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Abstract

This study constructs a Cournot duopoly model with production externality and shows that the Cournot reaction functions are unimodal. Focusing on the stable fixed point, it derives analytic particular as well as general solutions of the discrete-time nonlinear adjustment process of output. Since there are, in general, difficulties in finding analytical solutions in a discrete time system, it is worthwhile to preset the existence condition for analytical solutions.

JEL Classification: C72, D43

Key word: analytic solution, functional equation, nonlinear duopoly game. heterogenous agents

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1 Introduction

Seminal papers of Day [1982,83] shed light on nonlinearities existing in traditional economic models and indicate a possibility of chaotic fluctuations. Those focus on roles of nonlinearities played in dynamic process, nonlinearities which are necessarily involved in economic models and which the traditional economics rarely pays much attentions to. Since then, nonlinear analysis has received an increasing amount of concern. It has been demonstrated that any economic dynamic model can give rise complex dynamics involving chaos if it has strong nonlinearities. Main results obtained so far are summarized in Day [1994], Lorenze [1993], Majumder, Mitra and Nishimura [2000], and Rosser [2000], to name only a few.

In the literature of nonlinear economic dynamics, it is a usual procedure to perform numerical simulations to detect dynamic characteristics of such a nonlinear system since common analytical properties are limited. Computer simulations visualize how a dynamics process evolves over time even if analytical treatment is insufficient and thus are useful and helpful for nonlinear analysis. However, in spite of rapid development of speed and accuracy of computer, it is still possible that simulation results do not approximate a true solution of discrete-time system. According to Lorenze [1993], the deviation between the numerical simulations and analytical solutions is mainly due to two storage properties of computers; truncating a rational number and the binary representation of a number.[1] One way to deal with this immanent features of computers is to find an analytical solution of discrete time system. In this study, we construct an nonlinear dynamical system of Cournot duopoly and aim at finding its general analytical solution.

The paper is organized as follows. Section 2 constructs a nonlinear duopoly model with production externality and shows its stability condition. Section 3 consider the existence of particular as well as general analytic solutions and demonstrates explicit forms of those solutions. Section 4 is for concluding remarks.

2 Nonlinear Duopoly Model

In this section, we construct a discrete-time Cournot model to consider the existence of a fixed point and its stability properties. An inverse demand function is assumed to be linear and decreasing,

$$p = a - bQ, \quad a > 0 \text{ and } b > 0,$$

[1] See Appendix 4 of Lorenze [1993] for more detailed discussion.

where Q is the industry output, provided demand equals supply. To simplify a dynamic process, only two firms are considered in this study. Those duopolists, denoted by X and Y, produce homogenous output x and y so that the total supply is made up of supplies of two duopolists, $x + y = Q$. Each firm is assumed to have **production externality** in a sense that the choice of any one firm affects the production possibility of the other firm. Although externalities come in many variates, we confine our analysis to the case in which the externality affects the cost of production. In particular, we deal with the simplest presence of externality in which the production cost of a firm is linear with respect to its own production and nonlinear with respect to its rival's production. Namely, the cost functions of firm X and Y are given by

$$C^x(x, y^{ex}) = c_x(y^{ex})x \text{ and } C^y(y, x) = c_y(x^{ex})y.$$

where x^{ex} and y^{ex} are output expectations. This specification implies that the marginal cost of production of one firm depends on the amount of output produced by the other firm. We say it has a **positive externality** if the marginal cost is decreasing for an increment of the rival's output and a **negative externality** if it is increasing. Solving the profit maximization problems yields reaction functions of duopolists; for firm X

$$r^x(y^{ex}) = \arg \underset{x}{Max} \Pi^x(x, y^{ex}),$$

and for firm Y,

$$r^y(x^{ex}) = \arg \underset{y}{Max} \Pi^y(x^{ex}, y),$$

where the profit functions are defined by

$$\Pi^x(x, y^{ex}) = (a - b(x + y^{ex}))x - c_x(y^{ex})x,$$

and

$$\Pi^y(y, x^{ex}) = (a - b(x^{ex} + y))x - c_y(x^{ex})y.$$

Due to the presence of production externality, the reaction function can be up- or downward-sloping according to the external effect is positive or negative. A profile of the reaction function depends on a specification of the cost function. In this study, following Kopel [1996], we adopt the convenient form of the marginal cost functions,

$$c_x(y^{ex}, \alpha) = (a - b(1 + 2\alpha)y^{ex} + 2b\alpha(y^{ex})^2) \text{ and } c_y(x^{ex}, \beta) = c_x(x^{ex}, \beta).$$

that make the reactions functions to be identical with the logistic map. If duopolists have naive expectations, $x_t^{ex} = x_{t-1}$ and $y_t^{ex} = y_{t-1}$, the adjustment process of output is described by the double logistic system,

$$DS_1 : \begin{cases} x_{t+1} = \alpha y_t(1 - y_t), \\ \\ y_{t+1} = \beta x_t(1 - x_t) \end{cases}$$

where α and β are positive adjustment coefficients. Apparently, the system possesses only the trivial solution when those coefficients are less than unity and possibly generates negative solutions when either or both of those coefficients is greater than four. To eliminate those uninteresting cases, we assume

Assumption 1: $1 < \alpha < 4$ and $1 < \beta < 4$.

We consider the existence of the non-trivial fixed point and its stability properties. An intersection of those two reaction curves determines a fixed point (i.e., Cournot-Nash equilibrium). Since the curves are unimodal, multiple fixed points are possible. We thus clarify the parametric condition under which DS_1 possesses multiple fixed points.

