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with Time Delays

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# Heterogeneous Agents Model of Asset Price with Time Delays \*

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## Abstract

This study constructs a heterogeneous agents model of a financial market in continuous time framework. There are two types of agents, fundamentalists and chartists. The former follow the traditional efficiency market theory and have a linear demand function whereas the latter experience delays in the formation of price trends and possess a *S*-shaped demand function. The main feature of this study is a theoretical investigation on the effects caused by two time delays in a price adjustment process. In particular, two main results are demonstrated: one is that the stability switching curves are analytically derived and the other is that the stability losses and gains can repeatedly occur when the shape of the curves are meandering. These imply that multiple delays might generate price deviations from the equilibrium value.

**Keywords:** Heterogeneous agents model, Two time delays, Limit cycle, Hopf bifurcation, Stability switching curve

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# 1 Introduction

In a financial market persistent volatility or deviations of the asset price from the equilibrium value is often observed. Traditionally the efficient market hypothesis is adopted to analyze the behavior of fundamentals and, however, it fails to explain the discrepancy between the observed market price and the fundamental value of the asset. Recently various heterogeneous agents models are proposed to describe such fluctuations of asset price time evaluation. There are two types of traders, *fundamentalists* who follow the traditional efficient market hypothesis and determine the demand based on the difference between the observed and the fundamental values of the asset and *chartists* who base their decision on the past pattern on price changes. It is shown that these heterogeneities with some nonlinear behavior can cause dynamic phenomena that the fundamental analysis is unable to explain, that is, various dynamics ranging from cyclic fluctuations to erratic behavior. See Day and Huang (1990), Brock and Hommes (1998), Chiarella and He (2003), Chiarella *et al.* (2002, 2006), to name only a few. These models are considered in a discrete-time framework. In addition, continuous-time models with heterogeneous agents are also developed. Chiarella (1992) extends the model of Beja and Goldman (1980) and shows that it is capable of generating a rich form of price dynamics including sustainable fluctuations around the fundamental value when the chartist demand possesses some nonlinearities. Only recently Dibeh (2005) applies the continuous model to a case in which detection of price movements is less than perfect and presents the one-asset model with time delays in the price trends.

This study extends the scope of Dibeh's model one step further. His asset price dynamics is described by a general differential equation with  $n$ -different delays. Since the general model is mathematically intractable, numerically specified cases are examined in Dibeh (2005). In particular, convergence of three-delay models with various specified values of the delays and the birth of limit cycle in a one-delay model are numerically illustrated. Qu and Wei (2010) provide theoretical foundations for the one-delay model. This study constructs a theoretical underpinning of a two-delay model and concerns how the delays affect price time evolutions. The main results to be obtained are

- (1) The stability switching curve is theoretically derived in the two delays parameter plane.
- (2) Limit cycles (i.e., persistent divergence of the market price from the corresponding fundamental value) emerge via Hopf bifurcation when stability of the equilibrium price is lost.
- (3) Stability losses and gains can repeatedly take place as one delay becomes larger and the other delay is fixed at some length.

This paper is organized as follows. Section 2 recapitulates the basic elements of the one delay model and derives a benchmark dynamics of the one delay solution. Section 3 constructs a two delay model and theoretically derives stability

switching curves on which the birth and death of limit cycles occur. Section 4 presents some numerical simulations concerning the theoretical results. Section 5 contains concluding remarks and further research directions.

## 2 Model with One Delay

In this section we briefly summarize a time delay model of speculative asset prices proposed by Dibeh (2005). There are two types of agents, the fundamentalists and the chartists, and the time evolution of the asset price is controlled by a nonlinear delay differential equation,

$$\frac{\dot{p}(t)}{p(t)} = (1 - m)D^c(p(t), p(t - \tau)) + mD^f(p(t)) \quad (1)$$

where  $D^f(p)$  and  $D^c(p)$  are the demand functions of the fundamentalists and the chartists, respectively, and  $m \in (0, 1]$  is the market fraction of the fundamentalists. The functions are specified as follows:

$$\begin{aligned} D^f(p(t)) &= -(p(t) - \nu), \\ D^c(p(t), p(t - \tau)) &= \tanh[p(t) - p(t - \tau)]. \end{aligned} \quad (2)$$

The fundamentalists base their demand formation on the difference between the actual price of the asset,  $p$  and the fundamental price of the asset,  $\nu$ . Since fundamentalists believe that asset prices must converge instantaneously to  $\nu$ , they will sell (buy) the asset when  $p > (<) \nu$ . The chartists base their decisions of market participation on the price trend of the asset. If chartists believe that prices will rise (fall), then their demand for the assets will increase (decrease). The model becomes

$$\dot{p}(t) = (1 - m) \tanh[p(t) - p(t - \tau)] p(t) - m[p(t) - \nu] p(t) \quad (3)$$

that is a nonlinear delay differential equation where the first term is bounded from both sides. Clearly  $p^* = p(t) = p(t - \tau) = \nu$  is a unique positive equilibrium price.

If  $R_H^1(p(t), p(t - \tau))$  denotes the right hand side of equation (3), then the linear approximation in a neighborhood of an equilibrium  $\mathbf{p}^* = (\nu, \nu)$  is

$$\dot{p}_\delta(t) = \left. \frac{\partial R_H^1}{\partial p(t)} \right|_{\mathbf{p}^*} p_\delta(t) + \left. \frac{\partial R_H^1}{\partial p(t - \tau)} \right|_{\mathbf{p}^*} p_\delta(t - \tau)$$

where  $p_\delta = p - p^*, \nu$

$$\left. \frac{\partial R_H^1}{\partial p(t)} \right|_{\mathbf{p}^*} = (1 - 2m)\nu \text{ and } \left. \frac{\partial R_H^1}{\partial p(t - \tau)} \right|_{\mathbf{p}^*} = -(1 - m)\nu.$$

Therefore the linearized equation has the form

$$\dot{p}_\delta(t) = (1 - 2m)\nu p_\delta(t) - (1 - m)\nu p_\delta(t - \tau) \quad (4)$$

