

**Discussion Paper Series No.70**

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**July , 2004**

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July 2004

## Abstract

In this paper, we investigate the macroeconomic impact of the inflation targeting policy by using the analytical framework of a dynamic Keynesian model with a debt effect. We show that the monetary authority can stabilize an unstable economy by carrying out the sufficiently *credible* inflation targeting policy even in case of the liquidity trap, as long as the destabilizing Fisher debt effect is not extremely strong. We also show the existence of the cyclical fluctuation at some range of the parameter values by using the Hopf bifurcation theorem, and we provide some numerical examples which support our analysis.

**Key words** : inflation targeting policy, dynamic Keynesian model, Fisher debt effect, Mundell effect, liquidity trap, credibility, Hopf bifurcation.

**JEL codes** : E3, E4, E5

## 1 Introduction

In traditional textbook interpretation of the static Keynesian model, the monetary policy becomes ineffective in case of the 'liquidity trap' in which nominal rate of interest is stuck at its lower bound. If this is true, in the depressed Japanese economy in the late 1990s and the early 2000s, the monetary policy must be ineffective, because nominal rate of interest already fell to nearly zero. In contrast to this traditional view, however, Krugman(1998) constructed a formal model in which the 'inflation targeting policy' by the monetary authority is effective even in case of the liquidity trap. His paper had a practical purpose to present a policy recommendation to the Japanese central bankers. Although Japanese central bankers strongly opposed against the inflation targeting policy, this policy has been adopted by the central banks of several countries ( cf. Bernanke, Laubach, Mishkin and Posen(1999)), and recently this policy attracted attention of many economists in Japan and other countries partly due to the influence of Krugman's paper.

Krugman(1998)'s model is almost only existing formal model of the inflation targeting under liquidity trap, and his analytical framework is a microeconomically founded two period 'representative agent' approach, which presupposes the existence of a representative agent who tries to maximize the present value of the utility in period 1, conditional on the agent's expectation concerning the price level in period 2. The credibility or the believability of the central banker's behavior plays a crucial role for the effectiveness of the monetary policy in his model.

In this paper, we reconsider the macroeconomic impact of the inflation targeting policy by adopting another modeling strategy. Our analytical framework is a high dimensional dynamic Keynesian model with a debt effect, which was developed by Chiarella and Flaschel(2000), Chiarella, Flaschel, Groh and Semmler(2000), Chiarella, Flaschel and Semmler(2001), Asada, Chiarella, Flaschel and Franke(2003), and Asada(2004). As noted by Asada, Chiarella, Flaschel and Franke(2003), "macrodynamics must look for progress from at least two perspectives"(p. 334). One approach seeks, like Krugman(1998), solid microeconomic foundations which is based on the optimizing behavior of single agent (representative agent). Usually, this modeling strategy results

in relatively small linear or loglinear models for reasons of tractability. On the other hand, another approach tries to provide a full picture of the economic interdependency by constructing high dimensional macrodynamic system. Usually this approach is not based on the explicit treatment of the optimizing behavior of agents, but, this does not necessarily mean that it contradicts the optimizing behaviors. In this paper, we adopt the latter modeling strategy to investigate the effect of the inflation targeting policy. Our model integrates the ‘debt effect’ on the investment expenditure which is due to Fisher(1933), Keynes(1936) and Minsky(1986) ( cf. Nasica(2000) ) into high dimensional Keynesian dynamic model which was developed by Chiarella, Flaschel and others. The merit of our approach is that we can make explicit some important stabilizing ( negative feedback ) and destabilizing ( positive feedback ) causal chains which are embedded in the dynamic process. We show that the monetary authority can stabilize an unstable economy by carrying out the sufficiently *credible* inflation targeting policy even in case of the liquidity trap, as long as the destabilizing Fisher debt effect is not extremely strong. We also show the existence of the cyclical fluctuation at some range of the parameter values.

This paper is organized as follows. In section 2, our basic dynamic Keynesian model with a debt effect is constructed. The model is reduced to a system of five dimensional nonlinear differential equations. In section 3, the nature of the long run equilibrium solution is considered. In section 4, we investigate the local stability / instability of the long run equilibrium by using the Routh-Hurwitz criteria, and then detect the condition for the existence of the cyclical fluctuation by using the Hopf bifurcation theorem. In section 5, we provide some numerical examples which support our analysis. In section 6, the economic interpretation of the main analytical results are provided. The proofs of the main propositions are contained in the appendices.

## 2. Formulation of the model

Our model consists of the following system of equations, where a dot over a symbol denotes the derivative with respect to time.

$$\dot{d} = \phi(g(\beta y, \rho - \pi^e, d)) - s_f \{ \beta y - i(\rho, d) d \} - \{ g(\beta y, \rho - \pi^e, d) + \pi \} d \quad (1)$$

$$\dot{y} = \alpha[\phi(g(\beta y, \rho - \pi^e, d)) + (1 - s_f)\{\rho v + i(\rho, d)d\} - \{s_f + (1 - s_f)s_r\}\beta y] ; \alpha > 0 \quad (2)$$

$$\dot{e}/e = \dot{y}/y + g(\beta y, \rho - \pi^e, d) - n \quad (3)$$

$$\dot{m}/m = \mu - \pi - g(\beta y, \rho - \pi^e, d) \quad (4)$$

$$\dot{\pi}^e = \gamma\{\theta(\mu_0 - n - \pi^e) + (1 - \theta)(\pi - \pi^e)\} ; \gamma > 0, \quad 0 \leq \theta \leq 1 \quad (5)$$

$$\pi = \varepsilon(e - \bar{e}) + \pi^e ; \varepsilon > 0 \quad (6)$$

$$\rho = \rho_0 + (h_1 y - m)/h_2 \equiv \rho(y, m) ; \rho_0 \geq 0, \quad h_1 > 0, \quad h_2 > 0 \quad (7)$$

$$\mu = \mu_0 + \delta(\mu_0 - n - \pi) ; \mu_0 > 0, \quad \delta \geq 0 \quad (8)$$

The meanings of the symbols are as follows.  $d = D/pK$  = debt-capital ratio.  $y = Y/K$  = output-capital ratio, which is also called 'rate of capacity utilization'.  $D$  = nominal stock of firms' private debt.  $p$  = price level.  $K$  = real capital stock.  $Y$  = real output (real national income).  $g = \dot{K}/K$  = rate of capital accumulation.  $\rho$  = nominal rate of interest of interest-bearing safe assets.  $i$  = nominal rate of interest which is applied to firms' private debt.  $\pi = \dot{p}/p$  = rate of price inflation.  $\pi^e$  = expected rate of price inflation.  $e = N/N^s$  = rate of employment = 1 - rate of unemployment ( $0 \leq e \leq 1$ ).  $N$  = labor employment.  $N^s$  = labor supply.  $n = \dot{N}^s/N^s$  = growth rate of labor supply (natural rate of growth) = constant  $> 0$ .  $m = M/pK$  = money-capital ratio.  $M$  = nominal money supply.  $\mu = \dot{M}/M$  = growth rate of nominal money supply. The function  $\phi(g)$  is the adjustment cost function of investment which was introduced by Uzawa(1968) with the properties  $\phi'(g) \geq 1$  and  $\phi''(g) \geq 0$ . We shall explain the economic meanings of the parameters  $\alpha, \gamma, \theta, \varepsilon, h_1, h_2, \delta, \nu, s_f$ , and  $s_r$  later.

Next, we shall explain how these equations are derived. Dynamic law of the motion of private debt can be expressed as follows.

$$\dot{D} = \phi(g)pK - s_f(rpK - iD) \quad (9)$$

where  $r = P/K$  is the rate of profit ( $P$  is the real profit), and  $s_f \in (0, 1]$  is the rate of internal retention of firms, which is assumed to be constant. For simplicity, we assume that there is no issues of new shares, and we neglect the repayment of the principal of debt.

Differentiating the definitional relationship  $d = D/(pK)$ , we have

$$\dot{d}/d = \dot{D}/D - \dot{p}/p - \dot{K}/K = \dot{D}/D - \pi - g. \quad (10)$$

Substituting Eq. (9) into Eq. (10), we obtain

$$\dot{d} = \phi(g) - s_f(r - id) - (g + \pi)d. \quad (11)$$

For the dynamic adjustment in the goods market, we assume the following Keynesian / Kaldorian quantity adjustment process (cf. Asada(1991)(2001)).

$$\dot{y} = \alpha(c + h - y) \quad ; \quad c = C/K, \quad h = E/K \quad (12)$$

where  $C$  is real consumption expenditure,  $E = \phi(g)K$  is real investment expenditure including adjustment cost, and  $\alpha$  is a positive parameter which represents the speed of adjustment in the goods market. For consumption expenditures, we shall assume as follows, which is a Kaleckian formulation of the two class economy (cf. Kalecki(1971)).

$$C = C_w + C_r \quad (13)$$

$$C_w = W = Y - P \quad (14)$$

$$C_r = (1 - s_r)\{(1 - s_f)P + \rho(V/p) + i(D/p)\} \quad ; \quad s_r \in (0,1] \quad (15)$$

where  $C_w$ ,  $C_r$ , and  $W$  are workers' real consumption, capitalists' real consumption, and real wage income respectively,  $V$  is the nominal value of the interest-bearing safe assets, and  $s_r$  is the capitalists' propensity to save, which is assumed to be a constant. These equations imply the Kaleckian postulate that the workers do not save, while the capitalists save a part of their income.<sup>1</sup> Substituting equations (13), (14), and (15) into Eq. (12), we obtain the following expression.<sup>2</sup>

$$\dot{y} = \alpha[\phi(g) + (1 - s_r)\{\rho(V/pK) + id\} - \{s_f + (1 - s_f)s_r\}r] \quad (16)$$

Furthermore, we assume the following relationships.

$$i = \rho + \xi(d) \equiv i(\rho, d) \quad ; \quad \xi(d) \geq 0, \quad i_d = \xi'(d) > 0 \quad \text{for } d > 0, \\ i_d < 0 \quad \text{for } d < 0 \quad (17)$$

$$g = g(r, \rho - \pi^e, d) \quad ; \quad g_r = \partial g / \partial r > 0, \quad g_{\rho - \pi} = \partial g / \partial (\rho - \pi^e) < 0,$$

$$g_d = \partial g / \partial d < 0 \quad (18)$$

$$P/Y = \beta = \text{constant}, \quad 0 < \beta < 1 \quad (19)$$

$$V/pK = \nu = \text{constant} > 0 \quad (20)$$

Eq. (17) captures the fact that the rate of interest of the ‘risky’ asset  $i$  is greater than  $\rho$ , and the difference between them reflects the degree of risk. Eq.(18) is the investment function with Fisher debt effect. We can derive this type of investment function theoretically from the optimizing behavior of firms by using both of Uzawa(1969)’s hypothesis of increasing cost (Penrose effect) and Kalecki(1937)’s hypothesis of increasing risk of investment (cf. Asada(1999)(2001)).

