Delay Differential Nonlinear Economic Models*

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Abstract

Dynamic delay economic models are compared with fixed and continuously distributed information lags. With small delays and exponentially decreasing kernel functions, the two types of models generate identical local asymptotic behavior. In the case of large delays the asymptotic properties however become different. Three particular economic models (the business cycle model of Goodwin, Kaldorian macro dynamic model augumented with Kaleckian investment lag and the Cournot oligopoly model) are used to illustrate these theoretical results and computer simulation examples illustrate that with larger delays more complex dynamics may emerge.

1 Introduction

The asymptotical behavior of dynamic economic systems has been the focus of a large number of studies with both discrete and continuous time scales. They are based on the qualitative theory of difference or ordinary differential equations (Bellman (1969) and Goldberg (1958)). It has been shown by many authors that the introduction of information delay into the dynamic models significantly changes their asymptotical properties. There is a significant difference between models with fixed time lags and models with continuously distributed delays. In the first case there is an infinite spectrum, and in the second case with gamma-function type kernel functions, the spectrum is finite. An important special case of continuously distributed time lags is given by exponentially decreasing kernel functions.

In this paper we compare dynamics generated by fixed time lags and continuously distributed delay with exponential kernel function. We will first show that these two types of models generate the same local dynamics if the delay is sufficiently small. This is, however, not true if the delay becomes large.

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The theoretical findings are illustrated by three well known economic models: the Goodwin model, the Kaldor-Kalecki model and Cournot oligopoly.

This paper is organized as follows. Section 2 introduces the main mathematical results, and the particular models are discussed in Section 3. Conclusions are drawn in Section 4.

2 Mathematical Result

Consider first the general linear differential-difference equation

$$\sum_{k=0}^{n} \alpha_k y^{(k)}(t) + \sum_{k=0}^{n} \beta_k y^{(k)}(t+\theta) = 0 \tag{1}$$

with a single delay θ, where

$$y^{(k)}(t) = \frac{d^k}{dt^k} y(t) \text{ and } y^{(k)}(t+\theta) = \frac{d^k}{dt^k} y(t+\theta).$$

Assuming small θ, linearization with respect to θ gives the approximation

$$\left(\sum_{k=0}^{n} \alpha_k y^{(k)}(t) + \sum_{k=0}^{n} \beta_k y^{(k)}(t) \right) + \left(\sum_{k=0}^{n} \beta_k y^{(k+1)}(t) \right) \theta = 0.$$

This is a linear homogeneous equation. As usual, looking for the solution in an exponential form, $y(t) = ve^{\lambda t}$ gives

$$\sum_{k=0}^{n} (\alpha_k + \beta_k) \lambda^k e^{\lambda t} v + \sum_{k=0}^{n} \beta_k \lambda^{k+1} e^{\lambda t} v\theta = 0,$$

and after simplification the characteristic polynomial of the system becomes

$$\sum_{k=0}^{n} \alpha_k \lambda^k + \left(\sum_{k=0}^{n} \beta_k \lambda^k \right) (1 + \lambda\theta) = 0, \tag{2}$$

which is a polynomial of degree $k+1$ in λ.

Consider next the equivalent delayed equation,

$$\sum_{k=0}^{n} \alpha_k y^{(k)}(t-s) + \sum_{k=0}^{n} \beta_k y^{(k)}(t) = 0. \tag{3}$$

Assuming continuously distributed lag with exponential kernel function,

$$w(t-s) = \frac{1}{\theta} e^{-\frac{t-s}{\theta}}$$

and taking delay expectation, a Volterra-type integro-differential equation is obtained:

$$\int_0^t w(t-s) \sum_{k=0}^{n} \alpha_k y^{(k)}(s) ds + \sum_{k=0}^{n} \beta_k y^{(k)}(t) = 0. \tag{4}$$

In the first factor we can introduce the new variable $z = t - s$ to have

$$\int_0^t w(z) \sum_{k=0}^{n} \alpha_k y^{(k)}(t-z)dz + \sum_{k=0}^{n} \beta_k y^{(k)}(t) = 0.$$

