}{2b}$$

where $a > c$ is assumed for the positive equilibrium output. In the text book model, it is usually implicitly assumed that the monopoly firm has full information on the entire demand curve. As a result, such a monopoly firm can jump to the profit maximizing output. The more recent literature of dynamic processes raises doubts on the assumption of full rationality and adopts bounded rationality in a sense that a monopoly firm has only limited or partial information on the demand curve. Under such a circumstance, a more realistic approach is to formulate the dynamic process with the gradient method:

$$\dot{q} = \alpha(q) \frac{d\pi}{dq} \tag{1}$$

where $\alpha(q)$ is a positive function that gives the adjustment of the firm. According to (1), the firm changes its quantities at a rate proportional to the marginal profit. In constructing best response dynamics global information is required about the profit functions, however in applying gradient dynamics only local information is needed. We start with a simple case and assume a constant adjustment coefficient in (1):

Assumption 1. $\alpha(q) = \alpha > 0$.

The gradient dynamics with a fixed time lag $\tau > 0$ under Assumption 1 is presented by

$$\dot{q}(t) = \alpha [a - c - 2bq(t - \tau)]. \tag{2}$$

Introducing the new variable $x(t) = q(t) - q^M$ reduces (2) to

$$\dot{x}(t) = -\gamma x(t - \tau) \text{ with } \gamma = 2b\alpha > 0, \tag{3}$$

which is a first-order delay linear equation with a trivial solution $x(t) = 0$ (or $q(t) = q^M$) for all $t \geq 0$. If there is no time lag, equation (3) generates a solution, $x_0 e^{-\gamma t}$, that converges to the trivial solution as t approaches infinity. Even in the case of the delay equation, substituting the exponential solution $x(t) = x_0 e^{\lambda t}$ into (3) and then arranging terms yields its corresponding characteristic equation

$$\lambda + \gamma e^{-\lambda \tau} = 0. \tag{4}$$

By continuity, it can be supposed that (3) is stable for a small value of τ . As the length of the delay changes, the stability of the trivial solution may also change. Such phenomena is referred to as a *stability switch*. In order to understand the stability switches of system (3), it is crucial to determine a threshold value of τ at which (4) has a pair of conjugate pure imaginary roots. It is clear that $\lambda = 0$ is not a solution of (4). We then assume without loss of generality that $\lambda = iv$, $v > 0$, is a root of the transcendental equation and write the real and imaginary parts as follow:

$$\gamma \cos v\tau = 0$$

and

$$v - \gamma \sin v\tau = 0.$$

Adding up the squares of both equations yields $v^2 = \gamma^2$ from which we obtain

$$v = \gamma.$$

The characteristic equation is a function of the delay τ , and hence the roots of the characteristic equation are also functions of the delay. Differentiating the characteristic equation with respect to τ yields

$$(1 - \gamma\tau e^{-\lambda\tau}) \frac{d\lambda}{d\tau} = \gamma\lambda e^{-\lambda\tau}.$$

which is reduced to

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1 - \gamma\tau e^{-\lambda\tau}}{\gamma\lambda e^{-\lambda\tau}} \text{ and from (4), } e^{-\lambda\tau} = -\frac{\lambda}{\gamma}.$$

Thus

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau}\right)^{-1} \right] = \frac{1}{v^2} > 0.$$

This inequality implies that all roots that cross the imaginary axis at iv cross from left to right as τ increases. From the characteristic equation for $\lambda = iv$,

$$\cos v\tau = 0$$

and

$$\sin v\tau = \frac{v}{\gamma} = 1.$$

There is a unique $v\tau \in (0, 2\pi]$ such that $v\tau$ satisfies both equations, $v\tau = \pi/2$. We denote the threshold value of τ as

$$\tau^* = \frac{\pi}{2\gamma} \simeq \frac{1.57}{\gamma}. \quad (5)$$

This result is summarized as follow:

Theorem 1 *System (3) has a threshold value τ^* of delay: the stationary point is asymptotically stable when $0 < \tau < \tau^*$ and it is unstable when $\tau > \tau^*$ where*

$$\tau^* = \frac{\pi}{2\gamma}.$$

Theorem 1 deals with the case in which the monopoly firm uses a small neighborhood of one point on the demand curve to determine its dynamic behavior. It implies that a small delay does not violate stability of the system but a larger delay might destabilize it. We will next examine stability if the monopoly firm determines its rate of the output change by taking into account realized outputs at time periods, $t - \tau_1 > 0$ and $t - \tau_2 > 0$. The dynamic equation having two time delays depends on the linear combination of the past outputs,

$$\dot{q}(t) = \alpha [a - c - 2b(wq(t - \tau_1) + (1 - w)q(t - \tau_2))] \quad (6)$$

where $w \in (0, 1)$ and $1 - w$ are the weights of the past outputs. We assume that $w \geq 1 - w$, that is, more weight is given to the more recent output¹. The introduction of variable $x(t) = q(t) - q^M$ again reduces equation (6) to

$$\dot{x}(t) = -\gamma[w x(t - \tau_1) + (1 - w)x(t - \tau_2)]. \quad (7)$$

Looking for a solution $x(t) = x_0 e^{\lambda t}$ yields the characteristic equation

$$\lambda + \gamma w e^{-\lambda \tau_1} + \gamma(1 - w)e^{-\lambda \tau_2} = 0.$$

For notational simplicity, let

$$\bar{\lambda} = \frac{\lambda}{\gamma}, \quad \gamma_1 = \gamma \tau_1 \text{ and } \gamma_2 = \gamma \tau_2$$

which then leads to

$$\bar{\lambda} + w e^{-\bar{\lambda} \gamma_1} + (1 - w)e^{-\bar{\lambda} \gamma_2} = 0.$$

Dropping the bar from $\bar{\lambda}$, we obtain the normalized characteristic equation

$$\lambda + w e^{-\lambda \gamma_1} + (1 - w)e^{-\lambda \gamma_2} = 0. \quad (8)$$

When $\gamma_1 = \gamma_2$, equation (8) is essentially the same as equation (4). Thus $\gamma_1 \neq \gamma_2$ is assumed henceforth.

