

# Price and Quantity Competition in Dynamic Oligopolies with Product Differentiation\*

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## Abstract

This is a continuation of the earlier work of Matsumoto and Szidarovszky (2010A) that compares Bertrand and Cournot equilibria in a differentiated  $n$ -firm oligopoly from static and dynamic points of view. It is, among others, found that both equilibria can be locally unstable if the firms naively forms expectations and the number of the firms are strictly more than three. The main purpose of this paper provides economic circumstances under which such unstable equilibria can be stabilized. Three main results have been shown. First, the equilibria can be locally stable if the dynamic system with adaptive expectations or the dynamic system with adaptive adjustments is adopted. Second, these two dynamic systems are equivalent as far as the local stability is concerned. Third, the total output under the dynamic system with naive expectations exhibits a period-2 cycle.

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# 1 Introduction

This is a continuation of the earlier work of Matsumoto and Szidarovszky (2010A) that compares Bertrand and Cournot equilibria in a differentiated  $n$ -firm oligopoly with linear demands and asymmetric constant marginal costs. In the literature of the Bertrand and Cournot competitions, there has been the conventional view that price competition is more competitive than quantity competition in a sense that the price-adjusting firm charges a lower price and produces more quantity of output than the quantity-adjusting firm. It has been challenged by a number of theoretical models under various conditions whether this view is true or not. Singh and Vives (1984) find that the view is true in a linear duopoly framework. Häckner (2000) reconsiders general  $n$ -firm oligopoly and points out that the results of Singh and Vives are sensitive to the duopoly assumption. These are static results. In the dynamic context of a  $n$ -firm oligopoly without product differentiation, Theocharis (1960) reminds us of the controversial result that the non-differentiated Cournot equilibrium is locally asymptotically stable under the naive adjustment process if and only if the number of firms is *two*. Matsumoto and Szidarovszky (2010A) analyze the general  $n$ -firm oligopoly model of Häckner, which is a direct extension of the duopoly model used by Singh and Vives. In particular, they address two issues: one is comparing the differentiated Bertrand and Cournot equilibria from the static point of view and the other is examining the stability of the equilibria. After deriving the optimal strategies of output, price and profit, they provide positive support for Theocharis' result however negative support for the conventional view on comparison between the two equilibria:

- (i) Bertrand and Cournot equilibria can be locally unstable when the number of the firms is strictly greater than three.
- (ii) The conventional view is not always true in the  $n$ -firm oligopoly.

Looking at these results from a dynamic point of view, it is clear that several important issues remain unsolved. The first issue relates to controlling unstable equilibria. In consequence of those two results (i) and (ii), it is probable that the conventional view does not hold and the one of the equilibria becomes locally unstable. Namely, the Bertrand price can be higher than the Cournot price while it is locally unstable or the Cournot output can be larger than the Bertrand output while it is locally unstable. In such cases comparing equilibria does not make any economic sense. Applying the results obtained in Matsumoto and Szidarovszky (2010B) in which a general  $n$ -firm oligopoly is examined with isoelastic price function and linear cost under Cournot competition, we will provide economic circumstances under which the unstable quantity and price dynamics can be stabilized. The second issue relates to global stability of a locally unstable equilibrium. Our main findings are the followings: it is possible to stabilize an equilibrium if some learning process is introduced and a locally unstable trajectory converges to a period-2 cycle if the non-negativity constraint is explicitly taken into account.

The paper is organized as follows. The next section introduces a linear  $n$ -firm oligopoly model with product differentiation and obtain the Cournot and Bertrand equilibria. Section 3 considers the control of unstable Cournot

and Bertrand dynamics with naive expectations through partial and adaptive adjustments. Section 4 concludes the paper.

## 2 $n$ -Firm Oligopoly Model

We recapitulate the main structure of the general oligopoly model constructed in Matsumoto and Szidarovszky (2010A) and proceed to dynamic analysis. The linear inverse demand function or the price function of good  $k$  is given by

$$p_k = \alpha_k - q_k - \gamma \sum_{\ell \neq k}^n q_\ell \text{ for } k = 1, 2, \dots, n, \quad (1)$$

where  $q_k$  is quantity of good  $k$ ,  $p_k$  is its price,  $\gamma$  measures the degree of differentiation between the goods and  $\alpha_k$  measures the quality of good  $k$ .<sup>1</sup> In this study, we assume that  $|\gamma| < 1$  and  $\gamma \neq 0$ , to confine our analysis to the case in which the goods are imperfect substitutes or complements and are not independent. Solving (1) for quantities yields the direct demand of good  $k$ ,

$$q_k = \frac{(1 + (n-2)\gamma)(\alpha_k - p_k) - \gamma \sum_{\ell \neq k}^n (\alpha_\ell - p_\ell)}{(1 - \gamma)(1 + (n-1)\gamma)} \text{ for } k = 1, 2, \dots, n. \quad (2)$$

It is linear in the other firms' prices and its price-independent demand is assumed to be positive, that is,  $(1 + (n-2)\gamma)\alpha_k - \gamma \sum_{\ell \neq k}^n \alpha_\ell > 0$ . It is further assumed that there is no fixed cost and the cost function of firm  $k$  is linear. The marginal cost is denoted by  $c_k$ . To avoid negative optimal production, it is also assumed that  $\alpha_k - c_k$  is positive. We can call this difference *net quality* of good  $k$ .