If we seek the solution in the usual exponential form, $y(t) = ve^{\lambda t}$ and substitute it into the above equation, we get

$$\int_0^t \frac{1}{\theta} e^{-\frac{z}{\theta}} \sum_{k=0}^{n} \alpha_k \lambda^k e^{\lambda(t-z)} v dz + \sum_{k=0}^{n} \beta_k \lambda^k e^{\lambda t} v = 0.$$

By dividing both sides by $e^{\lambda t} v$ and letting $t \to \infty$ we have a simplified expression for the first term:

$$\begin{aligned} \int_0^\infty \frac{1}{\theta} e^{-z(\lambda+\frac{1}{\theta})} dz \textstyle\sum_{k=0}^{n} \alpha_k \lambda^k &= \frac{1}{\theta} \left. \frac{e^{-z(\lambda+\frac{1}{\theta})}}{-(\lambda+\frac{1}{\theta})} \right]_{z=0}^{\infty} \textstyle\sum_{k=0}^{n} \alpha_k \lambda^k \\ &= \frac{1}{\lambda\theta + 1} \textstyle\sum_{k=0}^{n} \alpha_k \lambda^k, \end{aligned}$$

so the equation further simplifies as

$$\frac{1}{\lambda\theta + 1} \sum_{k=0}^{n} \alpha_k \lambda^k + \sum_{k=0}^{n} \beta_k \lambda^k = 0, \tag{5}$$

which is equivalent to equation (2). Therefore the local asymptotic behavior of the two dynamics is identical. We summarize this result:

Theorem 1 *Local dynamics generated by the general delay differential equation with a single and small delay is the same as the dynamics by the general differential equation with continuously distributed time lag with exponential kernel function.*

In the case of the general kernel function

$$w(t-s) = \frac{1}{n!} \left(\frac{n}{\theta}\right)^{n+1} (t-s)^n e^{-\frac{n(t-s)}{\theta}},$$

we know that as $\theta \to \infty$ or $n \to \infty$, the function converges to the Dirac-delta function centered at $t - s = 0$ and $t - s = \theta$, respectively. Therefore, in this limiting case the integro-differerntial equation (4) converges to the deterministic case with fixed delay. It is very interesting that in the exponential kernel function ($n = 0$) case, the two processes are even equivalent concerning the local behavior of the equilibrium. This is not true however for larger values of n, as it is demonstrated in Matsumoto and Szidarovszky (2009).

3 Economic Examples

We confirm Theorem 1 by examining various delay economic models when the time delay is small and invesigate the global dynamics of the delay models with continuously distributed time delay when the time delay is large.

3.1 Goodwin Model with Investment Lag

Goodwin (1951) constructed a business cycle model with nonlinear acceleration principle of investment and showed that the model gives rise to cyclic oscillations when its stationary state is locally unstable. Goodwin's basic model is summarized as a 1D nonlinear differential equation,

$$\varepsilon\dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0,$$

where a time variable y is national income, α the marginal propensity to consume, which is a positive constant and less than unity, ε a positive adjustment coefficient of y and $\varphi(\dot{y}(t))$ denotes the induced investment that is dependent on the rate of change in national income. The dot stands for differentiation with respect to time t. Goodwin's model adopts the nonlinear acceleration principle, according to which investment is proportional to the change in national income in a neighborhood of the equilibrium income but becomes inflexible for the extremely larger or smaller values of income.