We can interpret Eq.(19) as follows. By definition, we have

$$p = z(wN/Y) = zw/a \quad ; \quad z > 1 \quad (21)$$

where  $z$  is the mark up and  $a = Y/N$  is average labor productivity. Therefore, we can express the share of profit in national income as the increasing function of the mark up, namely,

$$\beta = P/Y = (Y - W)/Y = 1 - (W/Y) = 1 - \{(w/p)N/Y\} = 1 - (1/z). \quad (22)$$

We assume that the mark up is a constant which reflects the ‘degree of monopoly’ in the sense of Kalecki(1971), so that  $\beta$  also becomes a constant. In this case, we have

$$r = P/K = \beta Y/K = \beta y. \quad (24)$$

Eq. (20) is merely the simplifying assumption to avoid the unnecessary complications.

Substituting equations (17), (18), (19), (20), and (24) into equations (11) and (16), we obtain equations (1) and (2).

We can derive Eq.(3) as follows. By definition, we have

$$N = \frac{(Y/K)K}{Y/N} = yK/a \quad (25)$$

so that we also have

$$e = N/N^s = yK/aN^s. \quad (26)$$

We abstract from technical progress, so that we assume that the average labor productivity  $a$  is constant. In this case, differentiating Eq.(26), we have the following equation, which is nothing but Eq.(3).

$$\dot{e}/e = \dot{y}/y + \dot{K}/K - \dot{N}^s/N^s = \dot{y}/y - g(\beta y, \rho - \pi^e, d) - n \quad (27)$$

Next, differentiating the definitional equation  $m = M/pK$ , we have

$$\dot{m}/m = \dot{M}/M - \dot{p}/p - \dot{K}/K = \mu - \pi - g(\beta y, \rho - \pi^e, d), \quad (28)$$

which is Eq. (4).

Eq. (5) formalizes the expectation formation hypothesis of the public's expected rate of price inflation, which is a mixture of the forward-looking and the backward-looking (adaptive) types of expectation formations.<sup>3</sup> In case of  $\theta = 0$ , Eq. (5) is reduced to  $\dot{\pi}^e = \gamma(\pi - \pi^e)$ , which is nothing but the standard formulation of adaptive (backward-looking) expectation hypothesis. On the other hand, in case of  $\theta = 1$ , it is reduced to  $\dot{\pi}^e = \gamma(\mu_0 - n - \pi^e)$ , which implies that the expected rate of inflation is adjusted toward the target rate  $\mu_0 - n$ . We shall see in the next section that this target rate is in fact the long run equilibrium rate of inflation in our model. We assume that this target rate is announced by the monetary authority (central bank), so that it affects the expectation formation of the private sectors in the forward-looking manner.

Next, let us consider the price dynamics. We assume the following standard type of the expectation-augmented wage Phillips curve.

$$\dot{w} = \varepsilon(e - \bar{e}) + \pi^e \quad (29)$$

where  $\varepsilon$  is the speed of wage adjustment, which is assumed to be a positive parameter. On the other hand, from Eq. (21) we have

$$\pi = \dot{p} / p = \dot{w} / w. \quad (30)$$

Therefore, we can transform the wage Phillips curve (29) into the price Phillips curve (6).

Eq. (7) is nothing but the standard type of the Keynesian 'LM equation' which describes the equilibrium condition for the money market. We specify the nominal demand function for money as

$L^D = h_1 pY + (\rho_0 - \rho)h_2 pK$ , where  $\rho_0$  is the lower bound of the nominal rate of interest.<sup>4</sup> In this case, the equilibrium condition for the money market  $M = L^D$  becomes as follows.

$$m = M / pK = h_1(Y / K) + (\rho_0 - \rho)h_2 = h_1 y + (\rho_0 - \rho)h_2 \quad (31)$$

Solving this equation with respect to  $\rho$ , we have Eq. (7).<sup>5</sup>

Eq. (8) formalizes the monetary policy rule of the monetary authority (the central bank). This is a type of the 'inflation targeting rule' (cf. Krugman(1998) and Bernanke et al. (1999)). Monetary authority announces the target rate of inflation  $\mu_0 - n$  to the public, and adjusts the



growth rate of the nominal money supply towards the realization of this target.

We can reduce the system of equations (1) – (8) to the following system of the five dimensional nonlinear differential equations.<sup>6</sup>

$$\begin{aligned}
\text{(i)} \quad \dot{d} &= \phi(g(\beta y, \rho(y, m) - \pi^e, d)) - s_f \{\beta y - i(\rho(y, m), d)d\} \\
&\quad - \{g(\beta y, \rho(y, m) - \pi^e, d) + \varepsilon(e - \bar{e}) + \pi^e\}d \equiv F_1(d, y, e, \pi^e, m; \varepsilon) \\
\text{(ii)} \quad \dot{y} &= \alpha[\phi(g(\beta y, \rho(y, m) - \pi^e, d)) + (1 - s_f)\{\rho(y, m)v + i(\rho(y, m), d)d\} \\
&\quad - \{s_f + (1 - s_f)s_r\}\beta y] \equiv F_2(d, y, \pi^e, m; \alpha) \\
\text{(iii)} \quad \dot{e} &= e[F_2(d, y, \pi^e, m; \alpha) / y + g(\beta y, \rho(y, m) - \pi^e, d) - n] \\
&\quad \equiv F_3(d, y, e, \pi^e, m; \alpha) \\
\text{(iv)} \quad \dot{\pi}^e &= \gamma\{\theta(\mu_0 - n - \pi^e) + (1 - \theta)\varepsilon(e - \bar{e})\} \equiv F_4(e, \pi^e; \varepsilon, \gamma, \theta) \\
\text{(v)} \quad \dot{m} &= m[(1 + \delta)\{\mu_0 - \varepsilon(e - \bar{e}) - \pi^e\} - \delta n - g(\beta y, \rho(y, m) - \pi^e, d)] \\
&\quad \equiv F_5(d, y, e, \pi^e, m; \varepsilon, \delta) \tag{32}
\end{aligned}$$

### 3. Nature of the equilibrium solution

First, let us study the nature of the equilibrium solution of the system (32) which satisfies the condition  $\dot{d} = \dot{y} = \dot{e} = \dot{\pi}^e = \dot{m} = 0$ . The equilibrium values of the endogenous variables are determined by the following set of equations, which defines the long run equilibrium (steady state) of our system.

$$\begin{aligned}
\text{(i)} \quad \phi(n) - s_f \{\beta y - i(\rho(y, m), d)d\} - \mu_0 d &= 0 \\
\text{(ii)} \quad \phi(n) + (1 - s_f)\{\rho(y, m)v + i(\rho(y, m), d)d\} - \{s_f + (1 - s_f)s_r\}\beta y &= 0 \\
\text{(iii)} \quad g(\beta y, \rho(y, m) - \mu_0 + n, d) &= n \\
\text{(iv)} \quad e &= \bar{e} \\
\text{(v)} \quad \pi &= \pi^e = \mu_0 - n \tag{33}
\end{aligned}$$

Eq. (33)(iii) and (iv) imply that at the long run equilibrium position, the rate of capital accumulation (the rate of investment) is equal to the exogenously determined 'natural rate of growth', and the rate of employment is equal to the exogenously determined 'natural rate of employment'. Because of this fact, at first glance, it seems that the monetary policy is irrelevant to the determination of the long run equilibrium. But, in fact, this is not true. Usually, a set of equations (33) can be considered as the determinant of the equilibrium values  $(d^*, y^*, m^*, e^*, \pi^{e*})$  for given long run target value of the growth rate of money supply  $\mu_0$ . These equilibrium values except  $e^*$  usually depend on  $\mu_0$ . In particular,  $(d^*, y^*, m^*)$  are determined by the subsystem (33)(i) – (iii) for given  $\mu_0$ , and the equilibrium value of real rate of interest is determined by

$$(\rho - \pi^e)^* = \rho(y^*, m^*) + n - \mu_0 \quad (34)$$

By the way, nominal rate of interest has the nonnegative lower bound  $r_0$ , so that the inequality

$$(\rho - \pi^e)^* \geq r_0 + n - \mu_0 \quad (35)$$

must be satisfied. Since the economically meaningful ranges of  $y$  and  $d$  are restricted, there may be the case in which relatively small real rate of interest is required to keep the natural rate of growth. This means that the long run equilibrium may not exist because of too high real rate of interest if the monetary authority chooses too small value of  $\mu_0$ . That is to say, the target rate of inflation  $\mu_0 - n$  cannot be chosen completely arbitrarily, and there are some restrictions on the choice of its value, even if there remains some degree of freedom for the choice of  $\mu_0$ .<sup>7</sup> In fact, it is quite likely that the mildly positive target rate of inflation, for example, 2 or 3 percent per year, rather than zero inflation is required to ensure the existence of the long run equilibrium (cf. Krugman(1998)). Henceforth, we assume that  $\mu_0$  is fixed at the level which ensures the existence of long run equilibrium, and monetary authority announces its value to the public. It is assumed that the public use this information for their expectation formation in the manner of Eq. (5).