In Cournot competition, firm  $k$  chooses a quantity of good  $k$  to maximize its profit  $\pi_k = (p_k - c_k)q_k$  subject to (1), taking the other firms' quantities given. Assuming an interior maximum and solving its first-order condition yield the best reply of firm  $k$ ,

$$q_k = R_k^C \left( \sum_{\ell \neq k}^n q_\ell \right) \text{ for } k = 1, 2, \dots, n.$$

where

$$R_k^C \left( \sum_{\ell \neq k}^n q_\ell \right) = \frac{\alpha_k - c_k}{2 - \gamma} - \frac{\gamma}{2} \sum_{\ell \neq k}^n q_\ell. \quad (3)$$

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<sup>1</sup>The price function (1) can be derived as a solution of the utility maximization of the following form

$$U = \sum_{i=1}^n \alpha_i q_i - \frac{1}{2} \left( \sum_{i=1}^n q_i^2 + 2\gamma \sum_{i=1}^n \sum_{j>i}^n q_i q_j \right) - \sum_{i=1}^n p_i q_i.$$

Here  $\alpha_i$  can be considered to be a proxy for the quality of good  $k$  because an increase in  $\alpha_i$  positively affects the utility level.

It is easily checked that the second-order condition is certainly satisfied. The Cournot equilibrium output and price of firm  $k$  are

$$q_k^C = \frac{\alpha_k - c_k}{2 - \gamma} - \frac{\gamma}{(2 - \gamma)(2 + (n - 1)\gamma)} \sum_{\ell=1}^n (\alpha_\ell - c_\ell) \quad (4)$$

and

$$p_k^C = \frac{\alpha_k + c_k - \gamma c_k}{2 - \gamma} - \frac{\gamma}{(2 - \gamma)(2 + (n - 1)\gamma)} \sum_{\ell=1}^n (\alpha_\ell - c_\ell). \quad (5)$$

The superscript "C" is attached to variables to indicate that they are evaluated at the Cournot equilibrium. Subtracting (4) from (5) yields  $p_k^C - c_k = q_k^C$  which is, then, substituted into the profit function to obtain the Cournot profit,

$$\pi_k^C = (q_k^C)^2. \quad (6)$$

In Bertrand competition, firm  $k$  chooses the price of good  $k$  to maximize the profit  $\pi_k = (p_k - c_k)q_k$  subject to (2), taking the other firms' prices given. Solving the first-order condition yields the best reply of firm  $k$ ,

$$p_k = R_k^B \left( \sum_{\ell \neq k}^n p_\ell \right), \text{ for } k = 1, 2, \dots, n,$$

where

$$R_k^B \left( \sum_{\ell \neq k}^n p_\ell \right) = \frac{\alpha_k + c_k}{2} - \frac{\gamma}{2[1 + (n - 2)\gamma]} \sum_{\ell \neq k}^n (\alpha_\ell - p_\ell). \quad (7)$$

The second-order condition

$$\frac{\partial^2 \pi_k}{\partial p_k^2} = -\frac{1 + (n - 2)\gamma}{(1 - \gamma)(1 + (n - 1)\gamma)} < 0,$$

for an interior optimum solution is definitely satisfied for  $\gamma > 0$ . However the sign of the second derivative is ambiguous when  $\gamma < 0$ . This gives rise to make the following assumption:

**Assumption 1**  $1 + (n - 1)\gamma > 0$  when  $\gamma < 0$ .

The Bertrand equilibrium price and output of firm  $k$  are given by

$$p_k^B = \frac{(2 + (n - 3)\gamma)[(1 + (n - 1)\gamma)(\alpha_k + c_k) - \gamma c_k] - \gamma(1 + (n - 2)\gamma) \sum_{\ell=1}^n (\alpha_\ell - c_\ell)}{(2 + (2n - 3)\gamma)(2 + (n - 3)\gamma)} \quad (8)$$

and

$$q_k^B = \frac{1 + (n - 2)\gamma}{(1 - \gamma)(1 + (n - 1)\gamma)} (p_k^B - c_k) \quad (9)$$

implying that

$$p_k^B - c_k = \frac{(2 + (n - 3)\gamma)(1 + (n - 1)\gamma)(\alpha_k - c_k) - \gamma(1 + (n - 2)\gamma) \sum_{\ell=1}^n (\alpha_\ell - c_\ell)}{(2 + (2n - 3)\gamma)(2 + (n - 3)\gamma)}. \quad (10)$$

The superscript "B" is attached to variables to indicate that they are computed at the Bertrand equilibrium. Due to (9), the Bertrand profit of firm  $k$  is given as

$$\pi_k^B = \frac{(1 - \gamma)(1 + (n - 1)\gamma)}{1 + (n - 2)\gamma} (q_k^B)^2. \quad (11)$$

The price, quantity and profit comparisons are summarized in Table 1, which were shown earlier by Matsumoto and Szidarovszky (2010A). Let  $\beta_k$  denote the ratio of the average difference net quality of all firms over the net quality of firm  $k$ ,

$$\beta_k = \frac{\frac{1}{n} \sum_{\ell=1}^n (\alpha_\ell - c_\ell)}{\alpha_k - c_k}.$$

Firm  $k$  is called *higher-* or *lower-qualified* according to  $\beta_k$  is greater or less than unity. The results with " $\leq$ " are newly obtained. Accordingly, it can be observed that as far as the outputs and the prices are concerned, higher-qualified Cournot firms charge higher prices and produces less outputs than higher-qualified Bertrand firms regardless of whether the goods are substitutes or complements. It is also observed that profitability of these firms are ambiguous. On the other hand, when the firms are lower-qualified, then the Cournot profit is larger than the Bertrand profit when the goods are substitutes and the inequality is reversed when the goods are complements. The same results holds in the duopoly framework. However the conventional view does not necessarily hold in  $n$ -firm oligopolies.