"In order to come close to reality" (p.11 of Goodwin (1951)), the production lag θ between decisions to invest and the corresponding outlays is introduced into the above model and then the modified model becomes

$$\varepsilon\dot{y}(t) - \varphi(\dot{y}(t-\theta)) + (1-\alpha)y(t) = 0. \tag{6}$$

This is a *neutral delay nonlinear differential equation* in which θ is the fixed time lag. Since it is difficult to analytically solve this delay nonlinear model, it is a natural way to use a tractable approximation of (6). In particular, to investigate dynamics, we rewrite the equation as

$$\varepsilon\dot{y}(t+\theta) - \varphi(\dot{y}(t)) + (1-\alpha)y(t+\theta) = 0,$$

and expands it with respect to θ around $\theta = 0$ to obtain the following second-order nonlinear differential equation:

$$\varepsilon\theta\ddot{y}(t) + [\varepsilon + (1-\alpha)\theta]\dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0.$$

Clearly, $y(t) = 0$ for all t is a stationary state of this equation. Its asymptotic behavior is determined by the eigenvalues, which are the solutions of the characteristic equation,

$$\varepsilon\theta\lambda^2 + [\varepsilon + (1-\alpha)\theta - \upsilon]\lambda + (1-\alpha) = 0, \tag{7}$$

where $\upsilon = \varphi'(0)$. The characteristic roots are

$$\lambda_{1,2} = \frac{-k \pm \sqrt{k_2 - 4\varepsilon\theta(1-\alpha)}}{2\varepsilon\theta}$$

where $k = \varepsilon + (1-\alpha)\theta - \upsilon$. It follows that the product of the characteristic roots is positive since $0 < \alpha < 1$ and both ε and θ are positive:

$$\lambda_1 \lambda_2 = \frac{1-\alpha}{\varepsilon\theta} > 0,$$

which excludes the possibility of saddle stationary point. It also follows that the sum of the characteristic roots can be of either sign,

$$\lambda_1 + \lambda_2 = -\frac{\varepsilon + (1-\alpha)\theta - \upsilon}{\varepsilon\theta} \gtreqless 0.$$

Given the values of α and ε, the indeterminacy of the sign of the last equation means that the (υ, θ)-space is divided into two parts by the partition line

$$\upsilon = \varepsilon + (1-\alpha)\theta.$$

For all υ above this line, the sum of the characteristic roots is positive, hence the stationary state is locally unstable. In the same way, the stationary state is locally asymptotically stable for all υ below this line.

Continuously distributed time delay is an alternative approach to deal with time delay in investment. If we adopt it and denote the expected change of national income at time t by $\dot{y}^e(t)$, then Goodwin's delayed equation (6) can be written as the system of Volterra-type integro-differential equations:

$$\begin{cases} \varepsilon\dot{y}(t) - \varphi(\dot{y}^e(t)) + (1-\alpha)y(t) = 0, \\ \dot{y}^e(t) = \int_0^t \frac{1}{\theta} e^{-\frac{t-s}{\theta}} \dot{y}(s)ds, \end{cases} \tag{8}$$

where θ is a positive real parameter which is associated with the length of the delay. The second equation of (8) shows that the weighting function of the past changes in national income gives the most weight to the most recent income change and the weight is exponentially declining afterwards. Before turning to a closer examination of this model, we rewrite it as a system of ordinary differential equations. The time-differentiation of the second equation of (8) gives a simple equation for the new variable $z = \dot{y}^e$:

$$\dot{z}(t) = \frac{1}{\theta}\left(\dot{y}(t) - z(t)\right). \tag{9}$$

Solving the first equation for $\dot{y}$, replacing $\dot{y}^e$ with z, replacing $\dot{y}$ in (9) with the new expression of $\dot{y}$ and then adding the new dynamic equation of z will transform the system of the integro-differential equations to the following 2D system of ordinary differential equations:

$$\begin{cases} \dot{y}(t) = -\frac{1-\alpha}{\varepsilon} y(t) + \frac{1}{\varepsilon}\varphi(z(t)), \\ \dot{z}(t) = \frac{1}{\theta}\left(-\frac{1-\alpha}{\varepsilon} y(t) + \frac{1}{\varepsilon}\varphi(z(t)) - z(t)\right). \end{cases} \tag{10}$$

The Jacobian matrix of this system at $y = z = 0$ has the form

$$\mathbf{J_G} = \begin{pmatrix} -\dfrac{1-\alpha}{\varepsilon} & \dfrac{\nu}{\varepsilon} \\ \\ -\dfrac{1-\alpha}{\varepsilon\theta} & \dfrac{1}{\theta}\left(\dfrac{\nu}{\varepsilon}-1\right) \end{pmatrix}. \tag{11}$$