It is worth to note that the values of the parameters  $\alpha, \varepsilon, \gamma, \theta$ , and  $\delta$  do not affect the long run equilibrium values of the main variables. However, this does not mean that these parameter values are irrelevant to the dynamic behavior of the system. In fact, the changes of these parameter values can affect the dynamic stability of the system. We investigate this theme in the next section.

#### 4. Local stability analysis and the detection of cyclical fluctuations

In this section, we study the local stability / instability of the equilibrium point by assuming that there exists an economically meaningful equilibrium point such that  $d^* > 0$ ,  $y^* > 0$ , and  $m^* > 0$ . We can express the Jacobian matrix of the five dimensional system (32) which is evaluated *at the equilibrium point* as follows.

$$J_5(1/h_2) = \begin{bmatrix} F_{11} & F_{12}(1/h_2) & F_{13}(\varepsilon) & F_{14} & F_{15}(1/h_2) \\ F_{21}(\alpha) & F_{22}(\alpha, 1/h_2) & 0 & F_{24}(\alpha) & F_{25}(\alpha, 1/h_2) \\ F_{31}(\alpha) & F_{32}(\alpha, 1/h_2) & 0 & F_{34}(\alpha) & F_{35}(\alpha, 1/h_2) \\ 0 & 0 & F_{43}(\varepsilon, \gamma, \theta) & F_{44}(\gamma, \theta) & 0 \\ F_{51} & F_{52}(1/h_2) & F_{53}(\varepsilon, \delta) & F_{54}(\delta) & F_{55}(1/h_2) \end{bmatrix} \quad (36)$$

where

$$F_{11} = \partial F_1 / \partial d = \underbrace{(\phi'(n) - d)}_{(+)} \underbrace{g_d}_{(-)} - \mu_0 + s_f \underbrace{(i_d d + i)}_{(+)}$$

$$F_{12}(1/h_2) = \partial F_1 / \partial y = \beta \{ \underbrace{(\phi'(n) - d)}_{(+)} \underbrace{g_r}_{(+)} - s_f \} + [ \underbrace{\{\phi'(n) - 1\}}_{(+)} \underbrace{g_{\rho-\pi}}_{(-)} + s_f \underbrace{i_\rho}_{(+)} d ] (h_1 / h_2),$$

$$F_{13}(\varepsilon) = \partial F_1 / \partial \varepsilon = -\varepsilon d < 0,$$

$$F_{14} = \partial F_1 / \partial \pi^e = - \underbrace{(\phi'(n) - d)}_{(+)} \underbrace{g_{\rho-\pi}}_{(-)} - d,$$

$$F_{15}(1/h_2) = \partial F_1 / \partial m = - [ \underbrace{\{\phi'(n) - 1\}}_{(+)} \underbrace{g_{\rho-\pi}}_{(-)} + s_f \underbrace{i_\rho}_{(+)} d ] (1/h_2),$$

$$F_{21}(\alpha) = \partial F_2 / \partial d = \alpha [ \underbrace{\phi'(n)}_{(+)} \underbrace{g_d}_{(-)} + (1 - s_f) \underbrace{(i_d d + i)}_{(+)} ],$$

$$F_{22}(\alpha, 1/h_2) = \partial F_2 / \partial y = \alpha \beta [ \underbrace{\phi'(n)}_{(+)} \underbrace{g_r}_{(+)} - \{ s_f + (1 - s_f) s_r \} ] + \alpha [ \underbrace{\phi'(n)}_{(+)} \underbrace{g_{\rho-\pi}}_{(-)}$$

$$+ (1 - s_f) (v + i_\rho d) ] (h_1 / h_2),$$

$$F_{24}(\alpha) = \partial F_2 / \partial \pi^e = -\alpha \underbrace{\phi'(n)}_{(+)} \underbrace{g_{\rho-\pi}}_{(-)} > 0,$$

$$F_{25}(\alpha, 1/h_2) = \partial F_2 / \partial m = -\alpha [ \underbrace{\phi'(n)}_{(+)} \underbrace{g_{\rho-\pi}}_{(-)} + (1 - s_f) (v + i_\rho d) ] (1/h_2),$$

$$F_{31}(\alpha) = \partial F_3 / \partial d = \bar{e}[F_{21}(\alpha) / y + g_d],$$

$$F_{32}(\alpha, 1/h_2) = \partial F_3 / \partial y = \bar{e}[F_{22}(\alpha, 1/h_2) / y + \beta g_r],$$

$$F_{34}(\alpha) = \partial F_3 / \partial \pi^e = \bar{e}[F_{24}(\alpha) / y - g_{\rho-\pi}],$$

$$F_{35}(\alpha, 1/h_2) = \partial F_3 / \partial m = \bar{e}[F_{25}(\alpha) / y - g_{\rho-\pi}(1/h_2)],$$

$$F_{43}(\varepsilon, \gamma_2) = \partial F_4 / \partial e = \gamma(1-\theta)\varepsilon > 0,$$

$$F_{44}(\gamma_1) = \partial F_4 / \partial \pi^e = -\gamma\theta < 0,$$

$$F_{51} = \partial F_5 / \partial d = -m g_d > 0,$$

$$F_{52}(1/h_2) = \partial F_5 / \partial y = -m[\beta g_r + g_{\rho-\pi}(h_1/h_2)],$$

$$F_{53}(\varepsilon, \delta) = \partial F_5 / \partial e = -m(1+\delta)\varepsilon < 0,$$

$$F_{54}(\delta) = \partial F_5 / \partial \pi^e = m[-(1+\delta) + g_{\rho-\pi}] < 0, \quad F_{55}(1/h_2) = \partial F_5 / \partial m = m g_{\rho-\pi}(1/h_2) \leq 0.$$

For a moment, we shall concentrate on the special case of  $1/h_2 = 0 (h_2 \rightarrow +\infty)$ , which corresponds to the case of the ‘liquidity trap’ in which the nominal rate of interest is fixed at its lower bound  $\rho_0$ . In this case, the Jacobian matrix at the equilibrium point becomes as follows.<sup>8</sup>

$$J_5(0) = \begin{bmatrix} F_{11} & F_{12}(0) & -\varepsilon d & F_{14} & 0 \\ \alpha G_{21} & \alpha G_{22} & 0 & \alpha G_{24} & 0 \\ \bar{e}[\alpha G_{21} / y + g_d] & \bar{e}[\alpha G_{22} / y + \beta g_r] & 0 & \bar{e}[\alpha G_{24} / y + g_{\rho-\pi}] & 0 \\ 0 & 0 & \gamma(1-\theta)\varepsilon & -\gamma\theta & 0 \\ F_{51} & 0 & F_{53}(\varepsilon, \delta) & F_{54}(\delta) & 0 \end{bmatrix} \quad (37)$$

where

$$F_{12}(0) = \beta\{(\phi'(n)-d)g_r - s_f\},$$

$$G_{21} = \phi'(n)g_d + (1-s_f)(i_d d + i),$$

$$G_{22} = \beta[\phi'(n)g_r - \{s_f + (1-s_f)s_r\}], \quad \text{and} \quad G_{24} = -\phi'(n)g_{\rho-\pi} > 0.$$

Throughout the paper, we posit the following assumptions.

**Assumption 1.**

$$F_{11} < 0, \quad F_{12}(0) > 0, \quad F_{14} > 0, \quad G_{21} < 0, \quad \text{and} \quad G_{22} > 0.$$

**Assumption 2.**

$$F_{11}G_{22} - F_{12}(0)G_{21} > 0.$$

We can interpret the economic meanings of these assumptions as follows. **Assumption 1** will be satisfied if  $\phi'(n)$ ,  $g_r$ ,  $|g_{\rho-\pi}|$ , and  $|g_d|$  are sufficiently large at the equilibrium point. In other words, **Assumption 1** will in fact be satisfied if the sensitivity of adjustment cost with respect to the changes of investment activity and the sensitivities of investment with respect to the changes of relevant variables are relatively large. On the other hand, it is easy to show that

$$\begin{aligned} \lim_{s_r \rightarrow 1} \{F_{11}G_{22} - F_{12}(0)G_{21}\} = & \beta \left[ \underbrace{-d}_{(-)} \underbrace{g_d}_{(-)} - \mu_0 + \underbrace{(d}_{(+)} \underbrace{g_r}_{(+)} + s_f) \underbrace{(i_d + i)}_{(+)} \right] \underbrace{\{\phi'(n) g_r - 1\}}_{(+)} \underbrace{\}_{(+)} \\ & + s_f \underbrace{\phi'(n)}_{(+)} \underbrace{g_r}_{(+)}. \end{aligned} \quad (38)$$

The right hand side of Eq. (38) becomes positive if  $\phi'(n)$ ,  $g_r$ ,  $|g_d|$ , and  $i_d$  are sufficiently large. This means that **Assumption 2** will be satisfied if  $\phi'(n)$ ,  $g_r$ ,  $|g_d|$ ,  $i_d$ , and  $s_r$  are sufficiently large at the equilibrium point. In other words, **Assumption 2** will in fact be satisfied if capitalists' propensity to save as well as the sensitivities of investment etc. with respect to the relevant variables are relatively large.