	Substitutes ( $\gamma > 0$ )	Complements ( $\gamma < 0$ )
Higher-qualified ( $\beta_k < 1$ )	$p_k^C > p_k^B$ $q_k^C < q_k^B$ $\pi_k^C \leq \pi_k^B$ $\geq$	$p_k^C > p_k^B$ $q_k^C < q_k^B$ $\pi_k^C \leq \pi_k^B$ $\geq$
Lower-qualified ( $\beta_k > 1$ )	$p_k^C > p_k^B$ $q_k^C \leq q_k^B$ $\geq$ $\pi_k^C > \pi_k^B$	$p_k^C \leq p_k^B$ $\geq$ $q_k^C < q_k^B$ $\pi_k^C < \pi_k^B$

Table 1. Comparison of Cournot and Bertrand strategies

### 3 Dynamic Adjustment Processes

In addition to the results summarized in Table 1, it is also demonstrated in Matsumoto and Szidarovszky (2010A), and will be reviewed shortly, that under the native expectation scheme, the Cournot output can be locally unstable when the goods are substitutes while the Bertrand price can be locally unstable when the goods are complements. In consequence, we naturally raise two questions:

- (1) Are there any other expectation schemes under which the Cournot and Bertrand equilibria are stable?
- (2) Where does an unstable trajectory of output or price go when it starts in a neighborhood of a locally unstable equilibrium?

We introduce two expectation schemes: partial adjustment towards best response with naive expectations and the best reply dynamics with adaptive expectation. Then we answer the first question by finding that the smaller values of the adjustment coefficients stabilize the otherwise unstable equilibrium. Concerning the second question, global dynamics of locally unstable Cournot equilibria is examined and the existence of a period-2 cycle is shown. The quantity-adjusting system is examined in Section 3-1 and the price-adjusting system is discussed in Section 3-2.

### 3.1 Cournot Dynamics

#### 3.1.1 Local Adjustment

We turn our attention to the adjustment process and assume that in period  $t + 1$ , each firm expects the total output of the rest of the industry and changes its output level to the best response accordingly. This process can be written as

$$q_k(t + 1) = R_k^C(Q_k^E(t + 1))$$

where  $Q_k^E(t + 1)$  is the expected output of firm  $k$ . The adjustment process depends on how the firms form their expectations. If firm  $k$  has naive belief that the other firms' output will remain unchanged, then the expected output is determined by

$$Q_k^E(t + 1) = \sum_{\ell \neq k}^n q_\ell(t).$$

This formation is called the naive expectation. So we call the dynamic adjustment process based on this most simple expectation formation the best reply dynamics with naive expectations that will be associated to *naive dynamics*

$$q_k(t + 1) = R_k^C\left(\sum_{\ell \neq k}^n q_\ell(t)\right), \quad k = 1, 2, \dots, n. \quad (12)$$

According to naive dynamics, firm  $k$  immediately jumps to its best reply level. However, concerning changes of the output levels of any firm in most industries, it is clear that the changes require time, new employments, purchase of new machinery, etc. Therefore there are a lot of circumstances in which output changes are made gradually. It may be plausible under such a circumstance that the firms adjust their previous output levels in the direction towards the optimal levels in the next period. If the new output level is a weighted average of the current output and the naively-determined optimal output, then the resulting adjustment process is called partial adjustment towards the best reply with naive expectations that will be abbreviated as *partial dynamics*. Firm  $k$  gradually moves into the direction towards its profit maximizing output. This process is described by the following  $n$ -dimensional dynamic system:

$$q_k(t + 1) = (1 - \mu_k)q_k(t) + \mu_k R_k^C\left(\sum_{\ell \neq k}^n q_\ell(t)\right), \quad k = 1, 2, \dots, n. \quad (13)$$

where  $\mu_k \in (0, 1]$  is the adjustment coefficient. The Jacobian of this system has the form

$$\mathbf{J}_C = \begin{pmatrix} 1 - \mu_1 & \theta_1^C \mu_1 & \cdots & \theta_1^C \mu_1 \\ \theta_2^C \mu_2 & 1 - \mu_2 & \cdots & \theta_2^C \mu_2 \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \theta_n^C \mu_n & \theta_n^C \mu_n & \cdots & 1 - \mu_n \end{pmatrix}$$

where

$$\theta_k^C = \frac{\partial R_k^C}{\partial q_\ell} = -\frac{\gamma}{2}.$$

Notice that in the case of  $\mu_k = 1$  for all  $k$ , partial dynamics reduces to naive dynamics and firm  $k$  reaches its profit maximizing output. The corresponding characteristic equation reads

$$\det(\mathbf{J}_C - \lambda \mathbf{I}) = (-1)^n \left( \lambda - \frac{\gamma}{2} \right)^{n-1} \left( \lambda + \frac{(n-1)\gamma}{2} \right) = 0$$

which implies that there are  $n - 1$  identical eigenvalues and one different eigenvalue. Since  $|\gamma| < 1$  is assumed, it is apparent that the naive dynamics are locally asymptotically stable if  $|\lambda_n^C| < 1$  where

$$\lambda_n^C = -\frac{(n-1)\gamma}{2}.$$

Under Assumption 1,  $|\lambda_n^C| < 1$  always if  $\gamma < 0$ . On the other hand, the following results are obtained when  $\gamma > 0$ :

$$|\lambda_2^C| = \left| -\frac{\gamma}{2} \right| < 1 \text{ for } n = 2, \quad |\lambda_3^C| = |\gamma| < 1 \text{ for } n = 3$$

and solving  $\lambda_n^C = -1$  yields the stability condition of the Cournot output for  $n > 3$ ,

$$n < 1 + \frac{\gamma}{2}.$$

Hence we have the benchmark result concerning the local dynamics of the Cournot output<sup>2</sup>:

**Theorem 1** *Under Assumption 1, the best reply dynamics of the Cournot output with naive expectations is locally asymptotically stable if the goods are complements (i.e.,  $\gamma < 0$ ) while the local stability can be lost if the goods are substitutes (i.e.,  $\gamma > 0$ ) and the number of the firms are strictly greater than three.*

We now turn our attention to the case of  $\mu_k < 1$ . Notice that

$$\mathbf{J}_C = \mathbf{D} + \mathbf{a}\mathbf{b}^T$$

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<sup>2</sup>This is essentially the same as Theorem 1 of Matsumoto and Szidarovszky (2010A).

with

$$\mathbf{a}^T = (\mu_1, \mu_2, \dots, \mu_n), \quad \mathbf{b}^T = \left(-\frac{\gamma}{2}, -\frac{\gamma}{2}, \dots, -\frac{\gamma}{2}\right)$$

and

$$\mathbf{D} = \text{diag} \left(1 - \mu_1 \left(1 - \frac{\gamma}{2}\right), 1 - \mu_2 \left(1 - \frac{\gamma}{2}\right), \dots, 1 - \mu_n \left(1 - \frac{\gamma}{2}\right)\right)$$

The characteristic polynomial of  $\mathbf{J}_C$  can be decomposed by using the simple fact that if  $\mathbf{x}, \mathbf{y} \in R^n$ , then

$$\det(\mathbf{I} + \mathbf{x}\mathbf{y}^T) = 1 + \mathbf{y}^T \mathbf{x} \quad (14)$$

where  $\mathbf{I}$  is the  $n \times n$  identity matrix. We can rewrite the determinant as

$$\begin{aligned} \det(\mathbf{J}_C - \lambda \mathbf{I}) &= \det(\mathbf{D} + \mathbf{a}\mathbf{b}^T - \lambda \mathbf{I}) \\ &= \det(\mathbf{D} - \lambda \mathbf{I}) \det(\mathbf{I} + (\mathbf{D} - \lambda \mathbf{I})^{-1} \mathbf{a}\mathbf{b}^T). \end{aligned}$$

The first determinant is diagonal, the second has the special structure of (14) with  $\mathbf{x} = (\mathbf{D} - \lambda \mathbf{I})^{-1} \mathbf{a}$  and  $\mathbf{y} = \mathbf{b}$ . Identity (14) gives the characteristic equation of  $\mathbf{J}_C$  as

$$\prod_{k=1}^n \left(1 - \mu_k \left(1 - \frac{\gamma}{2}\right) - \lambda\right) \left[1 + \sum_{k=1}^n \frac{\mu_k(-\gamma/2)}{1 - \mu_k(1 - \gamma/2) - \lambda}\right] = 0. \quad (15)$$

The roots of the first factor are  $1 - \mu_k(1 - \frac{\gamma}{2})$  which are inside the unit circle if and only if their absolute values are less than unity. Direct calculations show that it is the case if and only if

$$\mu_k(2 - \gamma) < 4.$$

The other eigenvalues are the roots of equation

$$g(\lambda) = 1 + \sum_{k=1}^n \frac{\mu_k(-\gamma/2)}{1 - \mu_k(1 - \gamma/2) - \lambda} = 0$$

with

$$\lim_{\lambda \rightarrow \pm\infty} g(\lambda) = 1, \quad \lim_{\lambda \rightarrow 1 - \mu_k(1 - \gamma/2) \pm 0} g(\lambda) = \pm\infty$$

and

$$g'(\lambda) = \sum_{k=1}^n \frac{\mu_k(-\gamma/2)}{(1 - \mu_k(1 - \gamma/2) - \lambda)^2} < 0.$$

All eigenvalues are inside the unit circle if and only if  $g(-1) > 0$  which can be rewritten as

$$\sum_{k=1}^n \frac{\gamma \mu_k}{4 - \mu_k(2 - \gamma)} < 1.$$

In summary, we can present conditions for the local asymptotic stability of the partial dynamics (13).<sup>3</sup>

<sup>3</sup>This is a modified version of the first half of Theorem 2.1 in Bischi et al. (2010).