The corresponding characteristic equation is quadratic in λ:

$$\lambda^2 + \frac{\varepsilon + (1-\alpha)\theta - \nu}{\varepsilon\theta}\lambda + \frac{1-\alpha}{\varepsilon\theta} = 0.$$

Notice that this characteristic equation is equivalent to the characteristic equation (7). It follows that the local stability conditions are also identical. This means that the two delay dynamic systems generate the same dynamics in the neighborhood of $\theta = 0$.

We now turn our attention to the dynamics of (8) when θ is large. It is well-known that the Goodwin model generates a limit cycle when its stationary point is locally unstable. Goodwin (1951) assumed a piecewise linear investment function in his simulations. We numerically confirm his result but for the sake of analytical convenience, we assume a hyperbolic tangent type investment function:

$$\varphi(\dot{y}) = \delta\left(\tanh(\dot{y} - a) - \tanh(-a)\right), \delta > 0 \text{ and } a = 1. \tag{12}$$

We perform numerical simulations with the parameter values $\varepsilon = 0.5$ and $\alpha = 0.6$ as in Goodwin (1951). To make the stationary point locally unstable, we take $\theta = 0.8$ and $\delta = (1 + a^2)(\varepsilon + (1-\alpha)\theta) + 0.01$. The numerical result is illustrated in Figure 1 in which two trajectories, one continuous line starting at point a and the other dotted line at point b, are seen to converge to the limit cycle.

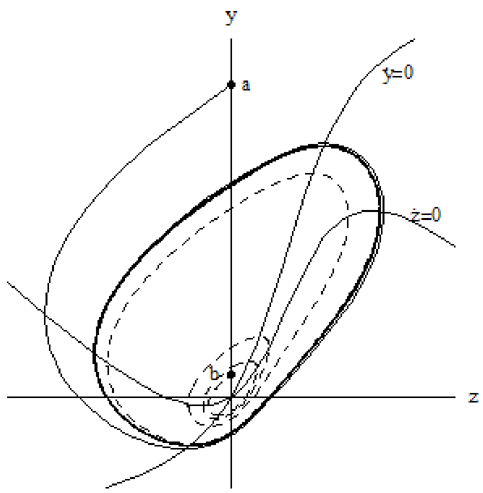

Figure 1. Existence of a stable limit cycle

Recently, Matsumoto (2009) reexamined Goodwin's model and showed the coexistence of multiple limit cycles, a stable cycle surrounding a unstable cycle when the stationary state is locally stable. This is illustrated in Figure 2 in which there are two limit cycles depicted as bold curves and the two trajectories starting at points a and b converge to the outer limit cycle whereas a trajectory starting at point c approaches the stable stationary point. A parametric difference between the first simulation and the second simulation is that only the value of δ is changed to $(1+a^2)(\varepsilon+(1-\alpha)\theta)-0.01$ from $(1+a^2)(\varepsilon+(1-\alpha)\theta)+0.01$.

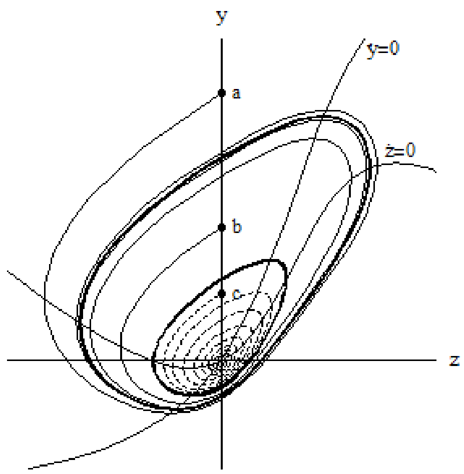

Figure 2. Co-existence of a stable and an unstable limit cycle

3.2 Kaldor-Kalecki Model with Investment Lag

Kaldor (1940) presented a business cycle model in which investment was positively related to the levels of income via a nonlinear relationship. Kalecki (1935) added a lag between the investment decision and the installation of investment goods. His model used a linear difference-differential equation to generate cyclic dynamics. The Kaldor-Kalecki model is a combination of nonlinear investment and a time lag in the capital accumulation. Let Y be the national income and K the capital stock. Then the Kaldor-Kalecki model can be written as