Symmetric weights: $w = \frac{1}{2}$

¹Mathematically, the case of $w > 1 - w$ is dual to the case of $w < 1 - w$. The results obtained in one case reduces to the results in the other if w and $1 - w$ are replaced with $1 - w$ and w , respectively.

Hale (1979) determines the geometry of the stability region for a linear differential equation with two delays when the weights of the delays are the same. If $\lambda = iv$ in equation (8) with $w = 1/2$, then the real and imaginary parts satisfy

$$0 = \frac{1}{2} \cos[v\gamma_1] + \frac{1}{2} \cos[v\gamma_2]$$

and

$$v = \frac{1}{2} \sin[v\gamma_1] + \frac{1}{2} \sin[v\gamma_2]$$

or, equivalently,

$$\begin{aligned} 0 &= \cos \left[\frac{v(\gamma_1 + \gamma_2)}{2} \right] \cos \left[\frac{v(\gamma_1 - \gamma_2)}{2} \right] \\ v &= \sin \left[\frac{v(\gamma_1 + \gamma_2)}{2} \right] \cos \left[\frac{v(\gamma_1 - \gamma_2)}{2} \right]. \end{aligned} \tag{9}$$

Notice that

$$\cos \left[\frac{v(\gamma_1 - \gamma_2)}{2} \right] \neq 0,$$

otherwise the two equations of (9) do not hold simultaneously. Setting the first equation of (9) equal to zero and solving it for v produce the main value solution²

$$v = \frac{\pi}{\gamma_1 + \gamma_2}$$

and from the second equation of (9),

$$\frac{\pi}{\gamma_1 + \gamma_2} = \cos \left[\frac{\pi(\gamma_1 - \gamma_2)}{2(\gamma_1 + \gamma_2)} \right].$$

This is a hyperbolic curve passing through the point $(\pi/2, \pi/2)$ and divides the (γ_1, γ_2) space into two parts as illustrated in Figure 1. All roots of equation (8) have strictly negative real parts for (γ_1, γ_2) in the gray region under the curve, and instability of system (7) occurs for (γ_1, γ_2) in the white region above the curve. We call it a *partition curve*. It is also observed that the curve is symmetric with respect to the diagonal and asymptotic to the line $\gamma_i = 1$, which then implies that the stationary state is asymptotically stable for any $\gamma_j > 0$ if $\gamma_i \leq 1$ for $i, j = 1, 2$ and $i \neq j$. Comparing this last result with Theorem 1 reveals that the two time delays increase the stability region when the weights are the same.

²System (9) has a general solutions

$$v = \frac{(2k+1)\pi}{\gamma_1 + \gamma_2}$$

and

$$(-1)^k \frac{(2k+1)\pi}{\gamma_1 + \gamma_2} = \cos \left((2k+1) \frac{\pi}{2} \frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2} \right) \quad k = 0, 1, 2, \dots$$

The main value solution means the solution for $k = 0$.

Theorem 2 System (7) with $w = \frac{1}{2}$ has a partition curve and its stationary state is asymptotically stable for (γ_1, γ_2) below the curve and unstable above it where the partition curve is defined by the locus (γ_1, γ_2) satisfying

$$\frac{\pi}{\gamma_1 + \gamma_2} = \cos \left[\frac{\pi(\gamma_1 - \gamma_2)}{2(\gamma_1 + \gamma_2)} \right].$$

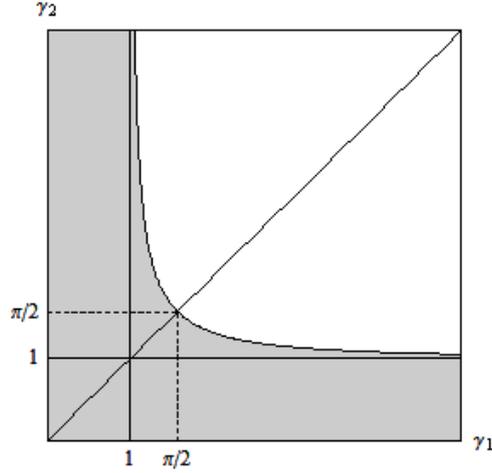


Figure 1. Division of the (γ_1, γ_2) space when $\omega = \frac{1}{2}$

Asymmetric weights: $\omega > \frac{1}{2}$

Differential equations with two time delays are very different from differential equations with a single time delay when $\gamma_1 \neq \gamma_2$. Following the analysis of Hale and Huang (1993), we give a geometric description of the steady state for system (7) in the (γ_1, γ_2) parameter space. We assume that $\lambda = iv$ with $v > 0$ is a root of equation (8) satisfying

$$iv + \omega e^{-iv\gamma_1} + (1 - \omega)e^{-iv\gamma_2} = 0.$$

Let \mathbb{C} denote the complex plane and

$$\Gamma_{|1-\omega|} = \{z \in \mathbb{C} : |z| = |1 - \omega|\}.$$

Denote γ_1 by r , γ_2 by σ and $vr = s$ for notational simplicity. For $r > 0$, define function $f_r : [0, 2\pi] \rightarrow \mathbb{C}$ by

$$f_r(s) = i\frac{s}{r} + \omega e^{-is}.$$

Let

$$\begin{aligned} g(r, s) &= r^2(|f_r(s)|^2 - |1 - \omega|^2) \\ &= s^2 - 2\omega r s \sin[s] + r^2(\omega^2 - (1 - \omega)^2) \end{aligned}$$