**Theorem 2** *The Cournot output is locally asymptotically stable if for all  $k$ ,*

$$\mu_k(2 - \gamma) < 4$$

and

$$\sum_{k=1}^n \frac{\gamma \mu_k}{4 - \mu_k(2 - \gamma)} < 1.$$

Let us now shift the emphasis of the stability analysis to two special cases. Consider first the case when the firms select identical adjustment coefficients (i.e.,  $\mu_k = \mu$  for all  $k$ ). The two conditions of Theorem 2 reduce, respectively, to

$$\mu(2 - \gamma) < 4 \text{ and } \mu(2 + (n - 1)\gamma) < 4$$

where the first inequality always holds for any  $\gamma \in (0, 1)$  and  $\mu \in (0, 1]$ . The corresponding characteristic equation can be written in the form

$$\left(1 - \mu\left(1 - \frac{\gamma}{2}\right) - \lambda\right)^n \left(1 + \frac{n\mu(-\gamma/2)}{1 - \mu(1 - \gamma/2) - \lambda}\right) = 0.$$

Assuming that the first  $n - 1$  eigenvalue are the same, we conclude that

$$\tilde{\lambda}_i^C = 1 - \mu\left(1 - \frac{\gamma}{2}\right) \text{ for } i = 1, 2, \dots, n - 1$$

which is less than unity and positive if  $\mu(2 - \gamma) < 4$ , that is, if the first condition of Theorem 2 is fulfilled. Making the second factor of the characteristic equation equal to zero and solving it for  $\lambda$  yield the remaining eigenvalue of  $\mathbf{J}_C$  with  $\mu_k = \mu$  for all  $k$  as

$$\tilde{\lambda}_n^C = 1 - \mu \left(1 + (n - 1)\frac{\gamma}{2}\right).$$

This is less than unity in absolute value if  $\mu(2 + (n - 1)\gamma) < 4$ , that is, if the second condition of Theorem 2 is fulfilled.

We can represent the second stability condition graphically. Given  $\mu$ , the  $\mu(2 + (n - 1)\gamma) = 4$  locus is a partition line dividing the positive  $(\gamma, n)$  plane into two parts; the stable region with  $\mu(2 + (n - 1)\gamma) < 4$  below the line and the unstable region with  $\mu(2 + (n - 1)\gamma) > 4$  above. In Figure 1 in which the red-hyperbola is the  $(n - 1)\gamma = 2$  locus (i.e., the partition line with  $\mu = 1$ ), the Cournot output is locally stable in the light-gray region and unstable in the darker-gray region under the naive dynamics, as Theorem 1 indicates. Three dotted loci are associated with three different values of  $\mu$  and the boundaries between the stable and instable regions. It can be seen that the boundary shifts upward as the value of  $\mu$  decreases. This means that the stability region of  $\tilde{\lambda}_n^C$  enlarges as  $\mu$  decreases to zero from unity. Any unstable Cournot output under naive dynamics can be stabilized with partial dynamics by selecting sufficiently

small values of  $\mu$ .

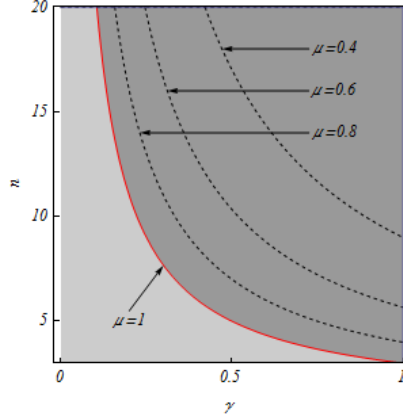


Figure 1 Enlargement of the stable region

Consider next a numerically-specified case with different values of  $\mu_k$ . Let us consider the set of parameters  $n = 4$ ,  $\mu_1 = 0.6$ ,  $\mu_2 = 0.5$ ,  $\mu_3 = 0.3$ ,  $\mu_4 = 0.2$ , and  $\gamma = 0.8$ .<sup>4</sup> The corresponding characteristic equation has the form

$$\prod_{k=1}^4 \left(1 - \mu_k \left(1 - \frac{\gamma}{2}\right) - \lambda\right) \left[1 + \sum_{k=1}^4 \frac{\mu_k (-\gamma/2)}{1 - \mu_k \left(1 - \gamma/2\right) - \lambda}\right] = 0$$

and the different  $\mu_k$ s indicate that  $1 - \mu_k \left(1 - \frac{\gamma}{2}\right)$  is not an eigenvalue of  $\mathbf{J}_C$ . All eigenvalues are the roots of equation  $g(\lambda) = 0$  with  $n = 4$ . Solving it yields four distinctive real roots,

$$\tilde{\lambda}_1^C \simeq 0.07, \tilde{\lambda}_2^C \simeq 0.67, \tilde{\lambda}_3^C \simeq 0.79 \text{ and } \tilde{\lambda}_4^C \simeq 0.87,$$

all of which are positive and less than unity. Hence the Cournot output is locally asymptotically stable under the partial dynamics.

Although the calculations are done with Mathematica (version 7), the roots are graphically obtained and confirmed to be less than unity in absolute value as follows. In Figure 2, the graph of  $g(\lambda)$  is illustrated. Since  $1 > \mu_1 > \mu_2 > \mu_3 > \mu_4 > 0$  and  $0 < \mu_i(1 - \gamma/2) < 1$ , the four dotted vertical lines pass through each of the points,  $1 - \mu_i(1 - \gamma/2)$  for  $i = 1, 2, 3, 4$ . As we have already seen that

$$\lim_{\lambda \rightarrow \pm\infty} g(\lambda) = 1, \quad \lim_{\lambda \rightarrow 1 - \mu_i(1 - \gamma/2) \pm 0} g(\lambda) = \pm\infty \text{ and } g'(\lambda) < 0$$

which imply that some parts of the graph of  $g(\lambda)$  are located between the dotted lines and cross the horizontal axis. Three of the four roots are found and depicted as the red points in the interval between the smallest  $1 - \mu_1(1 - \gamma/2)$  and the largest  $1 - \mu_4(1 - \gamma/2)$ . Furthermore

$$-1 < 1 - \mu_1(1 - \gamma/2) (= 0.64) \text{ and } g(-1) \simeq 0.63 > 0$$

implying that the first root, the most left red dot, found to be less than unity in absolute value, in particular it is about 0.07 in this case. It is thus confirmed

<sup>4</sup>Theorem 1 implies that 4 is the minimum integer number of the firms when the Cournot output is locally unstable under naive dynamics.

that all roots are real and inside the unit interval. Therefore the Cournot output is locally asymptotically stable under partial dynamics. Notice that the same Cournot output is locally unstable under the naive dynamics because  $\lambda_4^C = -6/5$ .