The characteristic equation of the Jacobian matrix (36) becomes as

$$\Delta_5(\lambda; 1/h_2) \equiv |\lambda I - J_5(1/h_2)| = 0. \quad (39)$$

In particular, in case of  $1/h_2 = 0$ , this equation becomes

$$\Delta_5(\lambda; 0) = |\lambda I - J_4(0)| \lambda = 0, \quad (40)$$

where

$$J_4(0) = \begin{bmatrix} F_{11} & F_{12}(0) & -\varepsilon d & F_{14} \\ \alpha G_{21} & \alpha G_{22} & 0 & \alpha G_{24} \\ \bar{e}[\alpha G_{21}/y + g_d] & \bar{e}[\alpha G_{22}/y + \beta g_r] & 0 & \bar{e}[\alpha G_{24}/y + g_{\rho-\pi}] \\ 0 & 0 & \gamma(1-\theta)\varepsilon & -\gamma\theta \end{bmatrix} \quad (41)$$

The characteristic equation (40) has a root  $\lambda_5 = 0$ , and other four roots are determined by the following equation.

$$\Delta_4(\lambda; 0) = |\mathcal{M} - J_4(0)| = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \quad (42)$$

where

$$a_1 = -\text{trace}J_4(0) = -F_{11} - \alpha G_{22} + \gamma\theta \equiv a_1(\alpha, \gamma, \theta), \quad (43)$$

$a_2 =$  sum of all principal second-order minors of  $J_4(0)$

$$\begin{aligned} &= \alpha \begin{vmatrix} F_{11} & F_{12}(0) \\ G_{21} & G_{22} \end{vmatrix} + \begin{vmatrix} F_{11} & -\varepsilon d \\ \bar{e}[\alpha G_{21}/y + g_d] & 0 \end{vmatrix} + \gamma\theta \begin{vmatrix} F_{11} & F_{14} \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} \alpha G_{22} & 0 \\ \bar{e}[\alpha G_{22}/y + \beta g_r] & 0 \end{vmatrix} \\ &+ \alpha\gamma\theta \begin{vmatrix} G_{22} & G_{24} \\ 0 & -1 \end{vmatrix} + \gamma \begin{vmatrix} 0 & \bar{e}[\alpha G_{24}/y + g_{\rho-\pi}] \\ (1-\theta)\varepsilon & -\theta \end{vmatrix} \\ &= \alpha \{ F_{11} G_{22} - F_{12}(0) G_{21} \} + \varepsilon d \bar{e} [\alpha G_{21}/y + g_d] \\ &+ \gamma [\theta (-F_{11} - \alpha G_{22}) - (1-\theta) \varepsilon \bar{e} \{ \alpha G_{24}/y + g_{\rho-\pi} \}] \equiv a_2(\alpha, \varepsilon, \gamma, \theta), \quad (44) \end{aligned}$$

$a_3 = -$ (sum of all principal third-order minors of  $J_4(0)$ )

$$\begin{aligned} &= -\alpha\gamma\varepsilon\bar{e} \begin{vmatrix} G_{22} & 0 & G_{24} \\ \alpha G_{22}/y + \beta g_r & 0 & \alpha G_{24}/y + g_{\rho-\pi} \\ 0 & 1-\theta & -\theta \end{vmatrix} - \gamma\varepsilon\bar{e} \begin{vmatrix} F_{11} & -d & F_{14} \\ \alpha G_{21}/y + g_d & 0 & \alpha G_{24}/y + g_{\rho-\pi} \\ 0 & 1-\theta & -\theta \end{vmatrix} \\ &- \alpha\gamma\theta \begin{vmatrix} F_{11} & F_{12}(0) & F_{14} \\ G_{21} & G_{22} & G_{24} \\ 0 & 0 & -1 \end{vmatrix} - \alpha\varepsilon d \bar{e} \begin{vmatrix} F_{11} & F_{12}(0) & -1 \\ G_{21} & G_{22} & 0 \\ \alpha G_{21}/y + g_d & \alpha G_{22}/y + \beta g_r & 0 \end{vmatrix} \\ &= \gamma [\alpha(1-\theta) \varepsilon \bar{e} (G_{22} g_{\rho-\pi} - G_{24} \beta g_r) + \varepsilon \bar{e} \{ (\theta d - (1-\theta) F_{14}) (\alpha G_{21}/y + g_d) \}] \end{aligned}$$

$$\begin{aligned}
& + (1-\theta) F_{11} (\alpha G_{24} / y + g_{\rho-\pi}) + \alpha \theta \{ F_{11} G_{22} - F_{12}(0) G_{21} \} \\
& + \alpha \varepsilon \bar{e} (G_{21} \beta g_r - G_{22} g_d) \equiv a_3(\alpha, \varepsilon, \gamma, \theta), \tag{45}
\end{aligned}$$

$$\begin{aligned}
a_4 = \det J_4(0) & = \alpha \gamma \bar{e} \begin{vmatrix} F_{11} & F_{12}(0) & -d & F_{14} \\ G_{21} & G_{22} & 0 & G_{24} \\ \alpha G_{21} / y + g_d & \alpha G_{22} / y + \beta g_r & 0 & \alpha G_{24} / y + g_{\rho-\pi} \\ 0 & 0 & 1-\theta & -\theta \end{vmatrix} \\
& = \alpha \gamma \bar{e} [\theta d (G_{21} \beta g_r - G_{22} g_d) - (1-\theta) \{ \beta g_r (F_{14} G_{21} - F_{11} G_{24}) \\
& + g_{\rho-\pi} (F_{11} G_{22} - F_{12}(0) G_{21}) + g_d (F_{12}(0) G_{24} - F_{14} G_{22}) \}] \equiv a_4(\alpha, \varepsilon, \gamma, \theta) \tag{46}
\end{aligned}$$

The characteristic equation (42) governs the local dynamics of the four dimensional subsystem (32)(i) – (iv) in case of  $1/h_2 = 0$ . Now, let us assume

**Assumption 3.**

$$F_{11} + \alpha G_{22} < 0, \quad \alpha G_{24} / y + g_{\rho-\pi} > 0, \quad \text{and} \quad G_{21} \beta g_r - G_{22} g_d > 0.$$

The economic interpretation of this assumption is as follows. This assumption implies that

$$F_{11} + \alpha G_{22} = (\phi'(n) - d) g_d - \mu_0 + s_f (i_d d + i) + \alpha \beta [\phi'(n) g_r - \{s_f + (1-s_f) s_r\}] < 0, \tag{47}$$

$$\alpha G_{24} / y + g_{\rho-\pi} = -g_{\rho-\pi} (\alpha \phi'(n) - y) / y > 0, \tag{48}$$

$$G_{21} \beta g_r - G_{22} g_d = (1-s_f) (i_d d + i) \beta g_r + \{s_f + (1-s_f) s_r\} g_d > 0. \tag{49}$$

The inequality (47) will be satisfied if the quantity adjustment speed in the goods market ( $\alpha$ ) is not extremely large. The inequality (48) will be satisfied if  $\alpha$  is not extremely small. The inequality (49) will be satisfied if the debt effect on investment ( $|g_d|$ ) is relatively small compared with the profit rate effect on investment ( $g_r$ ). Therefore, **Assumption 3** will in fact be satisfied if  $\alpha$  is only moderately large

and  $|g_d|$  is relatively small compared with  $g_r$ .

By the way, it is well known that the Routh-Hurwitz conditions for the local stability of the system (42) can be expressed as follows (cf. Gandolfo(1996)).

$$a_j > 0 \quad (j = 1, 2, 3, 4), \quad \Phi \equiv a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 > 0 \quad (50)$$

Let us denote the four dimensional subsystem (32)(i) – (iv) in case of  $1/h_2 = 0$  as the system  $(S_4)$ . By utilizing the criteria (50), we can prove the following results under **Assumptions 1 – 3**.

**Proposition 1.**

- (i) Suppose that  $\theta$  and  $\gamma$  are fixed at arbitrary values such that  $0 \leq \theta \leq 1$  and  $\gamma > 0$ . Then, the equilibrium point of the system  $(S_4)$  is locally unstable for all sufficiently large values of  $\varepsilon > 0$ .
- (ii) Suppose that  $\theta \in [0, 1]$  is fixed at a value which is sufficiently close to zero, and  $\varepsilon$  is fixed at an arbitrary positive value. Then, the equilibrium point of the system  $(S_4)$  is locally unstable for all sufficiently large values of  $\gamma > 0$ .
- (iii) Suppose that  $\theta \in [0, 1]$  is fixed at a value which is sufficiently close to 1, and  $\gamma$  is fixed at an arbitrary positive value. Then, the equilibrium point of the system  $(S_4)$  is locally asymptotically stable for all sufficiently small values of  $\varepsilon > 0$ .
- (iv) Suppose that the equilibrium point of the four dimensional system  $(S_4)$  is locally asymptotically stable. Then, the behavior of the variable  $m$  also becomes locally stable in the sense that we have  $\dot{m}/m \rightarrow 0$  as  $(e, \pi^e, g) \rightarrow (\bar{e}, \mu_0 - n, n)$ .

(Proof.) See **Appendix A**.

**Proposition 2.**

Suppose that  $\theta \in [0, 1]$  is fixed at a value which is sufficiently close to 1, and  $\gamma$  is fixed at an arbitrary positive value. Then, there exist some non-constant closed orbits at some intermediate range of the parameter value  $\varepsilon > 0$ .