Substituting the second equation of DS_1 into the first and equating it to x yields, after transposition and factorization, the following fourth-order equation where the time subscript is omitted for a while only for the notational simplicity,

$$-x(1 - \alpha\beta + \alpha\beta x + \alpha\beta^2 x - 2\alpha\beta^2 x^2 + \alpha\beta^2 x^3) = 0.$$

Apparently $x = 0$ is a trivial fixed point. To find another fixed point, we define

$$f_{\alpha,\beta}(x) = 1 - \alpha\beta + \alpha\beta x + \alpha\beta^2 x - 2\alpha\beta^2 x^2 + \alpha\beta^2 x^3.$$

This is a cubic polynomial and has either one real root and a conjugate pair of complex roots or three real roots according to combinations of α and β. As illustrated in Figure 1, $f_{\alpha,\beta}(x)$ with $\alpha = 2$ and $\beta = 3$ is depicted by the monotonically increasing dotted line and intersects the horizontal axis only once while $f_{\alpha,\beta}(x)$ with $\alpha = 3.5$ and $\beta = 3.8$ is depicted by the concave-convex real line and intersects the horizontal line three times.

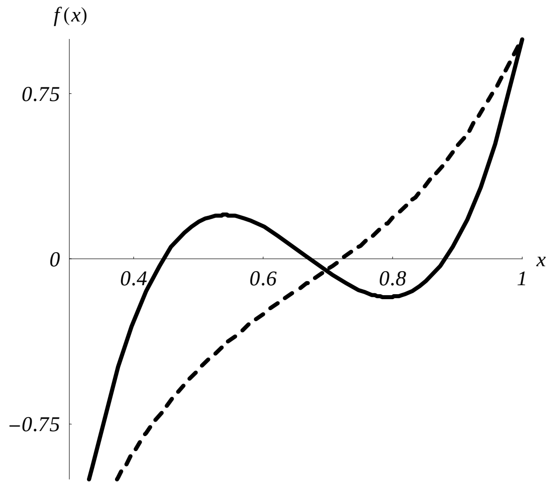

Figure 1. Two Profiles of $f(x)$

We find critical values of α and β that distinguish one case from the other. $f_{\alpha,\beta}(x)$ is defined on the unit interval and, for two end points, it has

$$f_{\alpha,\beta}(x) = 1 - \alpha\beta < 0 \text{ for } x = 0, \text{ and } f_{\alpha,\beta}(x) = 1 \text{ for } x = 1,$$

where the direction of inequality is due to Assumption 1. To see how the curvature of $f_{\alpha,\beta}(x)$ changes as x increases from zero to unity, we differentiate $f_{\alpha,\beta}(x)$ with respect to x to obtain its derivative which is, after arranging terms, given by

$$f'_{\alpha,\beta}(x) = \alpha\beta^2\{\frac{1}{\beta} + (3x-1)(x-1)\}.$$

$(3x-1)(x-1)$ has a minimum $-\frac{1}{3}$ at $x = \frac{2}{3}$. Thus, if $\beta < 3$ holds, then it must be the case that $f'_{\alpha,\beta}(x) > 0$ for $0 \leq x \leq 1$ regardless of a value of α. $f_{\alpha,\beta}(x)$ is monotonically increasing and crosses the horizontal line (i.e., the $x = 0$ locus) only once. Accordingly, the adjustment process DS_1 has a unique fixed point other than the trivial solution (i.e., $x = y = 0$) for $\beta < 3$.

$f'_{\alpha,\beta}(x) = 0$ has identical roots for $\beta = 3$ and two distinct roots in the unit interval for $\beta > 3$,

$$x_1 = \frac{2\sqrt{\beta} - \sqrt{\beta - 3}}{3\sqrt{\beta}} \text{ and } x_2 = \frac{2\sqrt{\beta} + \sqrt{\beta - 3}}{3\sqrt{\beta}}.$$

This implies that $f_{\alpha,\beta}(x)$ has a concave-convex-shaped profile and takes a local maximum at x_1 and a local minimum at x_2. Therefore $f_{\alpha,\beta}(x) = 0$ has three distinct roots if the maximum is positive and the minimum is negative. We then solve $f_{\alpha,\beta}(x_{\mathsf{max}}) = 0$ and $f_{\alpha,\beta}(x_{\mathsf{min}}) = 0$ for β to find two critical lines,

$$\begin{aligned} \alpha_{\mathsf{max}}(\beta) &= \frac{27}{6\sqrt{\beta-3}\sqrt{\beta}+9\beta-2\sqrt{\beta-3}\sqrt{\beta^3}-2\beta^2}, \\ \alpha_{\mathsf{min}}(\beta) &= \frac{27}{-6\sqrt{\beta-3}\sqrt{\beta}+9\beta+2\sqrt{\beta-3}\sqrt{\beta^3}-2\beta^2}, \end{aligned}$$

both which are defined for $\beta > 3$. Furthermore, $\alpha_{\mathsf{max}}(\beta) > \alpha_{\mathsf{min}}(\beta)$ and $\alpha'_{\mathsf{max}}(\beta) > 0$ and $a'_{\mathsf{min}}(\beta) > 0$ for $\beta > 3$ so that those lines divide a set of parameters where $3 < \alpha < 4$ and $3 < \beta < 4$ into three subregions. It can be checked that

$$f_{\alpha,\beta}(x_{\mathsf{max}}) > 0 \text{ for } (\alpha,\beta) \text{ such that } \alpha < \alpha_{\mathsf{max}}(\beta)$$

$$f_{\alpha,\beta}(x_{\mathsf{min}}) < 0 \text{ for } (\alpha,\beta) \text{ such that } \alpha > \alpha_{\mathsf{min}}(\beta).$$

Then we can define the subregion surrounded by those two lines and two straight lines, $\alpha = 4$ and $\beta = 4$

$$S = \{(\alpha,\beta) \mid 3 < \alpha < 4,\ 3 < \beta < 4,\ f_{\alpha,\beta}(x_{\mathsf{max}}) > 0 \text{ and } f_{\alpha,\beta}(x_{\mathsf{min}}) < 0\}.$$

$f_{\alpha,\beta}(x)$ has three distinct roots for $(\alpha,\beta) \in S$ and accordingly, DS_1 has three distinct fixed points.