Substituting an exponential solution into (4) and arranging the terms yield the corresponding characteristic equation,

$$\lambda - (1 - 2m)\nu + (1 - m)\nu e^{-\lambda\tau} = 0. \quad (5)$$

For  $\tau = 0$ , equation (5) is reduced to

$$\lambda - (1 - 2m)\nu + (1 - m)\nu = 0$$

with a unique negative root

$$\lambda = -m\nu < 0.$$

So the model with no delay is locally asymptotically stable. Further the model with a positive but short delay might be stable. However a longer delay could destabilize the model. It is clear that  $\lambda = 0$  is not a solution of (5). Hence if increasing the length of the delay changes stability to instability, then the real part of an eigenvalue must be zero. Suppose now that the eigenvalue is purely imaginary,  $\lambda = i\omega$  with  $\omega > 0$ .<sup>1</sup> Substituting it into (5) and breaking down the resultant characteristic equation into the real and imaginary parts present two equations,

$$-(1 - 2m)\nu + (1 - m)\nu \cos \omega\tau = 0, \quad (6)$$

$$\omega - (1 - m)\nu \sin \omega\tau = 0.$$

Moving the constant terms to the right hand side and adding the squares of the two equations give the relation

$$\omega^2 = \nu^2 m(2 - 3m).$$

If  $2 - 3m \leq 0$ , then there is no positive  $\omega$ , implying that there is no solution such as  $\lambda = i\omega$  with  $\omega > 0$ . Hence the negative real parts of the eigenvalues do not change their sign to positive for any value of  $\tau$  and therefore no stability switches occur. On the other hand, if  $2 - 3m > 0$ , then there is a positive solution,

$$\omega_0 = \nu\sqrt{m(2 - 3m)}.$$

Substituting it into the first equation of (6) and solving for  $\tau$  determines the values of  $\tau$  for which the characteristic equation has a purely imaginary solution, given  $v$ ,  $m$  and  $k$ ,

$$\tau_k(m, \nu) = \frac{1}{\omega_0} \left[ \cos^{-1} \left( \frac{1 - 2m}{1 - m} \right) + 2k\pi \right] \text{ for } k = 0, 1, 2, \dots$$

Hence the stability switch might occur when  $\tau$  increases and becomes any of these threshold values,

$$\tau_0 = \tau_0(m, v).$$

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<sup>1</sup>We will have the same results even if  $\omega < 0$ .

To check the direction of stability switches, we assume  $\lambda = \lambda(\tau)$  and differentiate equation (5) with respect to  $\tau$  to obtain,

$$\frac{d\lambda}{d\tau} + (1-m)\nu e^{-\lambda\tau} \left( -\frac{d\lambda}{d\tau}\tau - \lambda \right) = 0$$

that is solved for the inverse of the derivative

$$\left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{1}{\lambda[-\lambda + (1-2m)\nu]} - \frac{\tau}{\lambda}$$

where the relation obtained from the characteristic equation is used,

$$(1-m)\nu e^{-\lambda\tau} = -\lambda + (1-2m)\nu.$$

At  $\lambda = i\omega$ , the real part of the derivative is

$$\begin{aligned} \operatorname{Re} \left[ \left( \frac{d\lambda}{d\tau} \right)^{-1}_{\lambda=i\omega} \right] &= \operatorname{Re} \left[ \frac{1}{i\omega[-i\omega + (1-2m)\nu]} \right] \\ &= \frac{1}{\omega^2 + (1-2m)^2\nu^2} > 0. \end{aligned}$$

So we have the following result:

**Theorem 1** *The system is stable with  $m \geq 2/3$  for all  $\tau > 0$ . If  $m < 2/3$ , then stability is lost at the smallest stability switch curve,*

$$\tau_0(m, \nu) = \frac{\cos^{-1} \left( \frac{1-2m}{1-m} \right)}{\nu \sqrt{m(2-3m)}}$$

*and stability cannot be regained later. At  $\tau = \tau_0(m, \nu)$  Hopf bifurcation occurs with the possibility of the birth of limit cycles.*

Graphical representations are given in Figure 1. The yellow region in Figure 1(A) is the stable region. The upward-sloping black curve is the locus of  $(m, \tau)$  satisfying  $\tau = \tau_0(m, \nu)$  and asymptotic to the dotted line at  $m = 2/3$ . Thus this figure confirms first that the solution of (3) is locally asymptotically stable for any  $\tau \geq 0$  if  $m \geq 2/3$  and second that it is locally stable for  $\tau < \tau_0(m, \nu)$  and unstable for  $\tau > \tau_0(m, \nu)$  if  $m < 2/3$ . Since  $\partial\tau_0(m, \nu)/\partial\nu < 0$ , increasing the fundamental price shifts the stability switching curve downward and thus has a destabilizing effect in the sense that the stability region becomes smaller. Figure 1(B) depicts the bifurcation diagram with respect to  $\tau$  in which the local maximum and minimum of the trajectories are plotted against  $\tau$ . It shows that the trajectory converges to the stationary point  $v$  for  $\tau < \tau_0(m, \nu)$ , this stability is lost at  $\tau = \tau_0(m, \nu)$  and a limit cycle having one maximum and one minimum

emerges for  $\tau > \tau_0(m, \nu)$ .<sup>2</sup> It is also seen that the limit cycle is expanding as the length of the delay becomes larger.

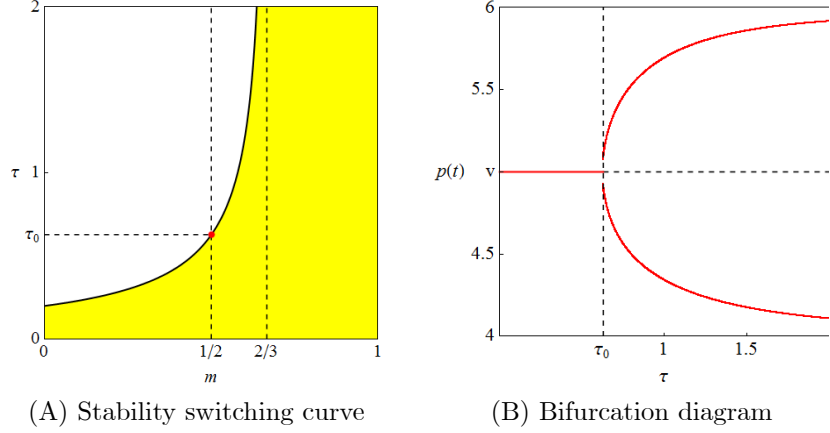


Figure 1. Dynamics of the one delay model

### 3 Model with Two Delays

We next extend the model one step further by introducing a second delay in the price trend. The price dynamic equation with two delays is

$$\dot{p}(t) = (1 - m) \tanh\{f[p(t), p(t - \tau_1), p(t - \tau_2)]\}p(t) - m[p(t) - \nu]p(t) \quad (7)$$

where

$$f[p(t), p(t - \tau_1), p(t - \tau_2)] = \sigma [p(t) - p(t - \tau_1)] + (1 - \sigma) [p(t - \tau_1) - p(t - \tau_2)]$$

with  $\sigma \in [0, 1]$ . Let  $R_H$  be right hand side of equation (7). The linear approximation of equation (7) evaluated at the stationary price vector  $\mathbf{p}^* = (\nu, \nu, \nu)$  is

$$\dot{p}_\delta(t) = \left. \frac{\partial R_H}{\partial p(t)} \right|_{\mathbf{p}^*} p_\delta(t) + \left. \frac{\partial R_H}{\partial p(t - \tau_1)} \right|_{\mathbf{p}^*} p_\delta(t - \tau_1) + \left. \frac{\partial R_H}{\partial p(t - \tau_2)} \right|_{\mathbf{p}^*} p_\delta(t - \tau_2)$$

where

$$\left. \frac{\partial R_H}{\partial p(t)} \right|_{\mathbf{p}^*} = \alpha =: [(1 - m)\sigma - m]\nu, \quad (8)$$

$$\left. \frac{\partial R_H}{\partial p(t - \tau_1)} \right|_{\mathbf{p}^*} = \beta_1 =: (1 - m)(1 - 2\sigma)\nu \quad (9)$$