(Proof.) See **Appendix B**.

We can summarize these propositions as follows.

- (1) If the wage adjustment speed in the labor market ( $\varepsilon$ ) is sufficiently large, the equilibrium point of the system ( $S_4$ ) becomes unstable irrespective of the parameter values concerning the price expectations. In other words, *the price flexibility tends to destabilize the system*.
- (2) If the value of a parameter which reflects the credibility of the inflation targeting policy by the monetary authority ( $\theta$ ) is so small that the public's formation of price expectation is highly adaptive ( backward-looking ), the high speed of the expectation adjustment ( $\gamma$ ) tends to destabilize the system.
- (3) Suppose that the wage adjustment speed in the labor market ( $\varepsilon$ ) is not excessively large. Suppose, furthermore, that the inflation targeting policy by the monetary authority is so credible that the public's formation of price expectation is sufficiently forward-looking ( $\theta$  is sufficiently close to 1 ). Then, the equilibrium point of the system ( $S_4$ ) becomes locally stable even if the speed of the expectation adjustment ( $\gamma$ ) is very large. If the subsystem ( $S_4$ ) which consists of the variables  $d$ ,  $y$ ,  $e$ , and  $\pi^e$  is locally stable, then, the behavior of money-capital ratio ( $m$ ) becomes also locally stable.
- (4) Under certain conditions, endogenous cyclical fluctuation around the equilibrium point occurs at some intermediate range of the speed of wage adjustment ( $\varepsilon$ ).

In the above formal analysis, we only considered the special case of  $1/h_2 = 0$  ( $h_2 \rightarrow +\infty$ ), which corresponds to the case of so called 'liquidity trap'. In this case, the five dimensional dynamical system (32) becomes decomposable in the sense that the behavior of money-capital ratio ( $m$ ) does not affect the dynamics of the remaining four variables  $d$ ,  $y$ ,  $e$ , and  $\pi^e$ , although the behavior of  $m$  depends on the behavior of other four variables. In this special case, the value of the monetary policy parameter  $\delta$  is irrelevant to the qualitative dynamics of the system. This fact may be related to the alleged 'ineffectiveness'

of monetary policy in case of the liquidity trap. However, as we already noted, *the monetary policy becomes effective through another channel of the ‘credibility effect’ which influences publics’ formation of price expectation, even in case of the liquidity trap.*

How dynamics of the system are modified if we consider the general case of  $1/h_2 > 0$  ( $0 < h_2 < +\infty$ ) instead of the case of liquidity trap? In this case, the five dimensional system (32) is no longer decomposable, and the value of the monetary policy parameter  $\delta$  can affect the qualitative dynamics of the system through the changes of the nominal interest rate  $\rho$ . Without committing to the formal analysis, we can see that the increase of the policy parameter  $\delta$  has a stabilizing effect at least potentially in this case of variable nominal rate of interest, because of the following reason.

In case of the variable nominal interest rate, the following stabilizing negative feedback effect, which is called ‘Keynes effect’, will work.

$$(y \downarrow) \Rightarrow e \downarrow \Rightarrow \pi \downarrow \Rightarrow m = (M / pK) \uparrow \Rightarrow \rho \downarrow \Rightarrow (\rho - \pi^e) \downarrow \Rightarrow g \uparrow \Rightarrow (y \uparrow) \quad (\text{KE})$$

The increase of the monetary policy parameter  $\delta$  will reinforce the part  $\pi \downarrow \Rightarrow m \uparrow$  of this process through another feedback chain  $\pi \downarrow \Rightarrow \mu \uparrow \Rightarrow m \uparrow$ , so that the increase of  $\delta$  will have a stabilizing effect. However, this stabilizing ‘Keynes effect’ will be almost negligible if the sensitivity of the *nominal* rate of interest with respect to the changes of the money-capital ratio ( $1/h_2$ ) is very small. In fact, this will be the case if the *nominal* rate of interest is already nearly zero, as in the case of the Japanese economy in the late 1990s and the early 2000s. Needless to say, the liquidity trap approximates this particular case in which the Keynes effect is very weak.

## 5.A numerical illustration

In this section, we present some numerical examples which support our analysis. The purpose of this section is not to present the quantitatively realistic numerical analysis, but to illustrate the qualitative conclusion of the mathematical analysis in the previous section. We assume the following parameter values and the functional forms.

$$\begin{aligned}
s_f = s_r = 1, \quad \beta = 0.2, \quad i = \rho + d^2, \quad \rho = 0.01, \quad \bar{e} = 0.97, \quad \phi(g) = g, \\
n = 0.03, \quad \alpha = 0.2, \quad \gamma = 0.3, \quad \varepsilon = 0.1, \\
g = 0.1\{1.8y^{0.8} - (\rho - \pi^e) - 0.9d - 0.19\} + n = 0.18y^{0.8} + 0.1\pi^e - 0.09d + 0.01 \quad (51)
\end{aligned}$$

We interpret  $100n$ ,  $100\rho$ , and  $100\pi^e$  as the annual percentages of the natural rate of growth, the nominal rate of interest, and the expected rate of price inflation respectively. This example corresponds to the so called 'liquidity trap' in which the nominal rate of interest is stuck at its lower bound of the 1 percent annual rate. In this case, a system of equations (32) (i)–(v) becomes as follows.

$$\begin{aligned}
\text{(i)} \quad \dot{d} &= (0.18y^{0.8} + 0.1\pi^e - 0.09d + 0.01)(1-d) - 0.2y + 0.01d + d^3 \\
&\quad - 0.1(e - 0.97)d - \pi^e d \\
\text{(ii)} \quad \dot{y} &= 0.2(0.18y^{0.8} + 0.1\pi^e - 0.09d + 0.01 - 0.2y) \\
\text{(iii)} \quad \dot{e} &= e[0.2(0.18y^{0.8} + 0.1\pi^e - 0.09d + 0.01 - 0.2y) / y \\
&\quad + 0.18y^{0.8} + 0.1\pi^e - 0.09d - 0.02] \\
\text{(iv)} \quad \dot{\pi}^e &= 0.3\{\theta(\mu_0 - 0.03 - \pi^e) + (1-\theta)0.1(e - 0.97)\} ; \quad 0 \leq \theta \leq 1 \\
\text{(v)} \quad \dot{m} &= m[(1+\delta)\{\mu_0 - 0.1(e - 0.97) - \pi^e\} - 0.03\delta \\
&\quad - (0.18y^{0.8} + 0.1\pi^e - 0.09d + 0.01)] \quad (52)
\end{aligned}$$

First, let us consider the long run equilibrium solution. In this case, a system of equations (33) (i)–(v), which determines the long run equilibrium values, becomes as follows.

$$\begin{aligned}
\text{(i)} \quad 0.03 - 0.2y + 0.01d + d^3 - \mu_0 d &= 0 \\
\text{(ii)} \quad y &= 0.15 \\
\text{(iii)} \quad 1.8y^{0.8} + \pi^e - 0.9d - 0.2 &= 0 \\
\text{(iv)} \quad e &= 0.97 \\
\text{(v)} \quad \pi = \pi^e = \mu_0 - 0.03 & \quad (53)
\end{aligned}$$

Substituting  $y = 0.15$  and  $\pi^e = \mu_0 - 0.03$  into Eq. (53)(i) and (iii), we have the following set of equations.

$$\text{(i)} \quad 0.01d + d^3 - \mu_0 d = 0$$

$$( \quad \quad \quad \text{ii} \quad \quad \quad ) \quad \quad \quad \mu_0 - 0.9d + 0.164589 = 0 \quad (54)$$

This is a set of simultaneous equations which determines the equilibrium values of  $d$  and  $\mu_0$ . This means that the equilibrium rate of growth of money supply  $\mu_0$  cannot be given exogenously, but it becomes an endogenous variable in the special case of the 'liquidity trap'. In other words, the central bank must choose the 'correct' value of  $\mu_0$  to sustain the long run equilibrium in this case ( see also footnotes (7) and (8) ). On the other hand, in this case of the liquidity trap, the equilibrium value of the money-capital ratio  $m$  becomes indeterminate. In fact,  $m$  becomes a 'path dependent' variable in the sense that  $\lim_{t \rightarrow \infty} m(t)$  depends on the initial value  $m(0)$  even if

the long run equilibrium point is stable.

From Eq. (54)(i), we have

$$\mu_0 = 0.01 + d^2. \quad (55)$$

Substituting this expression into Eq. (54)(ii), we obtain

$$d^2 - 0.9d + 0.174589 = 0. \quad (56)$$

Solving this equation, we have the following multiple solutions.

$$d_1^* \cong 0.282934, \quad d_2^* \cong 0.617066 \quad (57)$$

Corresponding to the small equilibrium value of debt-capital ratio  $d_1^*$ , we have the following equilibrium values.

$$\mu_{01}^* \cong 0.090052, \quad \pi_1^* = \pi^e_{1^*} = \mu_{01}^* - 0.03 \cong 0.060052 \quad (57)$$

On the other hand, we have the following equilibrium values corresponding to the large equilibrium value  $d_2^*$ .

$$\mu_{02}^* \cong 0.39077, \quad \pi_2^* = \pi^e_{2^*} = \mu_{02}^* - 0.03 \cong 0.36077 \quad (58)$$

In the former equilibrium point, the annual rate of price inflation is about 6 percent, which is a believable value. On the other hand, in the latter equilibrium point, the annual rate of price inflation is unbelievably high as the rate of inflation in modern advanced capitalist countries such as the United States, Japan, and the Euroland. Therefore, we assume that the central bank selects  $\mu_{01}^*$  which supports the lower equilibrium value  $\pi_1^*$ . In this case, the equilibrium

values of the relevant variables become

$$d^* \cong 0.282934, \quad y^* = 0.15, \quad e^* = 0.97, \quad \pi^* = \pi^{e^*} \cong 0.060052 \quad (59)$$

corresponding to the 'properly' selected value  $\mu_0 \cong 0.090052$ .