We can find the two combinations of r and s such that

$$g(r, s) = 0 \text{ and } \frac{dr}{ds} = -\frac{\partial g/\partial s}{\partial g/\partial r} = 0.$$

Solving the second condition for r yields

$$r = \frac{s}{\omega(\sin[s] + s \cos[s])}$$

which is substituted into the first condition to obtain

$$s^2 [\omega^2 (s^2 \cos^2[s] - \sin^2[s]) + (\omega^2 - (1 - \omega)^2)] = 0.$$

The graphical representation of this equation shows the existence of two solutions, s_0^* and s_2^* , and corresponding r_0 and r_2 ,

$$r_0 = \frac{s_0^*}{\omega(\sin[s_0^*] + s_0^* \cos[s_0^*])}$$

and

$$r_2 = \frac{s_2^*}{\omega(\sin[s_2^*] + s_2^* \cos[s_2^*])}.$$

As can be seen in Figure 2,

$$f_r([0, 2\pi]) \cap \Gamma_{|1-\omega|} \neq \emptyset$$

for $r \in [r_0, r_2]$. The two dotted points in Figure 2 indicate that $f_{r_i}([0, 2\pi]) \cap \Gamma_{|1-\omega|} = \{f_{r_i}(s_i)\}$ for $i = 0, 2$. Since the $f_r([0, 2\pi])$ curve shifts downward as r increases, there is a value $r_1 \in (r_0, r_2)$ such that $f_{r_1}(s)$ passes through the point $(-(1 - \omega), 0)$, that is,

$$r_1 = \frac{\cos^{-1} \left[-\frac{1-\omega}{\omega} \right]}{\omega \sin[\cos^{-1} \left[-\frac{1-\omega}{\omega} \right]]}$$

and

$$s_2(r_1) = r_1 \sqrt{\omega^2 - (1 - \omega)^2}.$$

Now for each $r \in (r_0, r_2)$, let $p_i = f_r(s(r))$ for $r = 1, 2$ and $\theta_i(r) \in [0, 2\pi]$ denote the angles from the negative real axis to the rays starting at the origin and passing through p_i in the clockwise direction respectively as illustrated in

Figure 2.

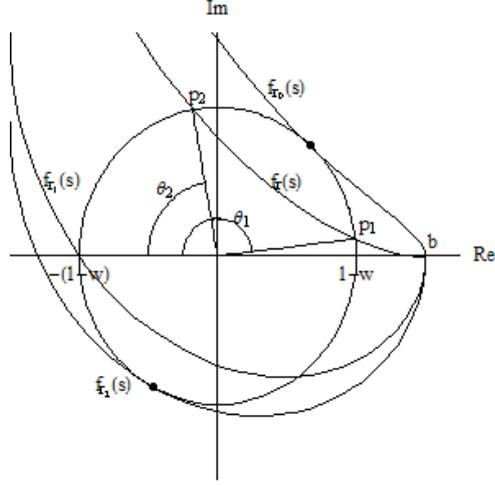


Figure 2. Illustration of $f_r(s)$ and Γ_{1-w} in \mathbb{C}

Let $\alpha = \text{Re}[f_r(s_i(r))]$ and $\beta = \text{Im}[f_r(s_i(r))]$. Then we have explicit forms of $\theta_1(r)$ and $\theta_2(r)$:

$$\theta_1(r) = \begin{cases} \pi - \tan^{-1} \left[\frac{\beta}{\alpha} \right] & \text{for } \alpha > 0 \text{ and } \beta \leq 0, \\ 2\pi - \tan^{-1} \left[\frac{\beta}{\alpha} \right] & \text{for } \alpha < 0 \text{ and } \beta < 0, \end{cases}$$

and

$$\theta_2(r) = \begin{cases} \pi - \tan^{-1} \left[\frac{\beta}{\alpha} \right] & \text{for } \alpha > 0 \text{ and } \beta > 0, \\ -\tan^{-1} \left[\frac{\beta}{\alpha} \right] & \text{for } \alpha < 0 \text{ and } \beta > 0, \\ 2\pi - \tan^{-1} \left[\frac{\beta}{\alpha} \right] & \text{for } \alpha < 0 \text{ and } \beta < 0. \end{cases}$$

Function $\theta_1(r)$ is continuous on $[r_0, r_2]$, and $\theta_2(r)$ is continuous on $[r_0, r_2] \setminus \{r_1\}$ with a jump at r_1 , since

$$\lim_{r \rightarrow r_1^-} \theta_2(r) = 0 \text{ and } \lim_{r \rightarrow r_1^+} \theta_2(r) = 2\pi.$$

Now let

$$\sigma_i(r) = \frac{r\theta_i(r)}{s_i(r)}.$$

We then select the value $w = 3/5$ to have the red curve of $\sigma = \sigma_1(r)$ and the blue curve of $\sigma = \sigma_2(r)$ shown in Figure 3A in which the dotted point corresponds to $(\pi/2, \pi/2)$ and the blue curve is discontinuous at $r = r_1$.³ Comparing

³The shape of $\sigma_i(r)$ depends on the value of w . Inequality $\sigma_1(r) > \sigma_2(r)$ on (r_1, r_2) can be reversed for a larger value of w .