Returning to DS_1, substituting the first equation into the second equation, and then following the same procedure to get $f_{\alpha,\beta}(x)$, we have the cubic polynomial for y which can be expressed by $f_{\beta,\alpha}(y)$ as it is symmetric with $f_{\alpha,\beta}(x)$ in parameters and output. Therefore the same argument used for distinguishing the number of roots $f_{\alpha,\beta}(x)$ possesses implies that $f_{\beta,\alpha}(y)$ has one real root if $\alpha < 3$ regardless of values of β, and a set of parameters that produces $f_{\beta,\alpha}(y_{\mathsf{max}}) > 0$ and $f_{\beta,\alpha}(y_{\mathsf{min}}) < 0$ is identical with S. To summarize, we have

Theorem 1 Under Assumption 1, the dynamical system DS_1 has three distinct fixed points if $(\alpha,\beta) \in S$ and one fixed point otherwise.

To examine the stability of a fixed point, we linearized DS_1 at the fixed point to have the Jacobi matrix,

$$J = \begin{pmatrix} 0 & \alpha(1-2y^e) \\ \beta(1-2x^e) & 0 \end{pmatrix}$$

where (x^e, y^e) is the fixed point. The eigen values λ_1 and λ_2 satisfy

$$\begin{cases} \lambda_1 + \lambda_2 = 0, \\ \lambda_1 \lambda_2 = -\alpha\beta(1-2x^e)(1-2y^e). \end{cases}$$

Set $\lambda = \lambda_1 = -\lambda_2 > 0$.The fixed point is stable if $\lambda^2 = \alpha\beta(1-2x^e)(1-2y^e)$ is less than unity. The stability apparently depends on values of x^e and y^e. Suppose that $\alpha < 3$ and $\beta < 3$. Then DS_1 has one fixed point according to Theorem 1. Since $f_{a,\beta}(x)$ as well as $f_{\beta,\alpha}(y)$ is a cubic polynomial, it is possible to obtain an explicit expression of the fixed point which turn out to be,[2]

$$\begin{cases} x^e = \dfrac{2}{3} + \dfrac{\sqrt[3]{2}(\alpha-3)\beta}{3\sqrt[3]{\Delta(\alpha,\beta)}} + \dfrac{\sqrt[3]{\Delta(\alpha,\beta)}}{3\sqrt[3]{2}\alpha\beta}, \\ y^e = \dfrac{2}{3} + \dfrac{\sqrt[3]{2}(\beta-3)\alpha}{3\sqrt[3]{\Delta(\beta,\alpha)}} + \dfrac{\sqrt[3]{\Delta(\beta,\alpha)}}{3\sqrt[3]{2}\alpha\beta}, \end{cases}$$

where

$$\Delta(\alpha,\beta) = \alpha\beta^2\{-27+9\alpha\beta\} - 2\alpha^2\beta + 3\sqrt{3}\sqrt{27-\alpha\beta(18-4(\alpha+\beta)+\alpha\beta)}.$$

Although both of x^e and y^e have complicated expressions, it is logically possible to derive a locus of (α, β) describing all parametric combinations that yield $\lambda^2 = 1$. However, instead of pursuing the logical possibility, we depict a 3D illustration of $\alpha\beta(1-2x^e)(1-2y^e)$ to confirm that the eigenvalues are less than unity in absolute value for $\alpha < 3$ and $\beta < 3$.

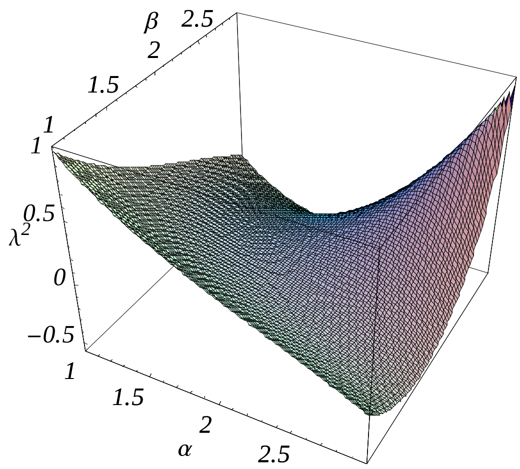

Figure 2. λ^2 surface

[2]Calculations are done with Mathematica, ver.4.1.

3 Analytical Solution

3.1 Transformation of Variable

We are ready to find an analytical solution of DS_1. As a starting point for a more complete study, we confine our analysis to a case in which the fixed point is stable. The stable fixed point provides a convenient heuristic setting to detect analytical properties of the Cournot output adjustment; it makes the formidable mathematical problem simpler and manageable; further, it enable us to derive rigorous results. We detect an analytic **particular** solution of the dynamical system, DS_1, in the first half of this section and then find an analytic **general** solution in the latter half. In Figure 2, we graphically confirm the stability of DS_1 at least for $1 < \alpha < 3$, and $1 < \beta < 3$. So it is safe to make

Assumption 2. $-1 < -\alpha\beta(1-2x^e)(1-2y^e) < 0$.