Figures 1 – 5 are the results of our simulation of the 'out of equilibrium' dynamics corresponding to the following initial values.

$$d(0) = 0.15, \quad y(0) = 0.13, \quad e(0) = 0.94, \quad \pi^e(0) = 0 \quad (60)$$

Insert Fig. 1 – Fig. 5 here.

In these figures, the following three alternative scenarios are shown.

Case D (Debt deflation) :  $\theta = 0$  for all  $t \geq 0$

Case R (Reflation) :  $\theta = 0$  for  $0 \leq t < 15$ , and  $\theta = 1$  for  $t \geq 15$

Case S (Stagflation) :  $\theta = 0$  for  $0 \leq t < 15$ , and  $\theta = 0.5$  for  $t \geq 15$

where  $t$  denotes 'time', and the unit time interval is interpreted as a year.<sup>9</sup>

It is worth to note that the dynamical system (52) is a decomposable system, and Eq. (52)(v) does not affect the dynamic behavior of the variables  $d$ ,  $y$ ,  $e$ , and  $\pi^e$ . This means that the changes of the value of the policy parameter  $\delta$  can not affect the dynamic behavior of real debt, real income, employment, and rate of price change. As we noted previously, this fact corresponds to the alleged 'ineffectiveness' of monetary policy in case of the liquidity trap. However, we have another root of the effectiveness of monetary policy through the influence on the publics' expectation formation even in this case. Figures 1 – 5 show this fact clearly.

Case D is a typical example of the debt deflation in which the expectation formation is purely adaptive for all times. In this case, initial prosperity which is due to the relatively low initial debt-capital ratio automatically transforms to the serious depression through the rapid increase of the debt-capital ratio and the serious price deflation. The long run equilibrium with  $e^* = 0.97$  and  $d^* = 0.282934$  is strongly unstable in this case.

In case R, it is supposed that the drastic ‘regime switching’ from  $\theta=0$  to  $\theta=1$  occurs at the period  $t=15$  because of the believable change of the attitude of central bankers. In this case, the long run equilibrium point becomes stable and the economy recovers rapidly. In this case, the debt deflation is not triggered off, but the price rate of inflation begins to rise toward the equilibrium level  $\pi^*=0.060052$  soon after the regime switching. This is the reason why we call this case ‘reflation’.<sup>10</sup>

In case S, it is supposed that the incomplete regime switching from  $\theta=0$  to  $\theta=0.5$  occurs at the period  $t=15$ . This means that the publics only incompletely believe the announcement by the central bank. In this case, the long run equilibrium is still unstable and the depression process continues in spite of the fact that the rate of price inflation begins to increase soon after the period  $t=15$ . This is the reason why we call this case ‘stagflation’. Is this incomplete regime switching meaningless? It is not necessarily so, because the decline of the rate of employment and the increase of the debt-capital ratio become less rapid compared with the case D, so that the depression is mitigated considerably by this regime switching. In other words, the long run equilibrium is relatively weakly unstable in this case.

It must be noted that in our three examples the structure of the economy is the same *except* the value of only one parameter  $\theta$ , which governs the publics’ expectation formation of prices. This implies that the so called ‘structural reform’ of the economy, which has nothing to do with the appropriate changes of the price expectation formation, is by no means necessary condition for the economic recovery from debt deflation. Our findings apparently contradict the usual assertion by the ‘structural reformists’ in Japan, aside from the fact that the term ‘structural reform’ is rather vaguely used as a rhetoric and usually its content is not well-defined in their argument.

## 6. Concluding remarks

The main destabilizing positive feedback mechanism in our dynamic Keynesian model is the so called ‘Fisher debt effect’, which can be represented schematically as follows.<sup>11</sup>

$$(y \downarrow) \Rightarrow e \downarrow \Rightarrow \pi \downarrow \Rightarrow d = (D/pK) \uparrow \Rightarrow g \downarrow \Rightarrow (y \downarrow) \quad (\text{FDE})$$

The strength of this effect will depend on the sensitivity of the rate of investment with respect to the changes of the debt-capital ratio ( $|g_d|$ ) and the speed of the price adjustment ( $\varepsilon$ ). The larger these parameter values, the more strong will be the Fisher debt effect.

If the publics' formation of price expectation is strongly backward-looking (adaptive), another destabilizing positive feedback effect through the changes of the expected *real* rate of interest, which is called 'Mundell effect', will also work. This effect can be represented as follows.

$$(y \downarrow) \Rightarrow e \downarrow \Rightarrow \pi \downarrow \Rightarrow \pi^e \downarrow \Rightarrow (\rho - \pi^e) \uparrow \Rightarrow g \downarrow \Rightarrow (y \downarrow) \quad (\text{ME})$$

The increase of the speed of the adaptation of price expectation ( $\gamma$ ) will reinforce this process by reinforcing the part  $\pi \downarrow \Rightarrow \pi^e \downarrow$ .

On the other hand, we have a stabilizing negative feedback effect if the publics' formation of price expectation is strongly forward-looking due to the credibility of the inflation targeting policy by the monetary authority (central bank). Even if the causal chain  $(y \downarrow) \Rightarrow e \downarrow \Rightarrow \pi \downarrow \Rightarrow \pi^e \downarrow$  works in the early stage of the depression process, the counteracting stabilizing process which is represented by

$$\mu_0 - n > \pi^e \Rightarrow \pi^e \uparrow \Rightarrow (\rho - \pi^e) \downarrow \Rightarrow g \uparrow \Rightarrow (y \uparrow) \quad (\text{ITE})$$

will begin to operate if the inflation targeting policy becomes to be sufficiently credible so that the weight of the forward-looking expectation ( $\theta$ ) becomes to be sufficiently close to 1. We shall call this stabilizing effect 'Inflation targeting effect'.

Even if the stabilizing Keynes effect is very weak because of the downward rigidity of the nominal rate of interest at its nearly zero level, the monetary authority can transform the depression process into the prosperity by carrying out the sufficiently *credible* inflation targeting policy, as long as the destabilizing Fisher debt effect is not extremely

strong. Subtle factors such as publics' expectation and credibility or believability of the attitude of central bankers play crucial roles which govern the dynamic behavior of the macro economy. This is the main conclusion of the present paper.

### Acknowledgment

This research was financially supported by the Japan Ministry of Education, Culture, Sports, and Technology ( Grant-in-Aid for Scientific Research (R) 15330037 ) and Chuo University Joint Research Grant 0382.

### Appendix A : Proof of Proposition 1.

(i) Differentiating Eq.(44) with respect to  $\varepsilon$ , we have

$$\partial a_2 / \partial \varepsilon = d\bar{e} \left[ \underset{(-)}{\alpha G_{21}} / y + \underset{(-)}{g_d} \right] - \gamma(1-\theta)\bar{e} \left\{ \underset{(+)}{\alpha G_{24}} / y + g_{\rho-\pi} \right\} < 0 \quad (\text{A1})$$

for all  $\theta \in [0,1]$  and  $\gamma > 0$  because of assumptions 1 and 3. This means that  $a_2$  becomes a linear decreasing function of  $\varepsilon$ , so that we have  $a_2 < 0$  for all sufficient large values of  $\varepsilon > 0$ . In other words, one of the Routh-Hurwitz conditions for stable roots (50) is violated for all sufficiently large values of  $\varepsilon > 0$ .

(ii) Suppose that  $\theta = 0$ . Then, we have

$$\partial a_2 / \partial \gamma = -\varepsilon\bar{e} \left\{ \underset{(+)}{\alpha G_{21}} / y + g_{\rho-\pi} \right\} < 0 \quad (\text{A2})$$

for all  $\varepsilon > 0$ . This means that we have  $a_2 < 0$  for all sufficiently large values of  $\gamma > 0$  even if  $\theta > 0$  as far as  $\theta$  is close to zero.

(iii) Suppose that  $\theta = 1$ . Then, the characteristic equation (42) becomes as

$$\Delta_4(\lambda; 0) = |\lambda I - J_3(0)|(\lambda + \gamma) = 0 \quad (\text{A3})$$

where

$$J_3(0) = \begin{bmatrix} F_{11} & F_{12}(0) & -\varepsilon d \\ \alpha G_{21} & \alpha G_{22} & 0 \\ \bar{e}[\alpha G_{21} / y + g_d] & \bar{e}[\alpha G_{22} / y + \beta g_r] & 0 \end{bmatrix} \quad (\text{A4})$$

The characteristic equation (A3) has a negative real root  $\lambda_4 = -\gamma$ , and other three roots  $\lambda_j (j=1,2,3)$  are determined by the following



equation.

$$|\lambda I - J_3(0)| = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \quad (\text{A5})$$

where

$$b_1 = -\text{trace}J_3(0) = -(F_{11} + \alpha G_{22}) > 0, \quad (\text{A6})$$

$b_2 =$  sum of all principal second-order minors of  $J_3(0)$

$$\begin{aligned} &= \begin{vmatrix} \alpha G_{22} & 0 \\ \bar{e}[\alpha G_{22}/y + \beta g_r] & 0 \end{vmatrix} + \begin{vmatrix} F_{11} & -\varepsilon d \\ \bar{e}[\alpha G_{21}/y + g_d] & 0 \end{vmatrix} + \alpha \begin{vmatrix} F_{11} & F_{12}(0) \\ G_{21} & G_{22} \end{vmatrix} \\ &= \varepsilon d \{ \alpha G_{21}/y + g_d \} + \alpha (F_{11}G_{22} - F_{12}(0)G_{21}), \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} b_3 &= -\det J_3(0) = \varepsilon d \alpha \bar{e} \begin{vmatrix} G_{21} & G_{22} \\ \alpha G_{21}/y + g_d & \alpha G_{22}/y + \beta g_r \end{vmatrix} \\ &= \varepsilon d \alpha \bar{e} (G_{21}\beta g_r - G_{22}g_d) > 0. \end{aligned} \quad (\text{A8})$$