Figure 2 with Figure 3A reveals the effects on the stability region caused by the asymmetric weights: the stability region is on the left side of the boundary curves, $\sigma = \sigma_1(r)$ and $\sigma = \sigma_2(r)$. In Figure 3B, we periodically extend $\sigma_i(r)$ by

$$\sigma_i^n(r) = \sigma_i(r) + \frac{2n\pi r}{s_i(r)} \text{ for } r \in [r_0, r_2] \text{ (} i = 1, 2\text{)}.$$

Dynamic system (7) is locally stable for (r, σ) in the gray region and it is locally asymptotically stable for any $\sigma > 0$ if $r < r_0$. Furthermore, the stability changes as (r, σ) crosses the $\sigma = \sigma_i(r)$ locus (i.e., stability switch occurs).

Theorem 3 *System (7) with $\omega > \frac{1}{2}$ is globally asymptotically stable for (r, σ) in the shaded region of Figure 3 and unstable otherwise.*

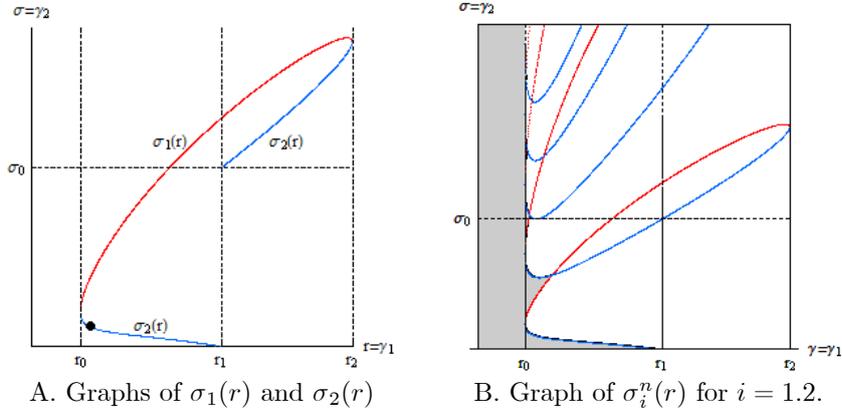


Figure 3. Stability region of (7) with $w = \frac{3}{5}$

Continuously distributed time delay is an alternative approach to deal with delays. If the expected deviation of the output from its equilibrium value is denoted by $x^\varepsilon(t)$ at time t and is based on the entire history of the actual changes of the deviations from zero to t , then the gradient dynamics with continuously distributed time lag can be written as the system of integro-differential equations

$$\begin{cases} \dot{x}(t) = -\gamma x^\varepsilon(t), \\ x^\varepsilon(t) = \int_0^t w(t-s, \tau, m) x(s) ds, \end{cases} \quad (10)$$

where the weighting function is defined by

$$w(t-s, \tau, m) = \begin{cases} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} & \text{if } m = 0, \\ \frac{1}{m!} \left(\frac{m}{\tau}\right)^{m+1} (t-s)^m e^{-\frac{m(t-s)}{\tau}} & \text{if } m \geq 1. \end{cases}$$

Here m is a nonnegative integer and τ is a positive real parameter, which is associated with the length of the delay. To examine local dynamics of system (10) in the neighborhood of the stationary point, we substitute the second equation of (10) into the first to get

$$\dot{x}(t) + \gamma \int_0^t w(t-s, \tau, m)x(s)ds = 0.$$

Looking for the solution in the usual exponential form $x(t) = x_0 e^{\lambda t}$ and substituting it into the above equation, we obtain

$$\lambda + \gamma \int_0^t w(t-s, \tau, m)e^{-\lambda(t-s)}ds = 0.$$

Introducing the new variable $z = t - s$ simplifies the integral as

$$\int_0^t w(t-s, \tau, m)e^{-\lambda(t-s)}ds = \int_0^t w(z, \tau, m)e^{-\lambda z}dz.$$

By letting $t \rightarrow \infty$ and assuming that $\text{Re}(\lambda) + \frac{m}{\tau} > 0$, we have

$$\int_0^\infty \frac{1}{\tau} e^{-\frac{z}{\tau}} e^{-\lambda z} dz = (1 + \lambda\tau)^{-1} \text{ if } m = 0$$

and

$$\int_0^\infty \frac{1}{m!} \left(\frac{m}{\tau}\right)^{m+1} z^m e^{-\frac{mz}{\tau}} e^{-\lambda z} dz = \left(1 + \frac{\lambda\tau}{m}\right)^{-(m+1)} \text{ if } m \geq 1.$$

That is,

$$\int_0^\infty w(z, \tau, m)e^{-\lambda z} dz = \left(1 + \frac{\lambda\tau}{\ell}\right)^{-(m+1)}$$

with

$$\ell = \begin{cases} 1 & \text{if } \ell = 0, \\ \ell & \text{if } \ell \geq 1. \end{cases}$$

Then the characteristic equations becomes

$$\lambda \left(1 + \frac{\tau}{\ell} \lambda\right)^{m+1} + \gamma = 0. \quad (11)$$

Expanding the characteristic equation presents the $(m+2)$ -th order equation

$$a_0 \lambda^{m+2} + a_1 \lambda^{m+1} + \dots + a_{m+1} \lambda + a_{m+2} = 0$$

where the coefficients a_i are defined by

$$a_k = \left(\frac{\tau}{\ell}\right)^{m+1-k} \binom{m+1}{k} \text{ for } 0 \leq k \leq m,$$

$$a_{m+1} = 1 \text{ and } a_{m+2} = \gamma.$$

According to the Routh-Hurwitz stability condition, the necessary and sufficient conditions that all roots of the characteristic equation have negative real parts are the following:

- (1) the coefficients are positive, $a_k > 0$ for $k = 0, 1, 2, \dots, m + 2$,
- (2) the principle minors of the Routh-Hurwitz determinant are positive,

$$D_2^m = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0, \quad D_3^m = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, \quad D_4^m = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix} > 0, \dots$$

Case 1. $m = 0$.