<sup>2</sup>In particular, we take  $m = 1/2$  and  $\nu = 5$ , which lead to  $\tau_0(1/2, 5) \simeq 0.628$ . If  $m$  is increased to 0.62 as in Dibeh's example,  $\tau_0(0.62, 5) \simeq 1.5304$ .

and

$$\left. \frac{\partial R_H}{\partial p(t - \tau_2)} \right|_{\mathbf{p}^*} = -\beta_2 =: -(1 - m)(1 - \sigma)\nu < 0. \quad (10)$$

Although  $\beta_2 > 0$  always, the signs of  $\alpha$  and  $\beta_1$  depend on values of  $m$  and  $\sigma$  in the following way,

$$\alpha \begin{cases} \geq 0 \\ \leq 0 \end{cases} \text{ according to } \sigma \begin{cases} \geq \frac{m}{1 - m} \\ < \frac{m}{1 - m} \end{cases}$$

and

$$\beta_1 \begin{cases} \geq 0 \\ \leq 0 \end{cases} \text{ according to } \sigma \begin{cases} \leq \frac{1}{2} \\ > \frac{1}{2} \end{cases}.$$

The linearized equation with these new parameters is rewritten as

$$\dot{p}_\delta(t) = \alpha p_\delta(t) + \beta_1 p_\delta(t - \tau_1) - \beta_2 p_\delta(t - \tau_2)$$

which is, with  $p_\delta(t) = e^{\lambda t} u$ , further converted into the characteristic equation,

$$\lambda - \alpha - \beta_1 e^{-\lambda \tau_1} + \beta_2 e^{-\lambda \tau_2} = 0. \quad (11)$$

This is essentially the same equation that Gu *et al.* (2005) investigate. Following their method, we determine the location of the eigenvalues. To this end, we divide the left hand side of (11) by  $\lambda - \alpha$  and denote the result as  $a(\lambda)$ :

$$a(\lambda) := 1 + a_1(\lambda)e^{-\lambda \tau_1} + a_2(\lambda)e^{-\lambda \tau_2} \quad (12)$$

with new functions

$$a_1(\lambda) = -\frac{\beta_1}{\lambda - \alpha}$$

and

$$a_2(\lambda) = \frac{\beta_2}{\lambda - \alpha}.$$

In the special case of  $\sigma = 1/2$ ,  $\beta_1 = 0$  implying that  $a_1(\lambda) = 0$ , so equation (11) becomes the one-delay equation

$$\lambda - \alpha + \beta_2 e^{-\lambda \tau_2} = 0$$

or

$$\lambda - \frac{1}{2}(1 - 3m)\nu + \frac{1}{2}(1 - m)\nu e^{-\lambda \tau_2} = 0.$$

This equation does not depend on  $\tau_1$ , so it can be arbitrary. Similarly to Theorem 1 we can prove that system is stable for all  $\tau_2$  as  $m \geq 1/2$ . If  $m < 1/2$ , then stability is lost at the smallest stability switching curve,

$$\tau_2(m, \nu) = \frac{\cos^{-1}\left(\frac{1 - 3m}{1 - m}\right)}{\nu \sqrt{m(1 - 2m)}}$$

and stability cannot be regained later. Furthermore at  $\tau_2 = \tau_2(m, \nu)$  Hopf bifurcation occurs. If  $\sigma \neq 1/2$ , then both  $\beta_1$  and  $\beta_2$  are nonzero, so equation (11) is a two-delay equation.



For  $\lambda = i\omega$  with  $\omega > 0$ ,  $a_1(\lambda)$  and  $a_2(\lambda)$  are written as

$$a_1(i\omega) = \frac{\alpha\beta_1}{\alpha^2 + \omega^2} + i\frac{\beta_1\omega}{\alpha^2 + \omega^2} \quad (13)$$

and

$$a_2(i\omega) = -\frac{\alpha\beta_2}{\alpha^2 + \omega^2} - i\frac{\beta_2\omega}{\alpha^2 + \omega^2}. \quad (14)$$

Their absolute values are

$$|a_1(i\omega)| = \frac{|\beta_1|}{\sqrt{\alpha^2 + \omega^2}} \quad (15)$$

and

$$|a_2(i\omega)| = \frac{\beta_2}{\sqrt{\alpha^2 + \omega^2}}. \quad (16)$$

We can consider the three terms in  $a(\lambda)$  as three vectors in the complex plane. If  $i\omega$  is a solution of  $a(\lambda) = 0$ , then putting these vectors head to tail forms a triangle. Solving  $a(i\omega) = 0$  analytically is equivalent to solving the following three inequality conditions graphically, each of which means that the length of the sum of two adjacent line segments of the triangle is not shorter than the length of the remaining line segment,

$$(i) \quad 1 \leq |a_1(i\omega)| + |a_2(i\omega)|,$$

$$(ii) \quad |a_1(i\omega)| \leq 1 + |a_2(i\omega)|,$$

$$(iii) \quad |a_2(i\omega)| \leq 1 + |a_1(i\omega)|.$$

Using the absolute values in (15) and (16), we rewrite these three conditions in terms of  $m$  and  $\sigma$ . First, substituting the absolute values into condition (i) yields

$$\sqrt{\alpha^2 + \omega^2} \leq |\beta_1| + \beta_2 \quad (17)$$

where the right-hand side has two expressions according to whether the value of  $\sigma$  is less than  $1/2$  or not,

$$|\beta_1| + \beta_2 = \begin{cases} (1-m)(2-3\sigma)\nu & \text{if } \sigma < \frac{1}{2}, \\ (1-m)\sigma\nu & \text{if } \sigma \geq \frac{1}{2} \end{cases}$$

which is nonnegative in both cases. So condition (17) is equivalent to

$$\omega^2 \leq \Delta_1(m, \sigma) := (|\beta_1| + \beta_2)^2 - \alpha^2. \quad (18)$$