Assumption 2 leads to $|\lambda_1| = |\lambda_2| < 1$. For a technical reason, we further assume time t to be a complex variable in the sequel, although it is usually used as a real number. To emphasize this alternation, we change the notation of variable dependency on time from x_t to $x(t)$ hereafter.

We transform the dynamical system DS_1 to the equilibrium point by changing variables by $u(t) = x(t) - x^e$ and $v(t) = y(t) - y^e$,

$$DS_2 : \begin{cases} u(t+1) = \alpha\{v(t)(1-v(t)) + 2v(t)y^e\}, \\ \\ v(t+1) = \beta\{u(t)(1-u(t)) + 2u(t)x^e\}. \end{cases}$$

For the sake of simplicity, we further transform this simultaneous system to a second-order difference equation of u. Let the first equation of DS_2 shift in one period ahead, and then substituting the second equation into the shifted first equation yields

$$DS_3 : \begin{cases} u(t+2) = \alpha(1-2y^e)\{\beta(1-2x^e)u(t) - \beta u(t)^2\} - \alpha\beta^2\{(1-2x^e)u(t) - u(t)^2\}^2, \\ \\ v(t) := \Phi(u(t+1)) = \dfrac{1-2y^e}{2}\{1 - \sqrt{1 - \dfrac{4\alpha u(t+1)}{\alpha(1-2y^e)^2}}\}. \end{cases}$$

For the notational simplicity in the latter analysis, we further change variables by setting $s(t) = u(t)$ and $w(t) = u(t+1)$ and denote the resultant expressions as F and G,

$$DS_4 : \begin{cases} s(t+1) = F(s(t), w(t)), \\ \\ w(t+1) = G(s(t), w(t)), \end{cases}$$

where

$$\begin{cases} F(s(t), w(t)) = w(t), \\ G(s(t), w(t)) = \alpha(1-2y^e)\{\beta(1-2x^e)s(t) - \beta s(t)^2\} - \alpha\beta^2\{(1-2x^e)s(t) - s(t)^2\}^2. \end{cases}$$

Any of those dynamics systems are conjugate to any other so that all generate qualitatively the same dynamics.

3.2 Existence of Analytic Solution

The first equation of DS_3 has a formal solution which is an infinite series of the form,

$$u(t) = \sum_{n=1}^{\infty} a_n \lambda^{nt}. \tag{1}$$

We, first, prove the existence of an analytic solution of DS_3 that has the expansion (1) in a half plane

$$\mid \lambda^t \mid = \mid e^{t \log \lambda} \mid \leq \eta, \ \ \log \lambda \cdot \mathrm{Re}[t] \leq \log \eta,$$

with constant $\eta > 0$, and then find an analytic general solution of DS_1. To get formal solution (1), we substitute (1) into the first equation of DS_3 and compare the coefficients of λ^{nt} on its right hand side and the ones of the left hand side to find

$$\begin{cases} a_1 D(\lambda) = 0, \\ a_k D(\lambda^{2k}) = C_k(a_1, a_2, ..., a_{k-1}) \text{ for } k \geq 2, \end{cases}$$

where $D(\lambda) = \lambda^2 - \alpha\beta(1-2y^e)\{(1-2x^e)$ and C_k polynomials of $a_1, a_2, ..., a_{k-1}$ are determined successively in calculations.[3] By the definition of the characteristic equation, $D(\lambda)$ is zero, and thus α_1 is arbitrary but not zero. Although (1) is a formal solution of DS_3, we can show that it is convergent and thus can be an exact solution of DS_3.

In order to simplify the notations, we set$s = u(t)$ and $z = u(t+2)$ in DS_3. Transposing $u(t+2)$ in the right hand side of the first equation of DS_3 to the right hand side and denote the resultant expression as an implicit form of s and z,

$$H(s,z) = -z + \alpha\beta(1-2y^e)\{(1-2x^e)s - s^2\} - \alpha\beta^2\{(1-2x^e)^2 s^2 - 2(1-2x^e)s^3 + s^4\}.$$

Since $H(s,z)$ is holomorphic in a neighborhood of $z = 0$, we get the following result,

[3] An illustraive example for determinations of coefficients and constructions of C_k are given in Appendix A.

Lemma 2 There exists a constant $\rho > 0$ and a holomorphic function ϕ in a neighborhood of $z = 0$ such that $\phi(0) = 0$ and

$$H(\phi(z), z) = 0 \text{ for } \mid z \mid \leq \rho.$$

Proof. Since $H(0,0) = 0$, and $\frac{\partial H(0,0)}{\partial s} = -\lambda_1\lambda_2 > 0$, the implicit function theorem confirms that a holomorphic function ϕ exists in a neighborhood of $z = 0$ such that $\phi(0) = 0$ and $H(\phi(z), z) = 0$. Furthermore, we have a constant K such that $\mid s \mid = \mid \phi(z) \mid \leq K \mid z \mid \text{for} \mid z \mid \leq \rho$. ■

Our main purpose is to show the existence of an analytic solution of $u(t)$ such that $u(t) = \phi(u(t+2))$. Since we have already defined a formal solution of $u(t)$ in (1), it suffices for our purpose to prove the convergence of the infinite series, (1).