The characteristic equation (11) is quadratic, $\tau\lambda^2 + \lambda + \gamma = 0$. Since all coefficients are positive, the stationary state is locally asymptotically stable for all $\tau > 0$, in which the delay is called *harmless*.

Case 2. $m = 1$.

The characteristic equation (11) is cubic and its coefficients are $a_0 = \tau^2 > 0$, $a_1 = 2\tau > 0$, $a_2 = 1 > 0$, $a_3 = \gamma > 0$ and $D_2^1 = \tau(2 - \tau\gamma)$. To obtain $D_2^1 > 0$, the delay τ should be less than the threshold value

$$\tau_1^* = \frac{2}{\gamma}. \quad (12)$$

Case 3. $m = 2$.

The characteristic equation (11) is 4th-order and its coefficients are $a_0 = \tau^3 > 0$, $a_1 = 6\tau^2 > 0$, $a_2 = 12\tau > 0$, $a_3 = 8 > 0$, $a_4 = 8\gamma > 0$, $D_2^2 = 64\tau^3 > 0$ and $D_3^2 = 32\tau^3(16 - 9\gamma\tau)$. To obtain $D_3^2 > 0$, the delay τ should be less than the threshold value

$$\tau_2^* = \frac{16}{9\gamma} \simeq \frac{1.78}{\gamma}. \quad (13)$$

Case 4. $m = 3$.

The characteristic equation (11) is 5th-order and its coefficients are $a_0 = \tau^4 > 0$, $a_1 = 12\tau^3 > 0$, $a_2 = 54\tau^2 > 0$, $a_3 = 108\tau > 0$, $a_4 = 81 > 0$, $a_5 = 81\gamma > 0$, $D_2^3 = 540\tau^5 > 0$, $D_3^3 = 972(48 + \gamma\tau) > 0$ and $D_4^3 = -6561\tau^6(\gamma^2\tau^2 + 336\gamma\tau - 576)$. To obtain $D_4^3 > 0$, the delay τ should be less than the threshold value

$$\tau_3^* = \frac{24(5\sqrt{2} - 7)}{\gamma} \simeq \frac{1.71}{\gamma}. \quad (14)$$

The relations (5), (12), (13) and (14) define the partition curves of (γ, τ) that divide the (γ, τ) space into stable and unstable parts. The three partition curves for $m = 1, 2, 3$ and one partition curve for the fixed time lag are depicted

in Figure 4. It can be seen that all curves are hyperbolic and the partition curves with the continuously distributed time lag are approaching the partition curves obtained under the fixed time lag from above. In addition to the result in Case I of $m = 0$, Figure 4 implies that the stable region becomes smaller as the value of m increases and converges to the region defined by the fixed time delay when m tends to infinity. The results obtained are natural if we notice the properties of the weighting function. For $m \geq 1$, zero weight is assigned to the most recent output, rising to maximum at $t - s$, and declining thereafter. As m increases, the function becomes more peaked around $t - s$ and tends to the Dirac delta function. In consequence for sufficiently large m , the weighting function may be regarded as very close to the Dirac delta function and the dynamic behavior under the continuously distributed time delay is very similar to the one under the fixed time delay. We can explain this phenomenon mathematically by noticing that the characteristic equation (11) of the continuously distributed case can be written as

$$\lambda + \gamma \frac{1}{\left(1 + \frac{\tau\lambda}{m}\right)^{m+1}} \frac{1}{\left(1 + \frac{\tau\lambda}{m}\right)} = 0,$$

and as $m \rightarrow \infty$, it converges to

$$\lambda + \gamma e^{-\lambda\tau} = 0$$

which is (3), the characteristic equation of the fixed time delay case. In short, under continuously distributed time delay, although we comprehensively use the delayed or past data of outputs, the stability domain is sensitive to the shapes of the weighting function. This result shows the sharp difference from the result obtained in the discrete delay model of Ahmad et al (2004) in which delay increases the stability region.

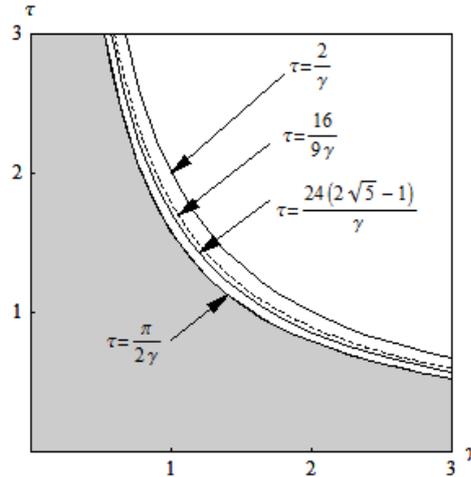


Figure 4. Stability regions

3 Delay Nonlinear Monopoly

In this section, we are concerned with cyclical behavior of delay monopoly, which is not observed in the linear framework. To this end, we consider the case in which the adjustment speed is a positive function of the output. In particular, we adopt the linear dependency:

Assumption 2. $\alpha(q) = \alpha q$ with $\alpha > 0$.