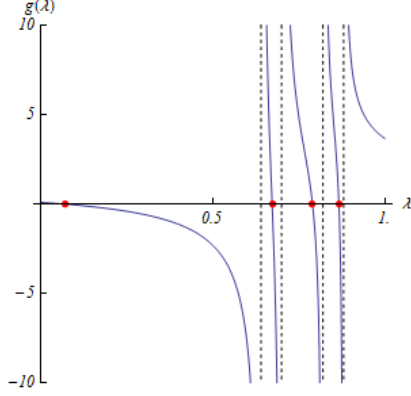


Figure 2 Graph of  $g(\lambda)$  with  $n = 4$

### 3.1.2 Global Adjustment

We are interested in global dynamics when the Cournot output is locally unstable. For the sake of simplicity, we focus on naive dynamics. Summing both sides of (12) over all values of  $k$  and taking into account the non-negativity of the total output yield the piecewise linear dynamic equation of the total output,  $Q = \sum_{\ell=1}^n q_\ell$ , as

$$Q(t+1) = \begin{cases} \frac{n(\bar{\alpha} - \bar{c})}{2} - \frac{\gamma(n-1)}{2}Q(t) & \text{if } Q(t) < Q_0, \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

where  $Q_0$  is the threshold value of the total output for which the first case of (16) generates zero output in the next period,

$$Q_0 = \frac{n(\bar{\alpha} - \bar{c})}{(n-1)\gamma}$$

and  $\bar{\alpha}$  and  $\bar{c}$  are the averages of  $\alpha_i$  and  $c_i$  defined by

$$\bar{\alpha} = \frac{1}{n} \sum_{\ell=1}^n \alpha_\ell \quad \text{and} \quad \bar{c} = \frac{1}{n} \sum_{\ell=1}^n c_\ell.$$

A fixed point of (16) is

$$Q_C = \frac{n(\bar{\alpha} - \bar{c})}{2 + (n-1)\gamma}$$

and it is locally stable if the slope of the first linear equation of (16) is less than unity in absolute value. When  $\gamma < 0$ ,  $Q_C$  is always stable in the region where Bertrand competition is feasible.<sup>5</sup> On the other hand, when  $\gamma > 0$ ,  $Q_C$

<sup>5</sup>Assumption 1 implies that

$$0 < -\frac{\gamma(n-1)}{2} < \frac{1}{2}.$$

can lose its stability. Hence we draw attention to the global dynamics of the locally unstable Cournot equilibrium of the total output by making the following assumption:

**Assumption 2.**  $\frac{\gamma(n-1)}{2} > 1$ .

Following Puu (2006), we can derive an asymptotic dynamic process of the individual output associated with changes of the total output. In particular, using (12), we first form an output difference dynamics by subtracting the dynamic equation of firm  $k$  from that of firm  $\ell$ ,

$$q_\ell(t+1) - q_k(t+1) = \frac{(\alpha_\ell - c_\ell) - (\alpha_k - c_k)}{2} - \frac{\gamma}{2} (q_\ell(t) - q_k(t)).$$

Given  $0 < \gamma < 1$ , the sequences of these differences are convergent, and the output difference eventually approaches a fixed quantity. After any transient has been passed, the equation can be rewritten as

$$q_\ell(t) = q_k(t) + \frac{(\alpha_\ell - c_\ell) - (\alpha_k - c_k)}{2 - \gamma}.$$

Summing this equation over all values of  $\ell$  and solving the resultant equation for  $q_k(t)$  gives the output dynamic equation of firm  $k$  depending on the total output as follows:

$$q_k(t) = \frac{Q(t)}{n} - \frac{(\bar{\alpha} - \bar{c}) - (\alpha_k - c_k)}{2 - \gamma}. \quad (17)$$

Substituting the last equation into the price function (1) yields the price dynamic equation associated with the total output,

$$p_k(t) = -\frac{1 + (n-1)\gamma}{n} Q(t) + \frac{\alpha_k(2-\gamma) - (1-\gamma)[(\alpha_k - c_k) - (\bar{\alpha} - \bar{c})]}{2-\gamma}. \quad (18)$$

It is clear that individual dynamics of  $q_k(t)$  and  $p_k(t)$  are synchronized with dynamics of the total output  $Q(t)$ . It, therefore, suffices for our purpose to confine attention to global dynamics of the total output.

Regardless of a choice of an initial point, the divergent quantity-adjustment process (16) with Assumption 2 sooner or later generates  $Q(t-1) > Q_0$  at period  $t-1$ . However the non-negativity constraint prevents the output in the next period from being negative and thus the total output is replaced with zero output.<sup>6</sup> Substituting  $Q(t) = 0$  into the dynamic equation (16) gives the total output at period  $t+1$ ,

$$Q(t+1) = \frac{n(\bar{\alpha} - \bar{c})}{2} > 0$$

so  $q_k(t+1) \geq 0$ .<sup>7</sup> Substituting  $Q(t+1)$  into the first equation of (16) yields the total output at period  $t+2$ ,

$$Q(t+2) = \frac{n(\bar{\alpha} - \bar{c})}{2} \left(1 - \frac{\gamma(n-1)}{2}\right) < 0,$$

<sup>6</sup>Since  $q_k(t)$  cannot be negative,  $Q(t) = 0$  implies  $q_k(t) = 0$  for all  $k$ .