Given a positive integer N, we can define a partial sum of the formal solution by $P_N(t) = \sum_{n=1}^{N} \alpha_n \lambda^{nt}$. If a convergent analytic solution $u(t)$ of (DS_3) that has the expression (1) exists, then we are able to derive a convergent infinite series, $p(t) = u(t) - P_N(t) = O(\mid \lambda^t \mid^{N+1})$. Conversely if there exists a function $p_N(t)$ such that $p_N(t) + P_N(t) = \phi(p_N(t+2) + P_N(t+2))$ and $p_N(t) = O(\mid \lambda^t \mid^{N+1})$ for $\mid \lambda^t \mid \leq \eta_N$ with some $\eta_N > 0$, then we can define a convergent series as a sum of $p(t)$ and $P_N(t)$, $u(t) = p(t) + P_N(t)$, which can be an exact solution of DS_3. We will show the existence of $u(t)$ with three steps: we first show the existence of p_N for $\mid \lambda^t \mid \leq \eta_N$ with some $\eta_N > 0$ in Lemma 3; then we prove the uniqueness of p_N in Lemma 4; finally we demonstrate that the infinite series $p_N(t) + P_N(t)$ is independent from a choice of N in Lemma 5. Before proceeding, we rewrite $p_N(t) = p(t)$, $\eta_N = \eta$ and

$$\begin{aligned} p(t) &= u(t) - P_N(t), \\ &= \{\phi(p(t+2) + P_N(t+2)) - \phi(P_N(t+2))\} + \{\phi(P_N(t+2) - P_N(t)\}, \\ &= g_1(t, p(t+2)) + g_2(t), \end{aligned}$$

where

$$g_1(t, p(t+2)) = \phi(p(t+2) + P_N(t+2)) - \phi(P_N(t+2)),$$

and

$$g_2(t) = \phi(P_N(t+2)) - P_N(t).$$

Now we define

$$S(\eta) = \{t \in C : \mid \lambda^t \mid \leq \eta\}.$$

Taking $A > 0$ and $0 < \eta < 1$ which is determined later, we also define

$$J(A,\eta) = \{p : p(t) \text{ is holomorphic and } \mid p(t) \mid \leq A \mid \lambda^t \mid^{N+1} \text{ for } t \in S(\eta)\}.$$

For $p(t) \in J(A,\eta)$, we further define map T such that

$$T[p](t) = g_3(t, p(t+2)) = g_1(t,\ p(t+2)) + g_2(t).$$

Lemma 3 Given N and $\rho > 0$, there exists a fixed point $p(t) = p_N(t) \in J(A,\eta)$ of map T for $t \in S(\eta)$.

Proof. Since ϕ is holomorphic on $\mid z \mid \leq \rho$, we have

$$\frac{d}{dz}\phi(z) = \frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{\phi(\xi)}{(\xi - z)^2} d\xi.$$

When $\mid z \mid \leq \frac{\rho}{2}$, we have $\mid \xi - z \mid \geq \mid \xi \mid - \mid z \mid \geq \rho - \frac{\rho}{2} = \frac{\rho}{2}$ and hence

$$\mid \frac{d}{dz}\phi(z) \mid \leq \frac{1}{\pi}\int_{|\xi|=\rho} \frac{\mid \phi(\xi) \mid}{(\frac{\rho}{2})^2} d\xi \leq \frac{1}{\pi}\int_{|\xi|=\rho} \frac{K}{(\frac{\rho}{2})^2} d\xi = \frac{8K}{\rho}. \tag{2}$$

Next we choose A and η such that $A\eta^{N+1} < \frac{\rho}{4}$. Then for sufficiently large t, we have

$$\mid p(t) \mid \leq A \mid \lambda^t \mid^{N+1} \leq A\eta^{N+1} < \frac{\rho}{4}.$$

The inequality still holds even for $t+2$,

$$\mid p(t+2) \mid \leq A \mid \lambda^{t+2} \mid^{N+1} = A \mid \lambda \mid^{2(N+1)} \mid \lambda^t \mid^{N+1} < \frac{\rho}{4}. \tag{3}$$

Consequently, for t large enough, $\mid P_N(t+2) \mid < \frac{\rho}{4}$ and then

$$\mid z \mid = \mid p(t+2) + P_N(t+2) \mid \leq \frac{\rho}{4} + \frac{\rho}{4} = \frac{\rho}{2}. \tag{4}$$

Since

$$\begin{aligned} g_1(t, p(t+2)) &= \int_0^1 \frac{d}{dr}\phi(rp(t+2) + P_N(t+2))dr \\ &= \int_0^1 p(t+2)\frac{d}{dr}\phi(rp(t+2) + P_N(t+2))dr, \end{aligned}$$

(2), (3) and (4) imply

$$\begin{aligned} \mid g_1(t, p(t+2)) \mid &\leq \textstyle\int_0^1 \mid p(t+2) \mid\mid \frac{d}{dr}\phi(rp(t+2) + P_N(t+2)) \mid dr, \\ &\leq \textstyle\int_0^1 A \mid \lambda^t \mid^{N+1} \mid \lambda \mid^{2(N+1)} \frac{8K}{\rho} dr, \\ &\leq \frac{8K}{\rho} A \mid \lambda \mid^{N+1} \mid \lambda^t \mid^{N+1}. \end{aligned} \tag{5}$$

From the definition of g_2, we have

$$| g_2(t) | \leq K_2 | \lambda^t |^{N+1} \tag{6}$$

where K_2 is a constant but its magnitude depends on N. Hence using (5) and (6), we have

$$| T[p](t) | \leq | g_1(t, p(t+1), p(t+2)) | + | g_2(t) | \leq (\frac{8K}{\rho} A | \lambda |^{N+1} + K_2) | \lambda^t |^{N+1} .$$

If we suppose N is so large that

$$\frac{8K}{\rho} A | \lambda |^{N+1} < \frac{1}{2},$$

then we have

$$| T[p](t) | \leq (\frac{1}{2} A + K_2) | \lambda^t |^{N+1} .$$

Furthermore, if we take A to be so large that

$$A > 2K_2,$$

then

$$| T[p](t) | \leq A | \lambda^t |^{N+1} .$$

Hence we find that T maps $J(A, \eta)$ into itself. The map T is obviously continuous if$J(A, \eta)$ is endowed with topology of uniform convergence on compact in $S(\eta)$. $J(A, \eta)$ is clearly convex and is relatively compact due to the theorem of Montel[??]. Since requirements of Schauder's fixed point theorem are all satisfied, we can show the existence of a fixed point $p(t) = p_N(t) \in J(A, \eta)$ of T which depends on N. This proves Lemma 3. ■

We turn to the uniqueness of the fixed point.