The gradient dynamics with discrete time lag $\tau > 0$ under Assumption 2 is given by

$$\frac{\dot{q}(t)}{q(t)} = \alpha [a - c - 2bq(t - \tau)].$$

This implies that the monopoly firm adjusts its growth rate of the output proportional to a change in profit. It is rewritten as

$$\dot{q}(t) = \bar{\gamma}q(t) \left(1 - \frac{q(t - \tau)}{q^M}\right) \quad (15)$$

where $\bar{\gamma} = 2b\alpha q^M$ with $q^M = (a - c)/2b$ as before. This is the delayed logistic equation with one discrete delay, which is called the Hutchinson equation. Introducing the new variable

$$y(t) = -1 + \frac{q(t)}{q^M},$$

rescaling the time $t = \tau \bar{t}$ and letting $y(t) = \bar{y}(\bar{t})$ reduce the dynamic equation (15) into the form

$$\dot{\bar{y}}(\bar{t}) = -\tau \bar{\gamma} \bar{y}(\bar{t} - 1)(1 + \bar{y}(\bar{t})).$$

Dropping the bars from \bar{t} and \bar{y} and denoting $\beta = \tau \bar{\gamma}$ yield

$$\dot{y}(t) = -\beta y(t - 1)(1 + y(t)). \quad (16)$$

We first turn attention to the stability of the zero solution $y(t) = 0$ of equation (16), which is equivalent to the stability of $q(t) = q^M$ in equation (15). Linearizing (16) at $y(t) = 0$ and letting y_δ denote the deviation of y from its stationary level give

$$\dot{y}_\delta = -\beta y_\delta(t - 1).$$

which is essentially the same as (3). Its corresponding characteristic equation is

$$\lambda + \beta e^{-\lambda} = 0.$$

Following the same procedure taken above, we can show that the trivial solution is asymptotically stable for $\beta < \pi/2$ (i.e., $\tau < \pi/2\bar{\gamma}$) and unstable for $\beta > \pi/2$ (i.e., $\tau > \pi/2\bar{\gamma}$). Furthermore, if we assume that ϕ is a continuous function in the interval $[-1, 0]$, its value is at least -1 everywhere, but at zero it has to be strictly greater than -1 , then we have the following global stability result due to Wright (1955)⁴.

⁴The following is Theorem 2.1 of Kuang (1993). Wright's conjecture that the global stability holds for $\beta < \pi/2$ still remains open.

Theorem 4 Let $\tau \leq \frac{3}{2\bar{\gamma}}$, and $y(t)$ be the solution of (16) with initial function ϕ . Then $\lim_{t \rightarrow \infty} y(t) = 0$.

If the monopolist takes into account of the several delayed outputs, the dynamic equation is

$$\begin{aligned} \dot{q}(t) &= \alpha q(t) [a - c - 2b \sum_{i=1}^n \omega_i q(t - \tau_i)] \\ &= 2\bar{\gamma} q(t) \left[1 - \sum_{i=0}^n \omega_i \frac{q(t - \tau_i)}{q^M} \right] \end{aligned} \quad (17)$$

where for the sake of simplicity the sum of the weight coefficients, ω_i , is assumed to be equal to unity

$$\sum_{i=0}^n \omega_i = 1.$$

Denoting

$$y(t) = -1 + \frac{q(t)}{q^M}$$

and letting $a_i = 2\bar{\gamma}\omega_i$, we see that the last expression of (17) can be rewritten as

$$\dot{y}(t) = -(1 + y(t)) \sum_{i=0}^n a_i y(t - \tau_i) \quad (18)$$

which is exactly the same as the delay logistic equation studied by Kuang (1991) with initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \quad \phi \in \mathbb{C} \text{ and } \phi(0) > -1,$$

where $\tau = \max\{\tau_i, i = 1, 2, \dots, n\}$. Applying Kuang's result (Corollary 4.1 of Kuang (1991)), we have the following:

Theorem 5 Consider (18) with the above initial conditions. If $\tau \sum_{i=1}^n a_i \leq 1$ then $\lim_{t \rightarrow \infty} y(t) = 0$.

Theorems 4 and 5 are concerned with the global asymptotical stability of the trivial solutions in the delay equations (16) and (18). Further, we can numerically examine global behavior when the local stability condition is violated. In Figure 5, we take the parameters values making the system locally unstable and perform simulations in the $(y(t), \dot{y}(t))$ space to show the birth of a limit cycle. In Figure 5A, a single delay is $\tau = 1$ and $\beta = 1.8$. In Figure 5B, we limit the number of delays to two. Two delays are $\tau_1 = 2$ and $\tau_2 = 2.5$ and other parameter values are $w_1 = 3/5$, $w_2 = 2/5$ and $\bar{\gamma} = 1$, which make $a_1 = w_1$ and $a_2 = w_2$. In both simulations the trajectories starting at the same initial point

$y(0) = 0.05$ converge to limit cycles.

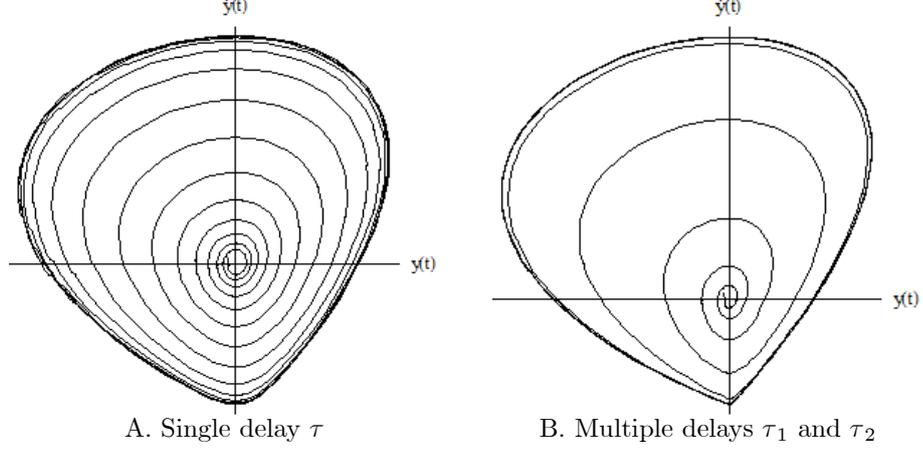


Figure 5. Birth of a limit cycle

The gradient dynamics with continuously distributed time lag is given by

$$\begin{cases} \dot{q}(t) = \alpha q(t) [a - c - 2bq^\varepsilon(t)] \\ q^\varepsilon(t) = \int_0^t w(t-s, \tau, m) q(s) ds \end{cases} \quad (19)$$