If  $\Delta_1(m, \sigma) \leq 0$ , then there is no  $\omega > 0$ , implying no stability switch for any delays. We then look for conditions under which  $\Delta_1(m, \sigma) > 0$  or

$$|\alpha| < |\beta_1| + \beta_2.$$

If  $\sigma < 1/2$ , then

$$|(1-m)\sigma - m| < (1-m)(2-3\sigma)$$

yields two inequality conditions,

$$\sigma < f_1(m) := \frac{2-3m}{2(1-m)} \quad (19)$$

and

$$\sigma < f_2(m) := \frac{2-m}{4(1-m)} \quad (20)$$

where the first condition is violated with all  $\sigma > 0$  for  $m \geq 2/3$ , and the second condition is ineffective as  $f_2(m) > 1/2$  for  $m \in [0, 1]$ . If  $\sigma \geq 1/2$ , then

$$|(1-m)\sigma - m| < (1-m)\sigma$$

generates two inequality conditions, one is

$$\sigma > f_3(m) := \frac{m}{2(1-m)} \quad (21)$$

and the other is  $m > 0$ , which is already assumed above. A graphical description of conditions (19) and (21) is given in Figure 2(A). The vertically-stripped region is surrounded by the curve of  $f_2(m)$  and  $\sigma = 1/2$  while the horizontally-stripped region is constructed by the curve of  $f_1(m)$  and  $\sigma = 1/2$ . Hence  $\Delta_1(m, \sigma) > 0$  in the union of these striped regions and  $\Delta_1(m, \sigma) \leq 0$  in the gray region including its boundaries where  $m \geq 2/3$  or  $f_1(m) \leq \sigma \leq f_3(m)$  hold. Denoting the striped region by  $\mathbf{U}$ ,

$$\mathbf{U} = \mathbf{U}_1 \cup \mathbf{U}_2$$

where

$$\mathbf{U}_1 = \left\{ (m, \sigma) \mid \sigma < \frac{1}{2} \text{ and } \sigma < f_1(m) \right\}$$

and

$$\mathbf{U}_2 = \left\{ (m, \sigma) \mid \sigma \geq \frac{1}{2} \text{ and } \sigma > f_3(m) \right\}.$$

We summarize the result concerning condition (i) as follows:

**Lemma 1** *If  $(m, \sigma) \in \mathbf{U}$ , then  $\Delta_1(m, \sigma) > 0$  and condition (i) holds for  $\omega < \omega_f = \sqrt{\Delta_1(m, \sigma)}$  which is violated for  $\omega > \omega_f$ , and also for all  $(m, \sigma) \notin \mathbf{U}$ .*

Second, condition (ii) is rewritten as

$$|\beta_1| - \beta_2 \leq \sqrt{\alpha^2 + \omega^2} \quad (22)$$

where the left-hand side has two expressions according to whether the value of  $\sigma$  is less than  $1/2$  or not,

$$|\beta_1| - \beta_2 = \begin{cases} (1-m)(-\sigma)\nu & \text{if } \sigma < \frac{1}{2}, \\ -(1-m)(2-3\sigma)\nu & \text{if } \sigma \geq \frac{1}{2}. \end{cases}$$

In the first case  $|\beta_1| - \beta_2 < 0$  always and in the second case, it can be of either sign,

$$|\beta_1| - \beta_2 \begin{cases} \geq 0 & \text{if } \sigma \geq \frac{2}{3}, \\ < 0 & \text{if } \sigma < \frac{2}{3}. \end{cases}$$

We see that if  $\sigma < 2/3$ , then the left-hand side of (22) is negative implying that the inequality is satisfied. On the other hand, if  $\sigma \geq 2/3$ , then (22) is equivalent to

$$\omega^2 \geq \Delta_2(m, \sigma) := (|\beta_1| - \beta_2)^2 - \alpha^2. \quad (23)$$

If  $\Delta_2(m, \sigma) \leq 0$ , then any  $\omega > 0$  satisfies (22). To have  $\Delta_2(m, \sigma) > 0$ , we then look for a condition for

$$||\beta_1| - \beta_2| > |\alpha|. \quad (24)$$

Since (22) is satisfied for  $\sigma < 2/3$  with any  $\omega > 0$ , we can focus on the case of  $\sigma \geq 2/3$ . Then (24) can be rewritten as

$$|(1-m)\sigma - m| < (1-m)(3\sigma - 2)$$

or

$$-(1-m)(3\sigma - 2) < (1-m)\sigma - m < (1-m)(3\sigma - 2)$$

which is two inequalities. The left hand side is

$$\sigma > f_2(m) \quad (25)$$

and the right hand side is

$$\sigma > f_1(m). \quad (26)$$

So let

$$\mathbf{A} = \left\{ (m, \sigma) \mid \sigma \geq \frac{2}{3}, \sigma > f_1(m) \text{ and } \sigma > f_2(m) \right\},$$

which is the vertically striped region in Figure 2(B). Hence we have the following result.

**Lemma 2** *If  $(m, \sigma) \in \mathbf{A}$ , then  $\Delta_2(m, \sigma) > 0$ , and condition (ii) holds if  $\omega > \omega_g = \sqrt{\Delta_2(m, \sigma)}$ , and violated if  $\omega \leq \omega_g$ . If  $(m, \sigma) \notin \mathbf{A}$ , then condition (ii) holds for all  $\omega > 0$ .*

Lastly, condition (iii) is rendered to

$$\beta_2 - |\beta_1| \leq \sqrt{\alpha^2 + \omega^2}. \quad (27)$$

It can be shown that  $\beta_2 - |\beta_1|$  is nonpositive if  $\sigma \geq 2/3$  and positive otherwise. Thus (27) is satisfied if  $\sigma \geq 2/3$ . So we focus only on the case of  $\sigma < 2/3$ . On the other hand, if the left hand side of (27) is positive, then it is required to satisfy the equivalent inequality,

$$\omega^2 \geq \Delta_2(m, \sigma) = (\beta_2 - |\beta_1|)^2 - \alpha^2 \quad (28)$$

which is the same as (23). If  $(\beta_2 - |\beta_1|)^2 - \alpha^2 \leq 0$ , then (27) holds for all  $\omega > 0$ . So we need to find condition such that  $\alpha^2 \leq (\beta_2 - |\beta_1|)^2$ .

If  $\sigma < 1/2$ , then this relation can be written as

$$-(1-m)\sigma < (1-m)\sigma - m < (1-m)\sigma.$$

The right hand side always holds, the left hand side can be rewritten as

$$\sigma > f_3(m). \quad (29)$$

Notice that

$$f_3(m) < \frac{1}{2} \text{ if } m < \frac{1}{2}.$$

The region

$$\mathbf{B} = \left\{ (m, \sigma) \mid \sigma < \frac{1}{2}, \sigma > f_3(m) \right\}$$

is the horizontally-striped region in Figure 2(B).