<sup>7</sup>Given  $Q(t+1)$ , equation (17) determines the output of firm  $k$  as

$$\frac{2(\alpha_k - c_k) - \gamma(\bar{\alpha} - \bar{c})}{2(2-\gamma)} = \frac{(\alpha_k - c_k)(2 - \gamma\beta_k)}{2(2-\gamma)}.$$

Notice that  $2 > \gamma\beta_k$  is necessary to have non-negative individual output.

where the inequality is due to Assumption 2. The non-negativity constraint replaces the negative value of  $Q(t+2)$  with zero output, and then this process repeats itself. In summary we have the following result on global dynamics:

**Theorem 3** *Given Assumption 2, the aggregate dynamic system (16) generates a period-2 cycle of the total output with periodic points*

$$Q_1 = 0 \text{ and } Q_2 = \frac{n(\bar{\alpha} - \bar{c})}{2}.$$

### 3.2 Bertrand Dynamics

If the Bertrand firms naively form their expectations on the prices, then the best reply dynamics of Bertrand price of firm  $k$  is obtained by lagging the variables,

$$p_k(t+1) = R_k^B \left( \sum_{\ell \neq k} p_\ell(t) \right).$$

We move one step forward and introduce a learning process in which each firm observes the other firms' choice of price and revises its price expectations based on earlier data. The most popular such learning process is obtained when the firms adjust their expectations adaptively according to

$$P_k^E(t+1) = P_k^E(t) + \mu_k \left( \sum_{\ell \neq k}^n p_\ell(t) - P_k^E(t) \right)$$

in which  $P_k^E(t+1)$  is the sum of the prices of the rest of the industry expected by firm  $k$  and the expectation is revised on the basis of the discrepancy between the observed value and the previously expected value. This adjustment process is called the best reply dynamics with adaptive expectations which will be abbreviated as *adaptive dynamics* and described by the  $2n$ -dimensional system

$$p_k(t+1) = R_k^B \left( \mu_k \sum_{\ell \neq k}^n p_\ell(t) + (1 - \mu_k) P_k^E(t) \right),$$

$$P_k^E(t+1) = \mu_k \sum_{\ell \neq k}^n p_\ell(t) + (1 - \mu_k) P_k^E(t),$$

for  $k = 1, 2, \dots, n$ . Notice that  $\mu_k = 1$  for all  $k$  reduces the adaptive dynamics to naive dynamics. The Jacobian of adaptive dynamics evaluated at the Bertrand price has the form

$$\mathbf{J}_B^A = \begin{pmatrix} 0 & \theta_1^B \mu_1 & \cdot & \theta_1^B \mu_1 & \theta_1^B (1 - \mu_1) & 0 & \cdot & 0 \\ \theta_2^B \mu_2 & 0 & \cdot & \theta_2^B \mu_2 & 0 & \theta_2^B (1 - \mu_2) & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \theta_n^B \mu_n & \theta_n^B \mu_n & \cdot & 0 & 0 & 0 & \cdot & \theta_n^B (1 - \mu_n) \\ 0 & \mu_1 & \cdot & \mu_1 & 1 - \mu_1 & 0 & \cdot & 0 \\ \mu_2 & 0 & \cdot & \mu_2 & 0 & 1 - \mu_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_n & \mu_n & \cdot & 0 & 0 & 0 & \cdot & 1 - \mu_n \end{pmatrix}$$

where

$$\theta_k^B = \frac{\partial R_k^B}{\partial p_\ell} = \frac{\gamma}{2[1 + (n-2)\gamma]}.$$

The eigenvalue equation of this matrix has the form

$$\mathbf{J}_B^A \mathbf{x} = \lambda \mathbf{x} \text{ with } \mathbf{x} = (p_1, \dots, p_n, p_1^e, \dots, p_n^e)$$

which is equivalently written as

$$\begin{cases} \theta_k^B \mu_k \sum_{\ell \neq k}^n p_\ell + \theta_k^B (1 - \mu_k) p_k^e = \lambda p_k, & 1 \leq k \leq n, \\ \mu_k \sum_{\ell \neq k}^n p_\ell + (1 - \mu_k) p_k^e = \lambda p_k^e, & 1 \leq k \leq n. \end{cases}$$

Subtracting the  $\theta_k^B$ -multiple of the second equation from the first one gives

$$\lambda(p_k - \theta_k^B p_k^e) = 0.$$

The value  $\lambda = 0$  cannot destroy stability, so we may assume  $\lambda \neq 0$ . Then  $p_k = \theta_k^B p_k^e$ , and by substituting it into the second equation, we have

$$\mu_k \sum_{\ell \neq k}^n \theta_\ell^B p_\ell + (1 - \mu_k) p_k^e = \lambda p_k^e, \quad 1 \leq k \leq n.$$

This equation with  $\theta_k^B = \theta_\ell^B$  is the eigenvalue problem of the  $n \times n$  matrix

$$\mathbf{J}_B = \begin{pmatrix} 1 - \mu_1 & \theta_1^B \mu_1 & \cdots & \theta_1^B \mu_1 \\ \theta_2^B \mu_2 & 1 - \mu_2 & \cdots & \theta_2^B \mu_2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \theta_n^B \mu_n & \theta_n^B \mu_n & \cdots & 1 - \mu_n \end{pmatrix}.$$