Lemma 4 The fixed point $p_N(t) \in J(A, \eta)$ of T is unique.

Proof. Suppose that there exists another fixed point, $p^*(t) = p_N^*(t) \in J(A^*, \eta^*)$. Put

$$\begin{aligned} A_0 &= \max(A, A^*),\ \eta_0 \leq \min(\eta, \eta^*), \\ u(t) &= p_N(t) + P_N(t),\ u^*(t) = p_N^*(t) + P_N(t) \end{aligned}$$

and

$$q(t) = p_N^*(t) - p_N(t).$$

Since the backward mapping $u^*(t) = \phi(u^*(t+2))u(t)$ and $u(t) = \phi(u(t+2))$ hold, we have

$$\begin{aligned} q(t) &= \{\phi(p_N^*(t+2) + P_N(t+2)) - \phi(P_N(t+2)) - P_N(t)\} \\ &\qquad -\{\phi(p_N(t+2) + P_N(t+2)) - \phi(P_N(t+2)) - P_N(t)\} \\ &= \phi(q(t+2) + u_N(t+2)) - \phi(u_N(t+2)) \\ &= \int_0^1 q(t+2)\frac{d}{dz}\phi(rq(t+2) + u_N(t+2))dr. \end{aligned}$$

If η_0 is sufficiently small, then

$$\mid \frac{d}{dz}\phi(rq(t+2) + u_N(t+2)) \mid< \frac{8K}{\eta}, \text{ and } \mid \lambda \mid^{N+1}< \frac{\eta}{32K}.$$

Consequently we have

$$\begin{aligned} \mid q(t) \mid \ &\le \textstyle\int_0^1 \frac{8K}{\rho} \mid q(t+2) \mid dr \\ &\le \textstyle\int_0^1 \frac{8K}{\rho} \mid \lambda \mid^{N+1} (2A_0 \mid \lambda^t \mid^{N+1})dr \\ &= \frac{16K}{\rho} A_0 \mid \lambda \mid^{N+1} \mid \lambda^t \mid^{N+1} \\ &< \frac{1}{2} A_0 \mid \lambda^t \mid^{N+1}, \end{aligned}$$

Hence

$$\mid q(t) \mid = \mid p_N^*(t) - p_N(t) \mid \le \frac{1}{2} A_0 \mid \lambda^t \mid^{N+1} = (\frac{1}{4})2A_0 \mid \lambda^t \mid^{N+1}.$$

Repeating this procedure k times, we obtain

$$\mid p_N^*(t) - p_N(t) \mid< (\frac{1}{4})^k (2A_0) \mid \lambda^t \mid^{N+1}, k = 1, 2, ...$$

where the inequality holds for any k. Letting $k \to \infty$, we have

$$p_N^*(t) = p_N(t) \text{ for } t \in S(\eta_0).$$

Since this implies that, $p_N^*(t) = p(t)$ and $p_N(t) = p(t)$ are holomorphic in $\mid \lambda^t \mid \le \min(\eta, \eta^*)$, . we concludes $p^*(t) \equiv p(t)$. ■

In Lemmas 3 and 4, the solution $u_N(t) = p_N(t) + P_N(t)$ has a subscript N and thus seems to sensitive to the value of N. Lemma 5 show that this is not the case.

Lemma 5 The solution $u_N(t) = p_N(t) + P_N(t)$ is independent of N.

Proof. Let $p_N(t) \in J(A_N, \eta_N)$ and $p_{N+1}(t) \in J(A_{N+1}, \eta_{N+1})$ be fixed points of T, and

$$\begin{aligned} u_{N+1}(t) &= p_{N+1}(t) + P_{N+1}(t) \\ &= p_{N+1}(t) + a_{N+1}\lambda^{(N+1)t} + P_N(t) \\ &= \tilde{p}_N(t) + P_N(t) \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{p}_N(t) &= \mid p_{N+1}(t) + a_{N+1}\lambda^{(N+1)t} \mid \\ &\leq A_{N+1}\lambda^{N+2} \mid \lambda^t \mid^{N+2} + \mid a_{N+1} \mid\mid \lambda^t \mid^{N+1} \\ &= (A_{N+1} \mid \lambda \mid + \mid a_{N+1} \mid) \mid \lambda^t \mid^{N+1} \\ &= A_N^* \mid \lambda^t \mid^{N+1} . \end{aligned}$$

By the uniqueness of the fixed point, $\tilde{p}_N(t) = p_N(t)$ for $t \in S(\eta_N) \cap S(\eta_{N+1})$. Thus

$$u_{N+1}(t) = u_N(t) \text{ for } t \in S(\eta_N) \cap S(\eta_{N+1}).$$

By analytic prolongation, both $u_N(t)$ and $u_{N+1}(t)$ are holomorphic in $S(\eta_N) \cup S(\eta_{N+1})$ and coincide. This completes the proof of Lemma 5. ■

Lemmas 3, 4 and 5 have demonstrated that a solution $u(t)$ of DS_3 is defined and holomorphic in $S(\eta)$ for $\eta > 0$, which has the expansion $u(t) = \sum_{n=1}^{\infty} a_n\lambda^{nt}$. To summarize those results, we have

Theorem 6 Under Assumption 2, .we have holomorphic solution of $u(t)$ of DS_3 in $S(\eta)$ for $\eta > 0$ that has the expansion $u(t) = \sum_{n=1}^{\infty} a_n\lambda^{nt}$.