Assume next that  $1/2 \leq \sigma < 2/3$ . Then  $\alpha^2 \leq (\beta_2 - |\beta_1|)^2$  holds if

$$-(1-m)(2-3\sigma) < (1-m)\sigma - m < (1-m)(2-3\sigma).$$

The left hand side can be rewritten as

$$\sigma < f_1(m)$$

and the right hand side as

$$\sigma < f_2(m).$$

Define the region

$$\mathbf{C} = \left\{ (m, \sigma) \mid \frac{1}{2} \leq \sigma < \frac{2}{3}, \sigma < f_1(m), \sigma < f_2(m) \right\}$$

which is diagonally-striped in Figure 2(B). Our result can be summarized as follows.

**Lemma 3** *If  $(m, \sigma) \in \mathbf{B} \cup \mathbf{C}$ , then  $\Delta_2(m, \sigma) > 0$ , and condition (iii) holds if  $\omega > \omega_g = \sqrt{\Delta_2(m, \sigma)}$  and violated if  $\omega \leq \omega_g$ . If  $(m, \sigma) \notin \mathbf{B} \cup \mathbf{C}$ , then condition (iii) holds for all  $\omega > 0$ .*

Notice that  $\Delta_2(m, \sigma) < \Delta_1(m, \sigma)$  for all values of  $m$  and  $\sigma$ . Lemmas 1, 2 and 3 imply the fulfillment of conditions (i), (ii) and (iii),  $\omega^2 - \Delta_1(m, \sigma) \leq 0$  and  $\omega^2 - \Delta_2(m, \sigma) \geq 0$ :

**Theorem 2** *If  $(m, \sigma) \in \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$ , then*

$$\Delta_2(m, \sigma) \leq \omega^2 \leq \Delta_1(m, \sigma) \text{ for } \omega \in [\omega_g, \omega_f]$$

*whereas if  $(m, \sigma) \in \mathbf{U} - \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$ , then*

$$0 \leq \omega^2 \leq \Delta_1(m, \sigma) \text{ for } \omega \in [0, \omega_f].$$

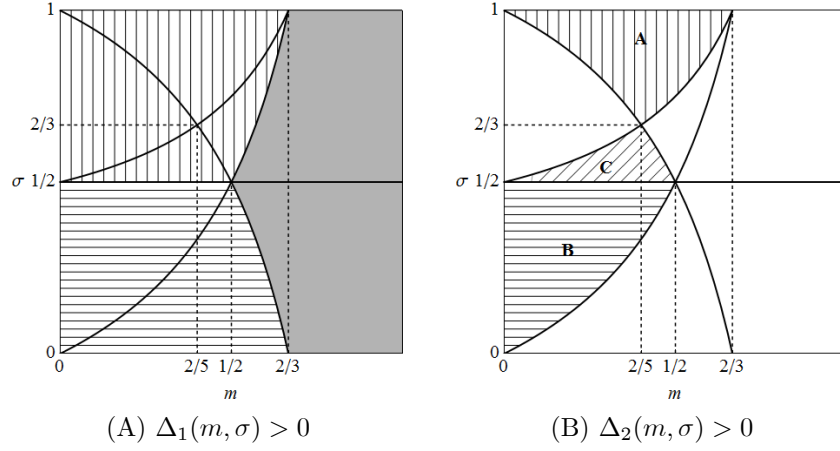


Figure 2. The triangle conditions (i), (ii) and (iii)

We will next find all pairs of delays  $(\tau_1, \tau_2)$  satisfying  $a(i\omega) = 0$ . Treating the three terms in (12) as three vectors constructing a triangle, we suppose that  $|1|$  is its base and denote the angle between  $|1|$  and  $|a_1(i\omega)|$  by  $\theta_1$  and the angle between  $|1|$  and  $|a_2(i\omega)|$  by  $\theta_2$ . Then by the law of cosine, these angles are

$$\theta_1(\omega) = \cos^{-1} \left( \frac{\alpha^2 + \omega^2 + \beta_1^2 - \beta_2^2}{2|\beta_1|\sqrt{\alpha^2 + \omega^2}} \right)$$

and

$$\theta_2(\omega) = \cos^{-1} \left( \frac{\alpha^2 + \omega^2 + \beta_2^2 - \beta_1^2}{2\beta_2\sqrt{\alpha^2 + \omega^2}} \right).$$

Since the triangle may be located above and below the horizontal axis in the complex plain, we have two possibilities,

$$\{\arg a_1(i\omega)e^{-\omega\tau_1} + 2k\pi\} \pm \theta_1(\omega) = \pi$$

and

$$\{\arg a_2(i\omega)e^{-\omega\tau_2} + 2n\pi\} \mp \theta_2(\omega) = \pi.$$

Solving these equations for  $\tau_1$  and  $\tau_2$  yields the threshold values of the delays,

$$\tau_1^\pm(\omega, k) = \frac{1}{\omega} \{\arg a_1(i\omega) + (2k - 1)\pi \pm \theta_1(\omega)\} \quad (30)$$

and

$$\tau_2^\mp(\omega, n) = \frac{1}{\omega} \{\arg a_2(i\omega) + (2n - 1)\pi \mp \theta_2(\omega)\}. \quad (31)$$

Notice that particular values of  $\arg a_1(i\omega)$  and  $\arg a_2(i\omega)$  depend on the choice of the parameter values. Let  $\Omega$  be the interval of  $\omega$  specified in Theorem 2. Then for  $\omega \in \Omega$  we can find the pairs of  $(\tau_1, \tau_2)$  constructing the stability switching curves consisting of two sets of parametric segments,

$$L_1(k, n) = \{\tau_1^+(\omega, k), \tau_2^-(\omega, n)\} \quad (32)$$

and

$$L_2(k, n) = \{\tau_1^-(\omega, k), \tau_2^+(\omega, n)\} \quad (33)$$

where the parameter  $\omega$  runs through interval  $\Omega$ .

## 4 Examples

In this section, we specify the parameter values and perform numerical simulations to confirm the shape of the stability switching curve. As seen in (13) and (14), the values of  $\arg(a_1(i\omega))$  and  $\arg(a_2(i\omega))$  depend on the signs of  $\alpha$  and  $\beta_1$ , the signs of which, in turn, depend on the values of  $m$  and  $\sigma$ . The  $(m, \sigma)$  plane is divided into four subregions by the loci of  $\alpha = 0$  and  $\beta_1 = 0$ . Accordingly we select four different combinations of  $m$  and  $\sigma$ , keeping  $\nu = 5$  in the following examples.