where the weighting function is assumed to be the same as it was specified in the previous section. To examine local dynamics of this system in a neighborhood of the equilibrium point $q(t) = q^M$, we need to consider the linearized version where $x(t) = q(t) - q^M$,

$$\begin{cases} \dot{x}(t) = -\bar{\gamma}x^\varepsilon(t), \\ x^\varepsilon(t) = \int_0^t w(t-s, \tau, m)x(s) ds. \end{cases}$$

which has exactly the same form as the linear system if γ is replaced by $\bar{\gamma}$ ($= \gamma q^M$). It then follows that the characteristic equations becomes

$$\lambda \left(1 + \frac{\lambda \tau}{\ell} \right)^{m+1} + \bar{\gamma} = 0.$$

When $m = 0$, the delay is harmless and the equilibrium point is locally asymptotically stable for any $\tau > 0$. When $m = 1$, it can be shown by applying

(12) that the parameters a_i of the corresponding characteristic equation satisfies the local stability condition $a_1 a_2 - a_0 a_3 > 0$ or

$$\tau < \frac{2}{\bar{\gamma}}.$$

We will show the possibility of the birth of limit cycles when loss of stability occurs. According to the Hopf bifurcation theorem, we have to show that the following two conditions are satisfied: (1) the characteristic equation of the dynamic system has a pair of pure imaginary roots and no other roots with zero real parts; (2) the real parts of these roots vary with a bifurcation parameter.

The $D_2^1 = a_1 a_2 - a_0 a_3 = 0$ line divides the parameter space into stable and unstable parts. Substituting $a_3 = a_1 a_2 / a_0$ into the characteristic equation and factoring it yield

$$(a_1 + a_0 \lambda)(a_2 + a_0 \lambda^2) = 0.$$

We have therefore three characteristic roots: two are purely imaginary and one is real and negative,

$$\lambda_{1,2} = \pm \sqrt{-\frac{a_2}{a_0}} = \pm i \frac{1}{\tau} \text{ and } \lambda_3 = -\frac{a_1}{a_0} = -\frac{2}{\tau} < 0.$$

The first condition of the Hopf theorem is satisfied. Next we select the delay τ as the bifurcation parameter and consider the roots of the characteristic equation as continuous function:

$$\tau^2 \lambda(\tau)^3 + 2\tau \lambda(\tau)^2 + \lambda(\tau) + \bar{\gamma} = 0$$

Differentiating it with respect to τ gives

$$\frac{d\lambda}{d\tau} = -\frac{2\tau \lambda^3 + 2\lambda^2}{3\tau^2 \lambda^2 + 4\tau \lambda + 1}.$$

Substituting $\lambda = i/\tau$, rationalizing the right hand side and noticing that the terms with λ and λ^3 are imaginary and the constant and λ^2 are real yield the following form of the real part of the derivative of λ with respect to τ

$$\text{Re} \left[\frac{d\lambda}{d\tau} \Big|_{\lambda = \frac{i}{\tau}} \right] = \frac{1}{5\tau^2} > 0$$

which indicates that the second condition is also satisfied. Figure 6 illustrates the birth of a limit cycle.⁵ We then summarize this result as follows:

⁵The parameters are specified as $a = 5$, $b = 1$, $c = 1$ and $\tau = 0.8$. The initial values of all variables are the same, $q^M - 0.1$. The dynamic system is

$$\begin{cases} \dot{q}(t) = -\bar{\gamma} q^e(t), \\ \dot{q}^e(t) = \frac{1}{\tau}(x(t) - q^e(t)), \\ \dot{x}(t) = \frac{1}{\tau}(q(t) - q^e(t)), \end{cases}$$

Theorem 6 *The monopolistic equilibrium point of the continuously distributed delay system (19) with $m = 1$ is destabilized through a Hopf bifurcation when the delay τ crosses the critical value*

$$\bar{\tau}_1^* = \frac{2}{\gamma}.$$

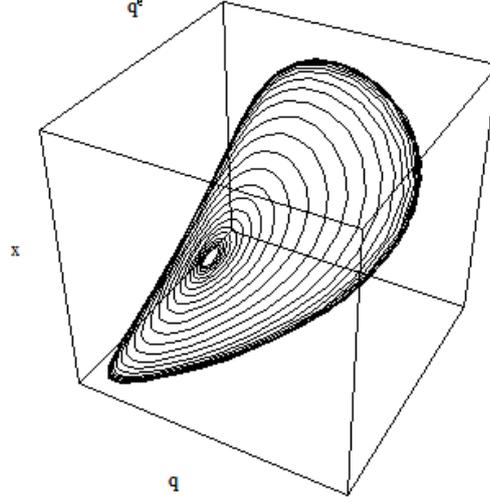


Figure 6. Birth of a limit cycle when $m = 1$

In the same way, we can show the existence of a limit cycle when $m = 2$. In particular, (13) implies that $\bar{\tau}_2^* = 16/(9\gamma)$ is the threshold value of the delay and from (Case 3), the curve of $a_1 a_2 a_3 - (a_0 a_3^2 + a_1^2 a_4) = 0$ is the partition curve between the stable and unstable regions. Solving the partition curve for a_4 , substituting it into the characteristic equation and factoring the resultant equation yield

$$(a_3 + a_1 \lambda^2)(a_1 a_2 - a_0 a_3 + a_1^2 \lambda + a_0 a_1 \lambda^2) = 0.$$