$$\begin{cases} \dot{Y}(t) = \alpha\left[I(Y(t), K(t)) - S(Y(t))\right], \\ \\ \dot{K}(t) = I(Y(t-\theta), K(t)) - \delta K(t), \end{cases} \tag{13}$$

where $I(Y,K)$ is an investment function and $S(Y)$ is the saving function. Investment depends positively on income and negatively on capital, so $dI/dY = I_Y > 0$ and $dI/dK = I_K < 0$. Furthermore, it takes a S-shaped profile with respect to Y indicating that investment becomes inflexible for low as well as high levels of income. Savings depends on income in the usual way, i.e., $0 < dS/dY = S_Y < 1$. We assume also that $I_Y - S_Y > 0$ at the fixed point of (13), that is, investment increases faster than savings as national income increase in a neighborhood of the fixed point, following Kaldor. In addition, $\alpha > 0$ is the adjustment coefficient and $\delta > 0$ is the depreciation rate of the capital. The first equation of (13) states that income changes proportionally to the excess demand in the goods market. The second equation is a standard capital accumulation equation but includes a time lag θ.

Consider first the local stability of (13) without time delay (i.e., $\theta = 0$), which is equivalent to the original Kaldor model. The Jacobian matrix has the form

$$\mathbf{J_K} = \begin{pmatrix} \alpha(I_Y - S_Y) & \alpha I_K \\ I_Y & I_K - \delta \end{pmatrix}$$

with the determinant

$$det\mathbf{J_K} = \alpha(I_Y - S_Y)(I_K - \delta) - \alpha I_K I_Y$$

and the trace

$$tr\mathbf{J_K} = \alpha(I_Y - S_Y) + (I_K - \delta).$$

Kaldor (1940) made two basic assumptions: $det\mathbf{J_K} > 0$ in order to exclude the possibility that a stationary point is saddle and $tr\mathbf{J_K} < 0$ to make the stationary point unstable. As seen in Chang and Smyth (1971), the gist of Kaldor's argument can be translated to show an existence of an endogenously persistent fluctuation by applying the Poincaŕe-Bendixson theorem. For this end, the local instability of the stationary point is the first requirement. Figure 3 illustrates the birth of a Kaldorian limit cycle with the following configuration of the model: The investment function is separable with respect to Y and K,

$$I(Y, K) = \phi(Y) + \beta K, \ \beta < 0,$$

where $\phi(Y)$ is assumed to be a symmetric S-shaped function,

$$\phi(Y) = \frac{A}{1 + e^{-BY}} - \frac{A}{2}, \ A > 0 \text{ and } B > 0,$$

and the parameters are specified as $A = 4$, $B = 1$, $c = 0.6$, $\alpha = 0.8$, $\beta = -0.2$ and $\delta = 0.05$. It can be seen that the limit cycle attracts two different trajectories, one starting at point a and the other starting at point b in the neighborhood of the stationary point.

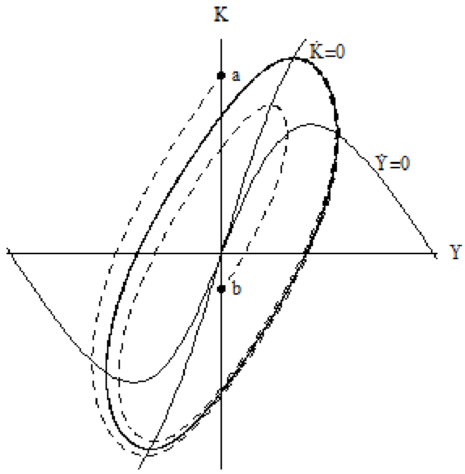

Figure 3. Existence of a Kaldorian limit cycle

Although we numerically confirm the existence of the Kaldorian limit cycle when the stationary point is locally unstable, we are interested in the destabilizing effect caused by a delay in investment so that the stationary point becomes asymptotically stable when $tr\mathbf{J_K} < 0$. In Figure 4, two trajectories belonging to the two different initial points a and b spiral toward the stationary point when $\beta = -0.4$ and $\delta = 0.2$ with the other parameters being unchanged.