**Example 1:**  $m = 1/4$ ,  $\sigma = 2/5$  and  $\nu = 5$

The specified point  $(m, \sigma)$  is in the set **B**. Under these specifications,  $\alpha > 0$  and  $\beta_1 > 0$  so

$$\arg a_1(i\omega) = \tan^{-1}\left(\frac{\omega}{\alpha}\right) \quad \text{and} \quad \arg a_2(i\omega) = \pi + \tan^{-1}\left(\frac{\omega}{\alpha}\right).$$

Substituting them into (30) and (31) gives

$$\tau_1^\pm(\omega, k) = \frac{1}{\omega} \left\{ \tan^{-1}\left(\frac{\omega}{\alpha}\right) + (2k - 1)\pi \pm \theta_1(\omega) \right\} \quad (34)$$

and

$$\tau_2^\mp(\omega, n) = \frac{1}{\omega} \left\{ \tan^{-1}\left(\frac{\omega}{\alpha}\right) + 2n\pi \mp \theta_2(\omega) \right\}. \quad (35)$$

The stability switching curve is shown in Figure 3(A) with  $n = 0$  and  $k = 0, 1, 2, 3$ . We see that  $L_1(k, 0)$  and  $L_2(k, 0)$  for  $k = 0, 1, 2, 3$  are the red and blue segments of the curve and shift rightward as  $k$  increases. The segment  $L_2(0, 0)$  is located in the second quadrant in which  $\tau_1^-(\omega, 0)$  is negative and thus is not depicted. So  $L_1(0, 0)$  is the left-most red segment, starts at the initial point  $(\tau_1^+(\omega_s, 0), \tau_2^-(\omega_s, 0))$  and arrives at the end point  $(\tau_1^+(\omega_e, 0), \tau_2^-(\omega_e, 0))$  as  $\omega$  increases from  $\omega_s$  to  $\omega_e$  where

$$\omega_s = \sqrt{(|\beta_1| - \beta_2)^2 - \alpha^2} \text{ and } \omega_e = \sqrt{(|\beta_1| + \beta_2)^2 - \alpha^2}.$$

$L_2(1, 0)$  is the left-most blue segment, starts at the initial point  $(\tau_1^-(\omega_s, 1), \tau_2^+(\omega_s, 0))$  and arrives at the end point  $(\tau_1^+(\omega_e, 1), \tau_2^-(\omega_e, 0))$ . It can be checked that

$$\tau_1^+(\omega_s, 0) = \tau_1^-(\omega_s, 1) \simeq 0.947$$

and

$$\tau_2^+(\omega_s, 0) = \tau_2^-(\omega_s, 0) \simeq 0.947.$$

The segments  $L_1(0, 0)$  and  $L_2(1, 0)$  have the same initial point located on the diagonal. The end point of the segment  $L_1(1, 0)$  is given by  $(\tau_1^+(\omega_e, 1), \tau_2^-(\omega_e, 0))$  where

$$\tau_1^+(\omega_e, 1) = \tau_1^-(\omega_e, 1) \simeq 1.548$$

and

$$\tau_2^-(\omega_e, 0) = \tau_2^-(\omega_e, 0) \simeq 0.495.$$

Hence the segments  $L_1(1, 0)$  and  $L_2(1, 0)$  have the same end points. In this way, one segment is connected with the other segment and the connecting segments for  $n = 0$  and  $k = 0, 1, 2, 3$  form the stability switching curve as shown in Figure 3(A). More generally, the following can be shown:

**Proposition 1** *Given the parameter specifications resulting in  $\alpha > 0$  and  $\beta_1 > 0$ , the segments  $L_1(k, n)$  and  $L_2(k+1, n)$  have the same initial point whereas the segments  $L_1(k, n)$  and  $L_2(k, n)$  have the same end point for any positive integer values of  $k$  and  $n$ .*

Fixing the value of  $\tau_2$  at  $\bar{\tau}_2 = 0.5$  and gradually increasing the value of  $\tau_1$  from zero to 8, the horizontal line at  $\tau_2 = \bar{\tau}_2$  crosses the stability switching curve five times with alternating stability losses and gains. The crossing points are denoted by the black dots and their  $\tau_1$ -coordinates are labelled by  $\tau_1^a (\simeq 1.548)$ ,  $\tau_1^b (\simeq 2.654)$ ,  $\tau_1^d (\simeq 3.650)$  and  $\tau_1^e (\simeq 5.443)$ , respectively, in the ascending order.<sup>3</sup> The bifurcation diagram of Figure 3(B) shows the stable and the unstable regions for  $\tau_1$  where two-cycles occur, the magnitude of the oscillations increases and then decreases back to the stable steady state when stability

<sup>3</sup>The fifth point  $\tau_1^e (\simeq 5.443)$  is not labelled in order to avoid graphical conjection.

is regained.

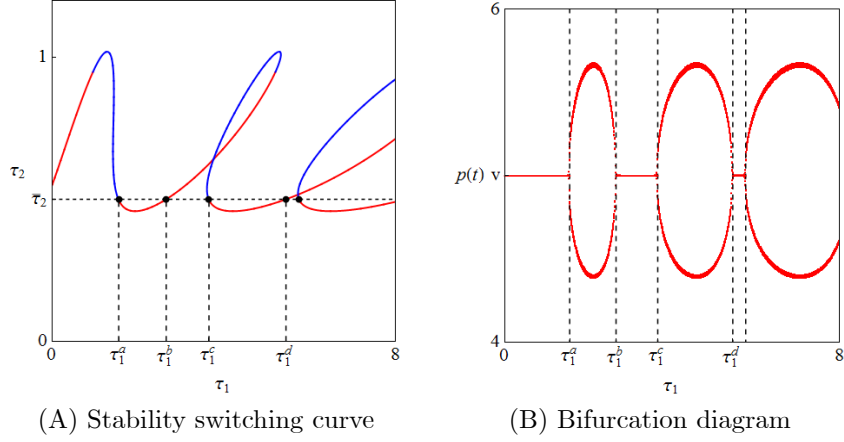


Figure 3. Illustration of Example 1

**Example 2:**  $m = 2/5$ ,  $\sigma = 7/12$  and  $\nu = 5$

The specified point  $(m, \sigma)$  is in the set  $\mathbf{C}$  and makes  $\alpha < 0$  and  $\beta_1 < 0$ . Therefore

$$\arg a_1(i\omega) = \pi + \tan^{-1} \left( \frac{\omega}{\alpha} \right) \text{ and } \arg a_2(i\omega) = \pi + \tan^{-1} \left( \frac{\omega}{\alpha} \right).$$

Substituting these expressions into (30) and (31) give

$$\tau_1^\pm(\omega, k) = \frac{1}{\omega} \left\{ -\tan^{-1} \left( -\frac{\omega}{\alpha} \right) + (2k + 1)\pi \pm \theta_1(\omega) \right\} \quad (36)$$

and

$$\tau_2^\mp(\omega, n) = \frac{1}{\omega} \left\{ -\tan^{-1} \left( -\frac{\omega}{\alpha} \right) + (2n + 1)\pi \mp \theta_2(\omega) \right\}. \quad (37)$$

Figure 4(A) shows the stability switching curve. With fixed value of  $\tau_2 = 1.5$ , we gradually increase  $\tau_1$  from 0 to 8. The horizontal line crosses the stability switching curve three times with two stability losses and one stability regain. As the bifurcation diagram in Figure 4(B) shows, two-cycles emerge in the unstable regions of  $\tau_1$  similarly to Example 1. It is possible to obtain the general result



similar to Proposition 1.