Two of the characteristic roots are purely imaginary,

$$\lambda_{1,2} = \pm \sqrt{-\frac{a_3}{a_1}} = \pm i \frac{2}{\tau \sqrt{3}}$$

and the other two roots are the solutions of the quadratic equation $(a_1 a_2 - a_0 a_3) + a_1^2 \lambda + a_0 a_1 \lambda^2 = 0$ and have negative real parts, since all coefficients are

where

$$q^e(t) = \int_0^t \left(\frac{1}{\tau}\right)^2 (t-s) e^{-\frac{t-s}{\tau}} q(s) ds$$

and

$$x(t) = \int_0^t \frac{1}{\tau} e^{-\frac{t-s}{\tau}} q(s) ds.$$

positive. Differentiating the characteristic equation with respect to τ gives

$$\frac{d\lambda}{d\tau} = -\frac{3\tau^2\lambda^4 + 12\tau\lambda^3 + 12\lambda^2}{4\tau^3\lambda^3 + 18\tau^2\lambda^2 + 24\tau\lambda + 8}.$$

Substituting $\lambda = i\frac{2}{\tau\sqrt{3}}$ and taking the real part, we have

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau} \Big|_{\lambda=i\frac{2}{\tau\sqrt{3}}} \right] = \frac{60}{439\tau^2} > 0.$$

Theorem 7 *The monopolistic equilibrium point of the continuously distributed delay system (19) with $m = 2$ is destabilized through a Hopf bifurcation when the delay τ crosses the critical value*

$$\bar{\tau}_2^* = \frac{16}{9\gamma}.$$

4 Concluding Remarks

This paper presents a delay monopoly model with linear price and cost function and examines its stability when dynamics is driven by the gradient process. The paper is divided into two parts and systems dynamics are compared with fixed and continuously distributed time delays in both parts. In the first part, we assume constant speed of adjustment and detect the threshold value of delay at which stability switch occurs. The monopoly with the single fixed delay has the minimum stable region while the continuously distributed delay becomes harmless when its weights are declining with the most weight given to the most current output. The stability region under the continuously distributed time delay converges to the stability region under the single fixed time delay as the weight function is approaching the Dirac delta function. It is confirmed that the stability region becomes complex when there are two fixed time delays. In the second part, we assume output-dependent speed of adjustment. This replacement of the adjustment speed makes the dynamic system nonlinear (i.e., a logistic equation). Local and global stability conditions are derived. Furthermore, the possibility of the birth of limit cycles is demonstrated when local stability is violated.

References

- [1] Ahmad, E., H. Agiza and S. Hassen, "On Modifications of Puu's Dynamical Duopoly," *Chaos Solitons and Fractals*, 11(2000), 1025-1028.
- [2] Bélair, J., and M. Mackey, "Consumer Memory and Price Fluctuations in Commodity Markets: An Integrodifferential Model," *Journal of Dynamics and Differential Equations*, 1 (1989), 299-325.
- [3] Bischi, G. I. and F. Lamantia, "Chaos Synchronization and Intermittency in a Duopoly Game with Sprillover Effects," in *Oligopoly Dynamics*, T. Puu and I. Sushko (eds), 195-217, 2002, Berlin/Heidelberg/New York, Springer-Verlag.
- [4] Bischi, G. I. and A. Naimuzada, "Global Analysis of a Duopoly Game with Bounded Rationality," *Advances in Dynamic Games and Applications*, 5 (1999), 361-385.
- [5] Chiarella, C., and A. Khomin, "An Analysis of the Complex Dynamic Behavior of Nonlinear Oligopoly Models with Time Lags," *Chaos, Solitons and Fractals*, 7 (1996), 2049-2066.
- [6] Chiarella, C., and F. Szidarovszky, "The Birth of Limit Cycles in Nonlinear Oligopolies with Continuously Distributed Information Lags," in *Modeling Uncertainty*, Dror, M., P. L'Ecuyer and F. Szidarovszky (eds), 249-268, 2001, Boston, Kluwer Academic Publisher.
- [7] Hale, J., "Nonlinear Oscillations in Equations with Delays," in *Nonlinear Oscillations in Biology*, K. C. Hoppenstadt (ed), Lectures in Applied Mathematics, 17 (1979), 157-185.
- [8] Hale, J., and W. Huang, "Global Geometry of the Stable Regions for Two Delay Differential Equations," *Journal of Mathematical Analysis and Applications*, 178(1993), 344-362.
- [9] Invernizzi, S., and A. Medio, "On Lags and Chaos in Economic Dynamic Models," *Journal of Mathematical Economics*, 20 (1991), 521-550.
- [10] Kuang, Y., *Delay Differential Equations with Applications in Population Dynamics*, Boston, Academic Press, 1993.
- [11] Puu, T. *Attractor, Bifurcations and Chaos*, 2003, Berlin/Heidelberg/New York, Springer-Verlag
- [12] An der Heiden, U., and M. Mackey, "The Dynamics of Production and Destruction: Analytic Insight into Complex Behavior," *Journal of Mathematical Biology*, 16 (1982), 75-101.