From these expressions, we obtain

$$\lim_{\varepsilon \rightarrow 0} b_2 = \alpha (F_{11}G_{22} - F_{12}(0)G_{21}) > 0, \quad (\text{A9})$$

$$\lim_{\varepsilon \rightarrow 0} (b_1 b_2 - b_3) = -\alpha (F_{11} + \alpha G_{22})(F_{11}G_{22} - F_{12}(0)G_{21}) > 0. \quad (\text{A10})$$

These inequalities imply that all of the Routh-Hurwitz conditions for stable roots of Eq. (A5) ( $b_1 > 0$ ,  $b_3 > 0$ ,  $b_1 b_2 - b_3 > 0$ ) are satisfied for all sufficiently small  $\varepsilon > 0$  if  $\theta = 1$ . This means, by continuity, that all of the Routh-Hurwitz conditions for stable roots of the four dimensional system ( $S_4$ ) are in fact satisfied for all sufficiently small  $\varepsilon > 0$  even if  $\theta < 1$ , as long as  $\theta$  is sufficiently close to 1.

(iv) If we substitute  $e = \bar{e}$ ,  $\pi^e = \mu_0 - n$ , and  $g = n$  into Eq. (32) (v), we have  $\dot{m}/m = 0$ . This implies that we have  $\dot{m}/m \rightarrow 0$  in case of  $(e, \pi^e, g) \rightarrow (\bar{e}, \mu_0 - n, n)$ .  $\square$

## Appendix B : Proof of Proposition 2.

Suppose that the premises concerning the parameter values  $\theta$  and  $\gamma$  are satisfied. Then, it follows from **Proposition 1** (i) and (iii) that the equilibrium point of the system ( $S_4$ ) is locally asymptotically stable for all sufficiently small values of  $\varepsilon > 0$ , and it is locally unstable for all sufficiently large values of  $\varepsilon > 0$ . Therefore, by continuity, there exists at least one ‘bifurcation point’ at which the local stability of the equilibrium point is lost as the parameter value  $\varepsilon$  increases. At such a bifurcation point, the characteristic equation (42) has at least one root with zero real part. By the way, from Eq.(46) we have

$$\lim_{\theta \rightarrow 1} a_4 = \lim_{\theta \rightarrow 1} (\lambda_1 \lambda_2 \lambda_3 \lambda_4) = \alpha \gamma \bar{\varepsilon} d (G_{21} \beta g_r - G_{22} g_d) > 0 \quad (\text{B1})$$

so that, by continuity, we have

$$\lambda_1 \lambda_2 \lambda_3 \lambda_4 > 0 \quad (\text{B2})$$

if  $\theta \in [0,1]$  is sufficiently close to 1, where  $\lambda_j (j=1,2,3,4)$  are four roots of the characteristic equation (42). The inequality (B2) means that the characteristic equation (42) does not have a root such that  $\lambda = 0$ . In this case, Eq.(42) must have a pair of pure imaginary roots

$$\lambda_1 = i\omega, \quad \lambda_2 = -i\omega \quad (i = \sqrt{-1}, \quad \omega > 0) \quad (\text{B3})$$

at the bifurcation point. Substituting Eq. (B3) into the inequality (B2), we obtain

$$\omega^2 \lambda_3 \lambda_4 > 0 \quad (\text{B4})$$

at the bifurcation point. On the other hand, it follows from the proof of **Proposition 1** (iii) that the characteristic equation (42) has a negative real root  $\lambda_4 = -\gamma$  when  $\theta = 1$ . This means that Eq. (42) has a negative real root  $\lambda_4 < 0$  even if  $\theta < 1$  as long as  $\theta$  is sufficiently close to 1. In this case, the remaining root  $\lambda_3$  also becomes real and negative from the inequality (B4).

In sum, the characteristic equation (42) has a set of pure imaginary roots and two negative real roots at the bifurcation point. Furthermore, the imaginary part of a pair of complex roots increases as the parameter  $\varepsilon$  increases passing through the bifurcation parameter value  $\varepsilon_0$ , because of the loss of stability. This means that the bifurcation

point in this case is in fact the Hopf bifurcation point, and we can apply Hopf bifurcation theorem to establish the existence of the closed orbits at some range of the parameter value  $\varepsilon > 0$  which are sufficiently close to the bifurcation value  $\varepsilon_0$  ( cf. Gandolfo(1996) Chap. 25 ).<sup>12</sup> □

### Notes

- (1) We can expect that in a ‘normal’ situation, corporate sector as a whole is a debtor and capitalists as a whole is a creditor. In this case, we have  $D > 0$ . However, the case  $D < 0$  is also possible.
- (2) In the formulae (12) – (16), income tax and government expenditure are neglected. We can introduce, however, these factors without changing the qualitative behavior of the model, at the cost of the complication of the notation, by assuming that the tax rates and the real government expenditure per capital stock are constant.
- (3) This type of expectation formation was studied by Asada, Chiarella, Flaschel, and Franke(2003).
- (4) This formulation is due to Asada, Chiarella, Flaschel, and Franke(2003).
- (5) Needless to say, we must suppose  $\rho = \rho_0$  in case of  $h_1 y - m < 0$ .

This corresponds to the case of the so called ‘liquidity trap’.

- (6) This system is in fact an extended version of the model which was presented by Asada(2004). In Asada(2004), simpler three dimensional model was studied.
- (7) We can show that in the special case of  $1/h_2 = 0 (h_2 \rightarrow +\infty)$ , monetary authority has no freedom for the choice of  $\mu_0$ . In this case, there is just one ‘correct’  $\mu_0$  which is consistent with the steady state with natural rate of growth. This case corresponds to the so called ‘liquidity trap’ in which the nominal rate of interest becomes insensitive to the changes of  $y$  and  $m$ .
- (8) Obviously, it is implicitly assumed that the monetary authority chooses the ‘correct’ value of  $\mu_0$  which is consistent with the existence of the long run equilibrium with natural rate of growth.
- (9) We adopted Euler’s algorithm and the time interval  $\Delta t = 0.1$  (years) for numerical simulations.

- (10) In this numerical simulation, the nominal rate of interest is fixed at its lower bound  $\rho_0 = 0.01$  for all time. But, in the real economy, the nominal rate of interest will begin to increase at the late stage of economic recovery. In this case, the speed of recovery will become less rapid at the late stage of economic recovery. However, the qualitative dynamics will not change seriously even if we introduce this effect explicitly.
- (11) Needless to say, this name is associated with Fisher(1933)'s classical paper on debt deflation.
- (12) For more elaborated treatment of the four dimensional Hopf bifurcations which are described here, see Asada and Yoshida(2003) and the mathematical appendix of Asada, Chiarella, Flaschel and Franke(2003).

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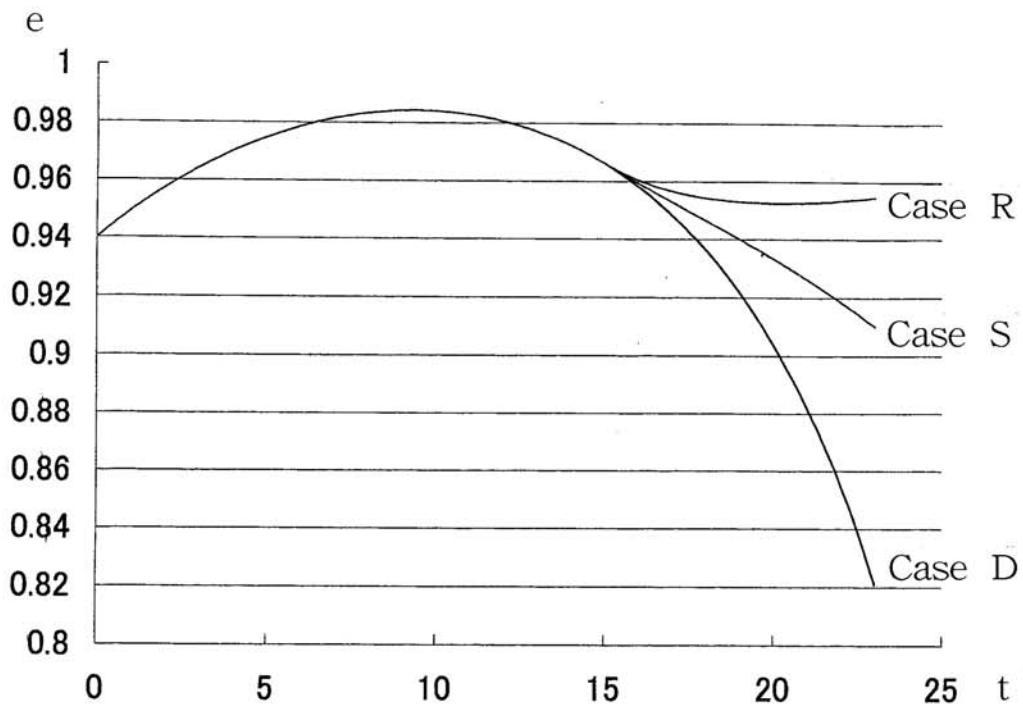