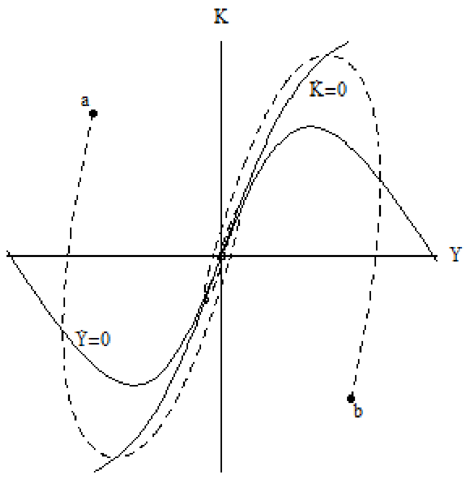

Figure 4, A stable Kaldorian stationary point

Now we are back to the delay Kaldor-Kalecki model (13). We first rewrite the capital accumulation equation as

$$\dot{K}(t+\theta) = I(Y(t), K(t+\theta)) - \delta K(t+\theta).$$

If the time lag is small enough, then linearizing it with respect to θ around $\theta = 0$ gives

$$\dot{K}(t) - \{I(Y(t), K(t)) - \delta K(t)\} + \left\{\ddot{K}(t) - I_K \dot{K}(t) + \delta \dot{K}(t)\right\}\theta = 0.$$

Introducing the new variable, $Z(t) = \dot{K}(t)$, the delayed Kaldor-Kalecki model is reduced to a 3D system of ordinary differential equations:

$$\begin{cases} \dot{Y}(t) & = \alpha(\ I(Y(t), K(t)) - S(Y(t))\ ) \\ \dot{K}(t) & = Z(t) \\ \dot{Z}(t) & = \dfrac{1}{\theta}\{I(Y(t), K(t)) - \delta K(t)\} + \{(I_K - \delta) - \dfrac{1}{\theta}\}Z(t). \end{cases} \tag{14}$$

The Jacobian matrix is

$$\mathbf{J_D} = \begin{pmatrix} \alpha(I_Y - S_Y) & \alpha I_K & 0 \\ 0 & 0 & 1 \\ \dfrac{1}{\theta} I_Y & \dfrac{1}{\theta}(I_K - \delta) & (I_K - \delta) - \dfrac{1}{\theta} \end{pmatrix}$$

with the determinant,

$$det\mathbf{J_D} = -\frac{det\mathbf{J_K}}{\theta} < 0$$

and the trace

$$tr\mathbf{J_D} = tr\mathbf{J_K} - \frac{1}{\theta} < 0,$$

where the inequalities are due to the assumptions $det\mathbf{J_K} > 0$ and $tr\mathbf{J_K} < 0$ in the Kaldor model. The characteristic equation of $\mathbf{J_D}$ is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{15}$$

where the coefficients are

$$\begin{aligned} a_1 &= -tr\mathbf{J_D} > 0, \\ a_2 &= \alpha(I_Y - S_Y)(I_K - \delta) - \frac{1}{\theta}\left(\alpha(I_Y - S_Y) + (I_K - \delta)\right), \\ a_3 &= -det\mathbf{J_D} > 0. \end{aligned}$$