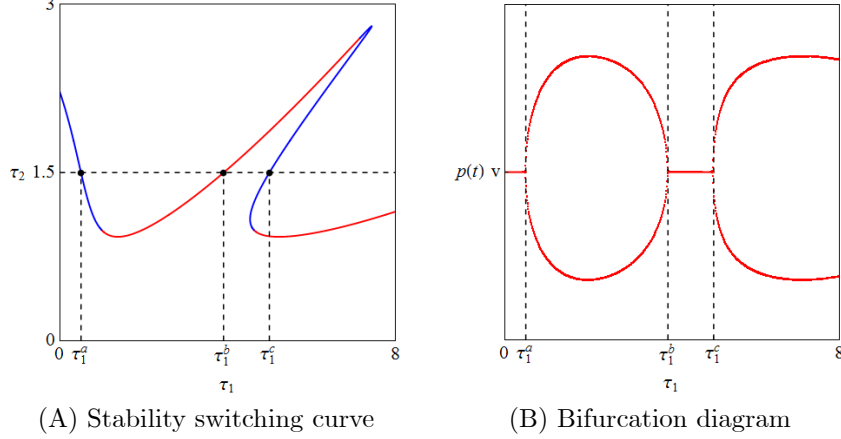


Figure 4. Illustration of Example 2

**Example 3:**  $m = 2/5$ ,  $\sigma = 5/6$  and  $\nu = 5$

The specified point  $(m, \sigma)$  belongs to the set  $\mathbf{C}$ . Under these specifications,  $\alpha > 0$  and  $\beta_1 < 0$  so

$$\arg a_1(i\omega) = \tan^{-1} \left( \frac{\omega}{\alpha} \right) \text{ and } \arg a_2(i\omega) = \pi + \tan^{-1} \left( \frac{\omega}{\alpha} \right).$$

Substituting these expressions into (30) and (31) give

$$\tau_1^\pm(\omega, k) = \frac{1}{\omega} \left\{ \tan^{-1} \left( \frac{\omega}{\alpha} \right) + 2k\pi \pm \theta_1(\omega) \right\} \quad (38)$$

and

$$\tau_2^\mp(\omega, n) = \frac{1}{\omega} \left\{ \tan^{-1} \left( \frac{\omega}{\alpha} \right) + 2n\pi \mp \theta_2(\omega) \right\}. \quad (39)$$

The stability switching curve is shown in Figure 5(A) with  $k = 0$  and  $n = 0, 1, 2$ . As in Example 1,  $L_1(0, n)$  and  $L_2(0, n)$  are the red and blue segments of the curve and shift upward as  $n$  increases.  $L_1(0, 0)$  is the lowest red segment crossing the horizontal line at  $\tau_1 \simeq 0.785$ . It starts at the initial point  $(\tau_1^+(\omega_s, 0), \tau_2^-(\omega_s, 0))$  and arrives at the end point  $(\tau_1^+(\omega_e, 0), \tau_2^-(\omega_e, 0))$  as  $\omega$  increases from  $\omega_s$  to  $\omega_e$ .<sup>4</sup>  $L_2(0, 0)$  is the lowest blue segment starting at the initial point  $(\tau_1^-(\omega_s, 0), \tau_2^+(\omega_s, 0))$  and arriving at the end point  $(\tau_1^-(\omega_e, 0), \tau_2^+(\omega_e, 0))$ . It can be checked that the end point is on the diagonal with

$$\tau_1^+(\omega_e, 0) = \tau_1^-(\omega_e, 0) \simeq 0.559$$

<sup>4</sup>In this example,  $\omega_s \simeq 1.414$  and  $\omega_e \simeq 2.449$ . Since  $\tau_2^-(\omega, 0) \leq 0$  for  $\omega \in [\omega_s, 2]$  where  $(\tau_1^+(\omega_s, 0), \tau_2^-(\omega_s, 0)) \simeq (0.870, -1.351)$  and  $(\tau_1^+(2, 0), \tau_2^-(2, 0)) \simeq (0, 0.785)$ ,  $L_1(0, 0)$  is depicted in Figure 5(A) for  $\omega \in [2, \omega_e]$

and

$$\tau_2^-(\omega_e, 0) = \tau_2^+(\omega_e, 0) \simeq 0.559.$$

These equalities imply that segments  $L_1(0, 0)$  and  $L_2(0, 0)$  have the same end point. The starting point of  $L_1(0, 1)$  is given by  $(\tau_1^+(\omega_s, 1), \tau_2^-(\omega_s, 0))$  where

$$\tau_1^+(\omega_s, 1) = \tau_1^-(\omega_s, 0) \simeq 0.870$$

and

$$\tau_2^-(\omega_s, 0) = \tau_2^+(\omega_s, 0) \simeq 3.092.$$

Hence segments  $L_2(0, 0)$  and  $L_1(0, 1)$  have the same initial point. More generally, the following can be shown:

**Proposition 2** *Given the parameter specifications resulting in  $\alpha > 0$  and  $\beta_1 < 0$ , the segments  $L_1(k, n)$  and  $L_2(k, n+1)$  have the same initial point whereas the segments  $L_1(k, n)$  and  $L_2(k, n)$  have the same end point for any positive integer values of  $k$  and  $n$ .*

A part of the stability switching curve is enlarged in Figure 5(B), and in Figure 6 two bifurcation diagrams are shown. With fixed value of  $\tau_1 = \tau_1^0$ , the value of  $\tau_2$  increases from 0 to 4, and the corresponding vertical line has three intersections with the stability switching curve with two stability losses and one regain, which is well shown in Figure 6(A). If the value of  $\tau_2$  is fixed at  $\tau_2 = \tau_2^0$  and  $\tau_1$  is increased from 0 to  $3/2$ , then three similar intersections are obtained with two stability losses and one stability regain. This is also verified in Figure 6(B). In all cases two-cycles are found in the cases of instability, where the magnitude increases from 0 to a maximum and then decreases back to zero when stability is regained. So the dynamics behavior of the system is relatively simple, limit cycles occur in the unstable regions.