Fig. 1. Alternative time paths of e

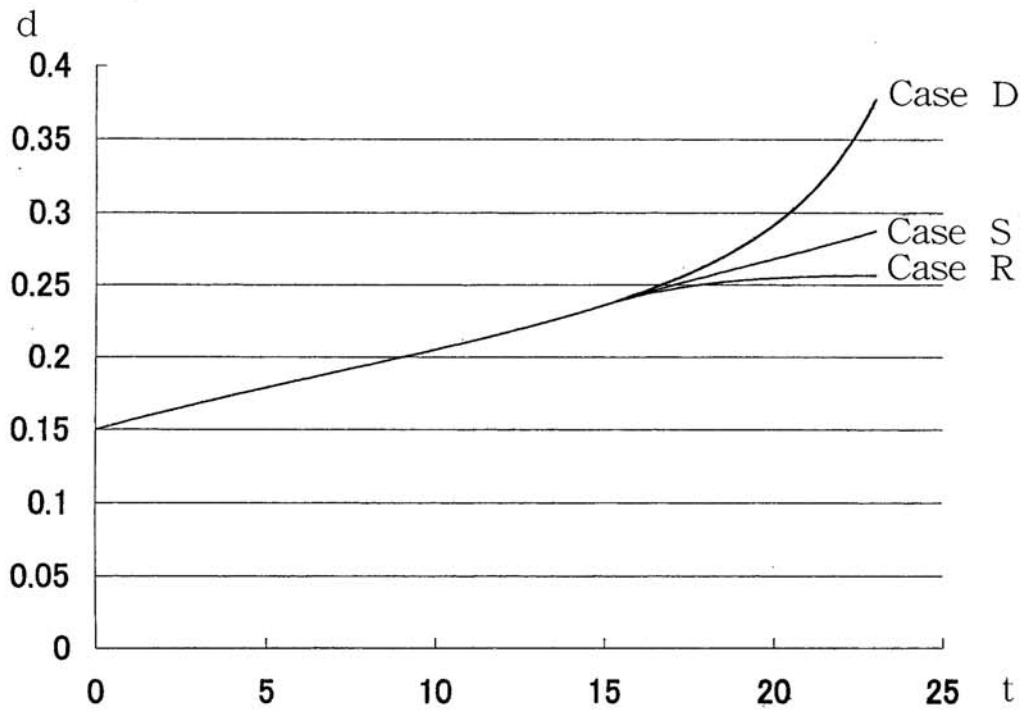


Fig. 2. Alternative time paths of d

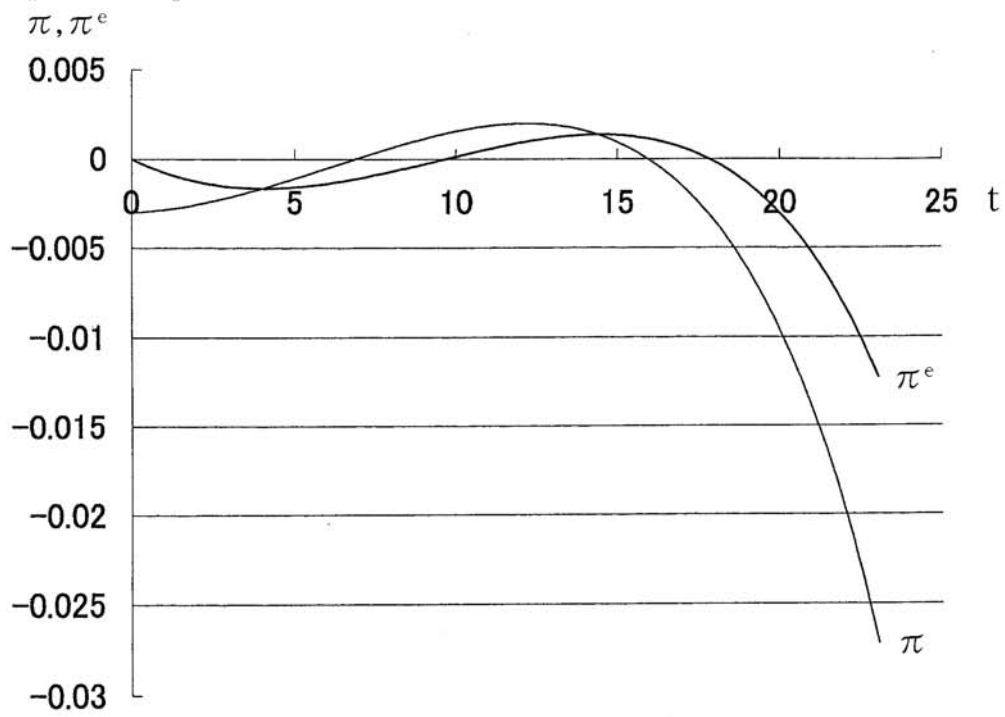


Fig. 3. Time paths of  $\pi$  and  $\pi^e$  (Case D)

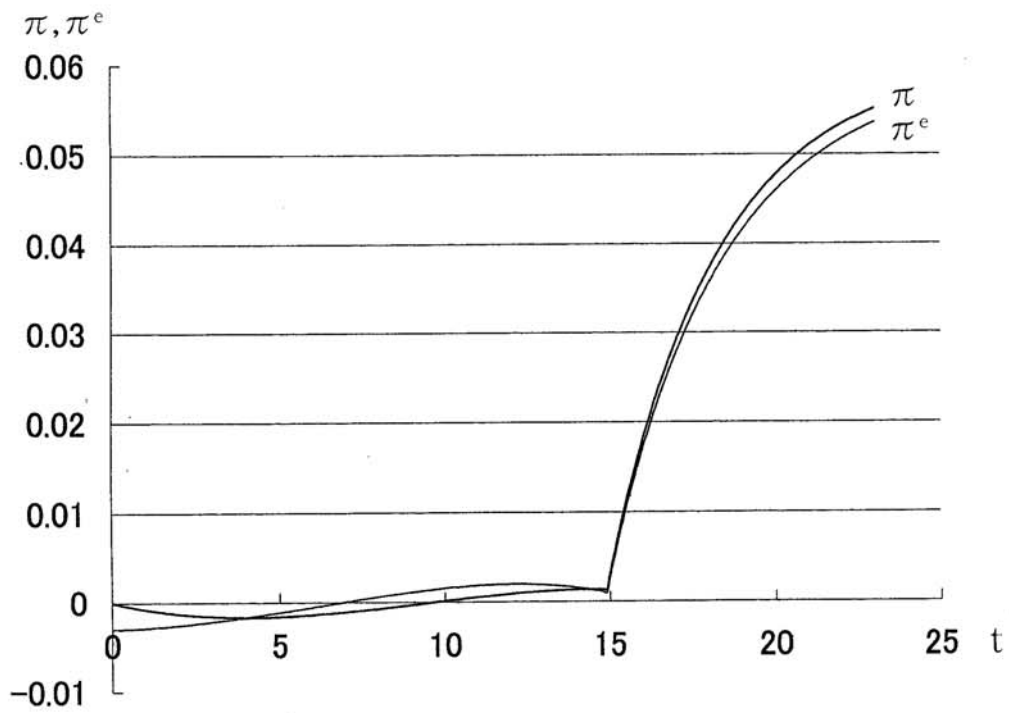


Fig. 4. Time paths of  $\pi$  and  $\pi^e$  (Case R)

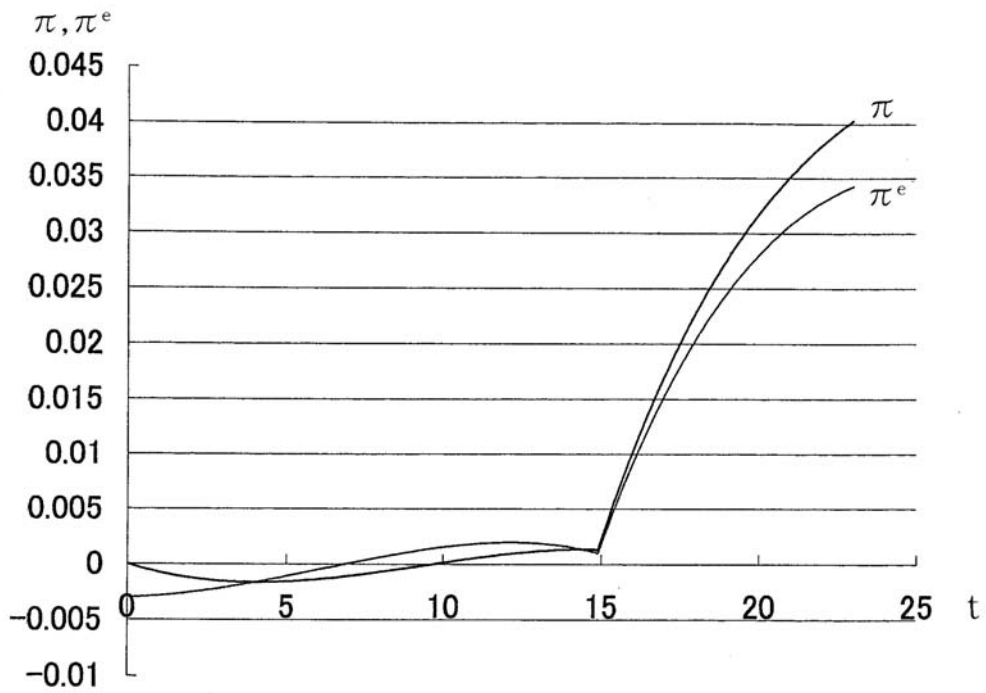


Fig. 5. Time paths of  $\pi$  and  $\pi^e$  (Case S)