If we assume continuously distributed time lag in the capital accumulation process, then $Y(t-\theta)$ is replaced by the expected income $Y^e(t)$, which is defined as the weighted average of the past realized incomes from zero to time t,

$$Y^e(t) = \int_0^t \frac{1}{\theta} e^{-\frac{t-s}{\theta}} Y(s) ds.$$

The delay 2D Kaldor-Kalecki model (13) can be reduced to a 3D system of ordinary differential equations:

$$\begin{cases} \dot{Y}(t) = \alpha \left[I(Y(t), K(t)) - S(Y(t)) \right], \\ \dot{K}(t) = I(Y^e(t), K(t)) - \delta K(t), \\ \dot{Y}^e(t) = \dfrac{1}{\theta} \left(Y(t) - Y^e(t) \right), \end{cases} \tag{16}$$

where the last equation is obtained by time differentiation of $Y^e(t)$. The Jacobian matrix at the stationary point is

$$\mathbf{J_C} = \begin{pmatrix} \alpha(I_Y - S_Y) & \alpha I_K & 0 \\ 0 & I_K - \delta & I_Y \\ \dfrac{1}{\theta} & 0 & -\dfrac{1}{\theta} \end{pmatrix}.$$

It can be easily checked that the Jacobian matrix $\mathbf{J_C}$ has the same characteristic equation as (15). Hence two different dynamic systems (14) and (16) generate the same dynamics in a neighborhood of the stationary point if θ is sufficiently small. According to the Routh-Hurwitz stability criterion, a necessary and sufficient condition that all roots of the cubic characteristic equation (15) have negative real parts is that $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 - a_3 > 0$. Notice that $a_1 > 0$ and $a_3 > 0$ are already shown to be positive due to Kaldor's assumptions. For sufficiently small θ, a_2 could be positive because its second term $-tr\mathbf{J_K}/\theta > 0$ is positive and can dominate the first term. By the same token $a_1 a_2 - a_3$ can be positive for a small θ. Hence it is safe to presume that the delay Kaldor-Kalecki system is stable when the investment delay is sufficiently small. Since a small θ means a small lag effect, this result is reasonable under the assumption that the original Kaldor model is stable as shown in Figure 4. The next question which we raise is whether or not the stability of the stationary state changes as the lengths of delays increase. We consider this question in model (16) only, since (14) is inappropriate for a large θ.

Now we turn our attention to the dynamic behavior of the delay Kaldor-Kalecki model with a large θ. As seen above, the coefficients a_1 and a_3 of the characteristic equation are positive. However, the sign of a_2 is not determined. Solving $a_2 = 0$ for θ yields the critical value of θ,

$$\theta_2 = \frac{\alpha(I_Y - S_Y) + (I_K - \delta)}{\alpha(I_Y - S_Y)(I_K - \delta)} > 0$$

implying that a_2 is positive for $\theta < \theta_2$. By the definitions of the coefficients of (15), we have

$$a_1 a_2 - a_3 = \frac{-A\theta^2 + B\theta - C}{\theta^2},$$

where

$$\begin{aligned} A &= \alpha(I_Y - S_Y)(I_K - \delta)\left[\alpha\left(I_Y - S_Y\right) + (I_K - \delta)\right] > 0, \\ B &= \left[\alpha(I_Y - S_Y) + (I_K - \delta)\right]^2 + \alpha I_K I_Y \gtreqless 0, \\ C &= \alpha(I_Y - S_Y) + (I_K - \delta) < 0. \end{aligned}$$

Let us denote the numerator of the last equation by $f(\theta)$. Since $f(\theta)$ is a concave quadratic polynomial, $f(0) = -C > 0$ implies that $f(\theta) = 0$ has one positive root, θ^*,

$$\theta^* = \frac{B + \sqrt{B^2 - 4AC}}{2A}.$$

Since $f(\theta^*) = 0$, $f(\theta) < 0$ for $\theta > \theta^*$. Furthermore $f(\theta_2) = (\theta_2)^2(-a_3) < 0$ and $\theta^* < \theta_2$ imply that $a_2 > 0$ at $\theta = \theta^*$. To emphasize the dependency of the coefficients on θ, we denote $a_i(\theta)$ for $i = 1, 2, 3$. For $\theta = \theta^*$, $a_1(\theta^*)a_2(\theta^*) - a_3(\theta^*) = 0$ By replacing $a_3(\theta^*)$ of the characteristic equation with $a_1(\theta^*)a_2(\theta^*)$, we are able to factor the characteristic equation,