However, we cannot assure the following condition

$$\frac{\partial H(s,z)}{\partial s} \neq 0 \text{ for all } z.$$

This indicates that a uniqueness of ϕ such that $s = \phi(z)$ is not confirmed globally. Hence when we have $s = \phi(z)$ from $w = u(t+1)$ and $z = u(t+2)$, there may be some branch points.

The analytic solution obtained in Theorem 6 is thought as a "particular solution" of the difference equation. A general analytic solutions of DS_3 is also derived..

Theorem 7 Assume the condition (D), and $\lambda = \lambda_1 = -\lambda_2 > 0$ where λ_i $(i = 1, 2)$ are solutions of $D(\lambda) = 0$. Suppose that u_τ be the analytic solution of DS_3 which has the expansion $u_t = \sum_{n=1}^{\infty} a_n\lambda^{nt}$. Furthermore, suppose that

χ_t is an analytical solution of DS_3 such that $\chi_{t+n} \to 0$ as $n \to +\infty$ uniformly on any compact set. Then there is a periodic entire function π_t such that

$$\chi_t = \sum_{n=1}^{\infty} a_n \lambda^{n(\frac{\log \pi(t)}{\log \lambda}+t)}$$

where $\pi(t)$ is an arbitrarily periodic function with period one.

Conversely if we put

$$\chi_t = \sum_{n=1}^{\infty} \alpha_n \lambda^{n(\frac{\log \pi(t)}{\log \lambda}+t)}$$

where π_t is a periodic function with period one, then χ_t is a solution of DS_3.

Proof. See Appendix B. ∎

Now we have a general solution $v_t = \Phi(\chi_t)$ of the second equation of DS_2. Therefore we can obtain stable analytic general solution (x_t, y_t) of DS_1 by

$$x_t = \chi_t + x^e, \text{ and } y_t = \Phi(\chi_t) + y^e.$$

4 Concluding Remarks

We have demonstrated that if the absolute values of eigen values are less than unity, it is possible to find particular as well as general analytic solutions of nonlinear Cournot Duopoly model. The principle and techniques utilized in this study can be applied to another discrete dynamic system.

In the theory of discrete dynamics, we have no existence theorem that causes difficulties in finding an analytical solution. Thus it is worthwhile to present the existence condition for analytical solution and a closed form solution. Our result applies to rigorously analyze and predict dynamic phenomenon observed in not only economics but also other areal such as ecology, biology so and so forth.

Appendix A

In this appendix we outline the procedure to determine values of coefficients a_n.

DS_3 is spelled out as

$$\begin{aligned} u(t+2) &= \alpha\beta(1-2x^e)(1-2y^e)u(t) - \alpha\beta\{(1-2y^e)+\beta(1-2x^e)^2\}u(t)^2 \\ &\quad +2\alpha\beta^2(1-2x^e)u(t)^3 - \alpha\beta^2 u(t)^4. \end{aligned}$$

By the definition of the formal solution, we $u(t)=\sum_{n=1}^{\infty} a_n\lambda^{nt}$ and $u(t+2)=\sum_{n=1}^{\infty} a_n\lambda^{n(t+2)}$, which are substituted into the above equation,

$$\begin{aligned} &a_1\lambda^2\lambda^t + a_2\lambda^4\lambda^{2t} + a_3\lambda^6\lambda^{3t} + \cdots \\ = \; &\alpha\beta^2(1-2x^e)(1-2y^e)\{a_1\lambda^t + a_2\lambda^{2t} + a_3\lambda^{3t} + \cdots\} \\ &-\alpha\beta\{(1-2y^e)+\beta(1-2x^e)^2\}\{a_1\lambda^t + a_2\lambda^{2t} + a_3\lambda^{3t} + \cdots\}^2 \\ &+2\alpha\beta^2(1-2x^e)\{a_1\lambda^t + a_2\lambda^{2t} + a_3\lambda^{3t} + \cdots\}^3 \\ &-\alpha\beta^2\{a_1\lambda^t + a_2\lambda^{2t} + a_3\lambda^{3t} + \cdots\}^4. \end{aligned}$$

We compare the coefficients of λ^{nt} in the left hand side of the above equation with the ones in the right hand side. For the coefficient of λ^t,we have

$$a_1\{\lambda^2 - \alpha\beta(1-2x^e)(1-2y^e)\} = a_1 D(\lambda)$$

where $D(\lambda) = \lambda^2 - \alpha\beta(1-2x^e)(1-2y^e)$ is the characteristics polynomial of DS_3. Since $D(\lambda)=0$ identically, a_1 can be arbitrary. For the coefficients of λ^{2t}, we have

$$a_2 D(\lambda^2) = -\alpha\{\beta(1-2y^e)+\alpha^2(1-2x^e)^2\}a_1^2.$$

Since $\lambda^2 \neq \lambda_1, \lambda_2$, we have $D(\lambda^k)\neq 0$ $(k\geq 2)$. So we can determine a_2 by dividing the both side by $D(\lambda^2)$,

$$a_2 = \frac{C_2(\alpha_1)}{D(\lambda^2)}$$

where $C_2(\alpha_1) = -\alpha\{\beta(1-2y^e)+\alpha^2(1-2x^e)^2\}a_1^2$. By the same token, for the coefficient λ^{3t}, we have

$$a_3 D(\lambda^3) = -2\alpha\{(\beta(1-2y^e)+\alpha^2(1-2x^e)^2)a_1a_2 - \alpha\beta(1-2x^e)a_1^3\}$$

and then

$$a_3 = \frac{C_3(a_1,a_2)}{D(\lambda^3)},$$

where $C_3(a_1, a_2) = -2\alpha\{(\beta(1-2y^e) + \alpha^2(1-2x^e)^2)a_1a_2 - \alpha\beta(1-2x^e)a_1^3\}$. Similarly, for the coefficient of $\lambda^{nt}(n \geq 4)$, α_n is sequentially determined by $a_1, a_2, ...a_{n-1}$ by

$$a_n = \frac{C_n(a_1, a_2, ...a_{n-1})}{D(\lambda^n)},$$

where $C_n(a_1, a_2, ...a_{n-1})$ is defined accordingly.