As in the Cournot competition, we first deal with the case of  $\mu_k = 1$ . The Jacobian  $J_B$  has the same structure of  $J_C$  with  $\mu_k = 1$  for all  $k$ . Replacing  $\theta_k^C$  with  $\theta_k^B$  yields the eigenvalues

$$\lambda_1^B = \cdots = \lambda_{n-1}^B = -\frac{\gamma}{2[1 + (n-2)\gamma]} \text{ and } \lambda_n^B = \frac{(n-1)\gamma}{2[1 + (n-2)\gamma]}.$$

The stability of the Bertrand price under the naive dynamics is summarized as follows<sup>8</sup>:

**Theorem 4** *Under Assumption 1, the best reply dynamics of the Bertrand price with naive expectations is locally asymptotically stable if the goods are substitutes (i.e.,  $\gamma > 0$ ) while it can be locally unstable if the goods are complements (i.e.,  $\gamma < 0$ ) and the number of the firms are strictly greater than three.*

Even in the case of  $\mu_k < 1$ ,  $\mathbf{J}_B$  is exactly the same as  $\mathbf{J}_C$  if the derivatives of the Cournot best reply (i.e.,  $\theta_k^C$ ) are replaced with the derivatives of the Bertrand best reply (i.e.,  $\theta_k^B$ ). From these facts we can conclude first that the local stability conditions of the adaptive dynamics are identical with those of the partial dynamics regardless of whether the quantities or prices are adjusted, and second that the stability conditions of the Bertrand prices are obtained by replacing the derivatives in Theorem 2 that provides the stability conditions of the Cournot output under partial dynamics. In summary asymptotically stable conditions for the Bertrand prices under adaptive dynamics as well as for partial dynamics are given as follows:

<sup>8</sup>This is the same as Theorem 2 of Matsumoto and Szidarovszky (2010).

**Theorem 5** *The Bertrand price is locally asymptotically stable under adaptive dynamics if for all  $k$ ,*

$$\mu_k \left( 1 + \frac{\gamma}{2[1 + (n-2)\gamma]} \right) < 2$$

and

$$\sum_{k=1}^n \frac{\mu_k \gamma}{2[1 + (n-2)\gamma] - \mu_k(2 + (2n-3)\gamma)} > -1.$$

The first condition of Theorem 4 is always fulfilled since  $\mu_k \in (0, 1]$ ,  $|\gamma| < 1$  and  $n > 3$ . The second condition is satisfied if the  $\mu_k$  values are sufficiently small. In particular Figure 2 illustrates how the Bertrand unstable price is stabilized by selecting the smaller values of  $\mu$  when the adjustment coefficients are assumed to be identical (i.e.,  $\mu_k = \mu$ ). The four dotted curves are associated with the four different values of  $\mu$  and divides the  $(\gamma, n)$  plane with  $\gamma < 0$  into the stability region under the curve and the instability region above. The Bertrand price is locally stable in the light-gray region and locally unstable in the dark-gray region when naive dynamics is adopted. The unstable region is surrounded by the two red loci: the  $1 + (n-1)\gamma = 0$  locus and the partition locus with  $\mu = 1$ . It can be seen that as the value of  $\mu$  becomes smaller, the dotted curves shift upward enlarging the stability region. In summary, the stability region under adaptive dynamics as well as partial dynamics enlarges as the speed of adjustment gets smaller.

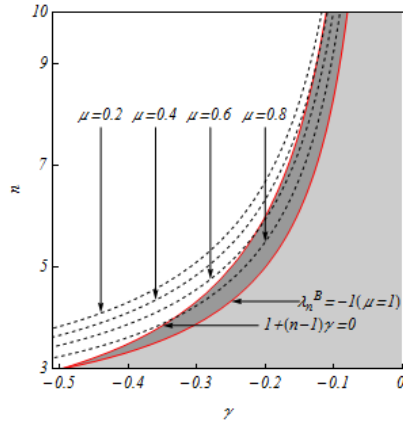


Figure 3. Enlargement of the stability region

## 4 Concluding Remarks

The main purpose of this paper is to reconsider, from the dynamic point of view, the conventional wisdom that price competition is more competitive than quantity competition. To this end, we shed light on the asymptotic behavior of Bertrand and Cournot equilibria. For the sake of mathematical simplicity we use a  $n$ -firm oligopoly with product differentiation in a linear framework in which price and demand functions are linear and so are cost functions. The best replies

and the equilibria in Cournot and Bertrand competitions are determined first and then it is shown that differentiated Bertrand and Cournot equilibria can be locally unstable if the firms have naive belief that all other firms' behavior will remain unchanged. The asymptotic behavior of both equilibria and the global behavior of locally unstable Cournot equilibrium are examined to obtain the following three results:

- 1) The local stability conditions of the best reply dynamics with naive expectations (i.e., adaptive dynamics) are identical with those of the partial adjustment towards the best reply with naive expectations (i.e., partial dynamics);
- 2) As a consequence of the second result, if the firms have either partial dynamics or adaptive dynamics, the smaller adjustment coefficient leads to larger stable region in which the equilibrium is locally asymptotically stable;
- 3) In Cournot competition, the total output as well as the individual outputs generates a period-2 cycle if the best reply dynamics with naive expectation is locally unstable.

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