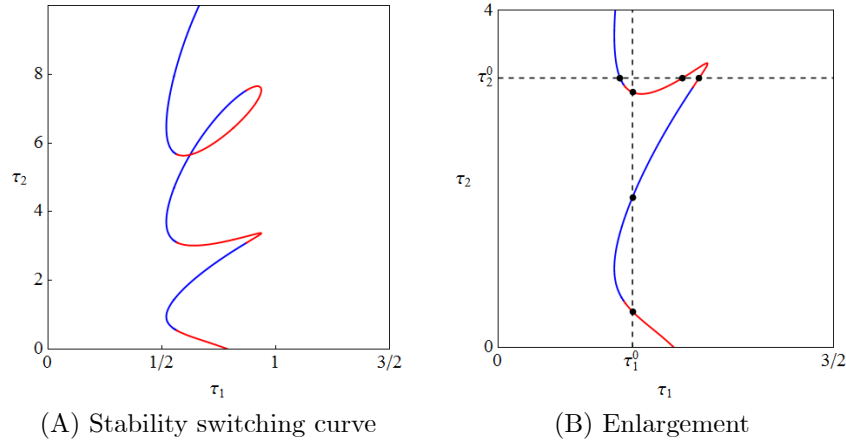


Figure 5. Stability switching curves in Example 3

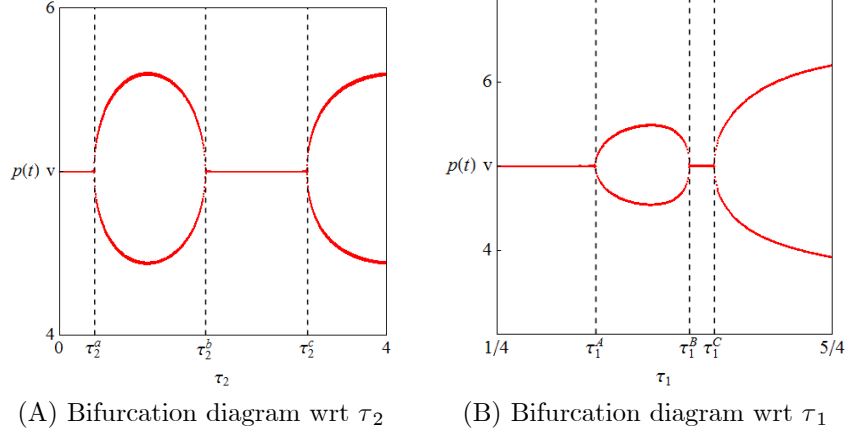


Figure 6. Bifurcation diagrams of Example 3

**Example 4:**  $m = 2/5$ ,  $\sigma = 2/5$  and  $\nu = 5$

The specified point  $(m, \sigma)$  is in the set  $\mathbf{B}$  as in Example 1. Under these specifications, we have  $\alpha < 0$  and  $\beta_1 < 0$  that are different from those in Example 1. As  $\alpha\beta_1 < 0$ ,

$$\arg a_1(i\omega) = \pi + \tan^{-1}\left(\frac{\omega}{\alpha}\right) \quad \text{and} \quad \arg a_2(i\omega) = 2\pi + \tan^{-1}\left(\frac{\omega}{\alpha}\right).$$

Substituting these expressions into (30) and (31) give

$$\tau_1^\pm(\omega, k) = \frac{1}{\omega} \left\{ \tan^{-1}\left(\frac{\omega}{\alpha}\right) + 2k\pi \pm \theta_1(\omega) \right\} \quad (40)$$

and

$$\tau_2^\mp(\omega, n) = \frac{1}{\omega} \left\{ \tan^{-1}\left(\frac{\omega}{\alpha}\right) + (2n+1)\pi \mp \theta_2(\omega) \right\}. \quad (41)$$

Figure 7(A) shows the stability switching curve and Figure 7(B) is the bifurcation diagram where  $\bar{\tau}_2 = 1.2$  and  $\tau_1$  increases from 0 to 8. Stability losses and

gains also repeatedly occur in this example.

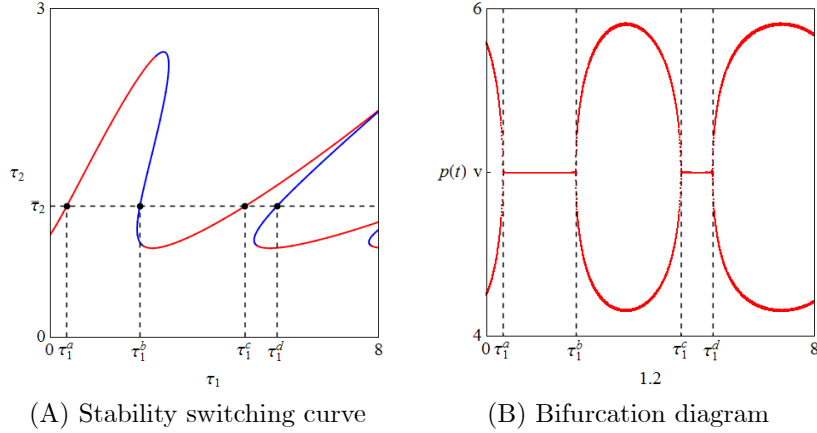


Figure 6. Illustration of Example 4

## 5 Concluding Remarks

We develop a heterogeneous agents model in which the chartists estimate the price trend with delayed information. It is analytically shown that the delay plane is divided into two regions by a stability switching curve and the equilibrium price is stable in the region including the origin of coordinates even if the delays are involved while it is unstable in the other region. Concerning global dynamics, it is numerically shown that fixing one delay at a positive length, the equilibrium price loses stability when the length of the other delay increases to arrive at the stability switching curve and it bifurcates to a limit cycle emerges via Hopf bifurcation when the delay is further increased. Stability regains never occur in the one delay model while stability losses and regains might repeatedly occur in the two delay model.

There is a number of directions for further research. Firstly, we should provide a theoretical analysis of the numerical examples in three-delay models conducted in Dibeh (2005) in which the convergent damped oscillations are given and the following conjecture is addressed: the longer is the time delay, the longer is the convergence time of the asset price to the fundamental value. Furthermore, the construction of the stability switching surface in the three delay space is an interesting extension. Secondly, since the model we deal with is an aggregative descriptive model, there is a need to construct a microeconomic underpinning. Third, based on the results obtained in the one-asset model, its extension to a multi-asset model is necessary as the key factor in financial theory is portfolio diversification.

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