Appendix B

In this appendix we give formal proof of Theorem 1.

Let $u(\tau)$ be a particular solution of DS_3, and $\chi(t)$ be a solution of DS_3 such that $\chi(t+n) \to 0$ as $n \to \infty$ uniformly on any compact set. We define

$$U(\lambda^t) = \sum_{n=1}^{\infty} a_n \lambda^t, \quad a_1 \neq 0.$$

It can be seen that U as well as χ is an open set. Since $U(0) = \chi(0) = 0$, for any $\eta_1 > 0$, there is some constant $\eta_2 > 0$ such that

$$U(\mid \tau \mid < \eta_1) \supset \{\mid \chi \mid < \eta_2\}.$$

So there is a large R such that if $\mid t' + n \mid > R$, then $\mid \chi(t'+n) \mid > \eta_2$. Hence there is a $\tau = \lambda^\sigma$ such that

$$\chi(t'+n) = U'(\tau) = U(\lambda^\sigma).$$

Since $\alpha_1 \neq 0$, we have a U^{-1}, with a help of the implicit function theorem, such that

$$\lambda^\sigma = U^{-1}(\chi(t'+n)).$$

Put $t = t' + n$, then $\lambda^\sigma = U^{-1}(\chi(t))$, and taking logarithm its both sides yields

$$\sigma = l(t) = \log_\lambda U^{-1}(\chi(t)).$$

When DS_3 has the solution $\chi(t)$, we can prove, according to Suzuki [1999], the existence of Ψ such that

$$\psi(F(\chi, \Phi(\chi))) = G(\chi, \Phi(\chi))$$

Then we have the first-order difference equation from DS_3,

$$\chi(t+1) = \Psi(\chi(t)).$$

Therefore we have

$$\chi(t+1) = \Psi(\chi(t)) = \Psi(U(\lambda^\sigma).) = \Psi(u(\sigma)) = u(\sigma+1),$$

and

$$\sigma + 1 = l(t+1), \quad l(t) + 1 = l(t+1).$$

Hence we obtain

$$l(t) = t + \pi(t)$$

where $\pi(t)$ is an periodic function with period one. Then

$$\sigma = t + \pi(t)$$

and

$$\chi(t) = U(\lambda^{\sigma}) = \sum_{n=1}^{\infty} a_n (\lambda^{t+\pi(t)})^n = \sum_{n=1}^{\infty} a_n (\lambda^{\pi(t)} \lambda^t)^n$$

Now we put $\lambda^{\pi(t)}$ into $\pi(t)$ so that $\chi(t)$ can be written as

$$\chi(t) = \sum_{n=1}^{\infty} a_n \lambda^{n(\frac{\log \pi(t)}{\log \lambda} + t)}$$

This completes the proof.

References

[1] Bischi, G., and M. Kopel, "Equilibrium selection in a nonlinear duopoly game with adaptive expectations," Journal of Economic Behavior and Organization, 46, 2001, 73-100

[2] Day, R., "Irregular Growth Cycles," American Economic Review, 72, 406-414, 1982.

[3] Day, R., "The Emergence of Chaos from Classical Economic Growth," Quarterly Journal of Economics, 98, 201-213, 1983.

[4] Kopel, M., "Simple and Complex adjustment dynamics in Cournot duopoly model," Chaos, Solitions and Fractals, 7, 2031-2048, 1996.

[5] Lorenze, H-W, Nonlinear Dynamical Economics and Chaotic Motions, Springer-Verlag, Berlin, 1993.

[6] Majumdar, M., Mitra, T., and K. Nishimura, Optimization and Chaos, Springer, 2000.

[7] Matsumoto, A., "Let It Be; Chaotic Price Instability can be Beneficial", Discussion Paper 25, Institute of Economic Research, Chuo University, forthcoming in Chaos, Solitons and Fractals.

[8] Matsumoto, A., and Y. Nonaka, "Profitable Chaos in Nonlinear Duopoly Market with Asymmetric Production Externality," Discussion Paper 37, Institute of Economic Research, Chuo University, 2002.

[9] Rosser, B., From Catastrophe to Chaos: A General Theory of Economic Discontinuities, Kluwer Academic Publishers, Boston, 2000.

[10] Smart, D.R., Fixed Point Theorem, Cambridge University Press, 1974

[11] Suzuki, M., "Holomorphic Solutions of Some Functional Equations," Nihonkai Mathematical Journal 5, 1994, 109-114.

[12] Suzuki, M., "On Some Difference Equations in Economic Model," Mathematica Japonica, 43, 1996, 129-134.

[13] Suzuki, M., "Holomorphic Solutions of Some System of n functional Equations with n variables related to difference systems," Aequationes Mathematicase, 57, 1999.

[14] Suzuki, M., "Difference Equation for a Population Model," Discrete Dynamics in Nature and Society, 5, 9-18, 2000.