$$(\lambda + a_1(\theta^*))(\lambda^2 + a_2(\theta^*)) = 0$$

that can be explicitly solved for λ. One of the three roots is real and negative whereas the other two are pure imaginary,

$$\lambda_1 = -a_1(\theta^*) < 0 \text{ and } \lambda_{2,3} = \pm i\sqrt{a_2(\theta^*)} = \pm i\xi.$$

In order to apply the Hopf bifurcation theorem, we have to show that the real parts of the complex roots are sensitive to a change in the bifurcation parameter, θ. Suppose that λ is a function of θ. By implicitly differentiating the characteristic equation with respect to θ we have

$$\frac{d\lambda(\theta^*)}{d\theta} = \frac{1}{\theta^{*2}} \frac{\lambda(\theta^*)^2 - tr\mathbf{J_K}\lambda(\theta^*) + det\mathbf{J_K}}{3\lambda(\theta^*)^2 + 2a_1(\theta^*)\lambda(\theta^*) + a_2(\theta^*)}.$$

Substituting $\lambda = \pm i\xi$ and arranging terms yield

$$\mathrm{Re}\left(\frac{d\lambda(\theta^*)}{d\theta}\right) = \frac{1}{\theta^{*2}} \frac{\xi^2 - det\mathbf{J_K} - a_1(\theta^*)tr\mathbf{J_K}}{2(\xi^2 + a_1(\theta^*)^2)},$$

where the denominator is positive. We can show that the numerator is never zero. Substituting

$$a_1(\theta^*) = -tr\mathbf{J_K} + \frac{1}{\theta^*}$$

and

$$\xi^2 = a_2(\theta^*) = det\mathbf{J_K} + \alpha I_K I_Y - \frac{1}{\theta} tr\mathbf{J_K}$$

into the numerator and assuming that the resultant expression is zero yield

$$(tr\mathbf{J_K})^2 + \alpha I_K I_Y - \frac{2}{\theta} tr\mathbf{J_K} = 0.$$

However $a_2(\theta^*)a_1(\theta^*) = a_3(\theta^*)$ means that

$$\left(\frac{1}{\theta^*}\right)\left(det\mathbf{J_K} + \alpha I_K I_Y - \frac{1}{\theta^*}tr\mathbf{J_K}\right) = \frac{1}{\theta^*}det\mathbf{J_K}$$

which can be rewritten as

$$tr\mathbf{J_K}\left[\frac{1}{\theta^{*2}} - \alpha\left(I_Y - S_Y\right)\left(I_k - \delta\right)\right] = 0,$$

where the equality is impossible, since $tr\mathbf{J_K} < 0$, $I_Y - S_Y > 0$, $I_k - \delta < 0$ and $\theta^* > 0$. Therefore we have

$$\mathrm{Re}\left(\frac{d\lambda(\theta^*)}{d\theta}\right) \neq 0.$$

This implies that the real parts of the complex roots change signs as $\theta - \theta^*$ changes from negative to positive values. That is, it guarantees the existence of Hopf bifurcation.

Theorem 2 *The Kaldor-Kalecki model with continuously distributed lags having an exponential kernel function is locally asymptotic stable for* $0 \leq \theta < \theta^*$ *while it loses the stability at* $\theta = \theta^*$ *via a Hopf bifurcation.*

It is uncertain whether the limit cycle is subcritical or supercritical. In Figure 5, simulation results are shown with $\theta = 0.7$ and parameter values $c = 0.6$, $\alpha = 0.8$, $\beta = -0.4$ and $\delta = 0.2$, which are the same as in the simulation study presented in Figure 4. The critical value is $\theta^* \simeq 0.37$. The delay Kaldor-Kalecki model generates a supercritical limit cycle due to the destabilizing effect of the investment lag.

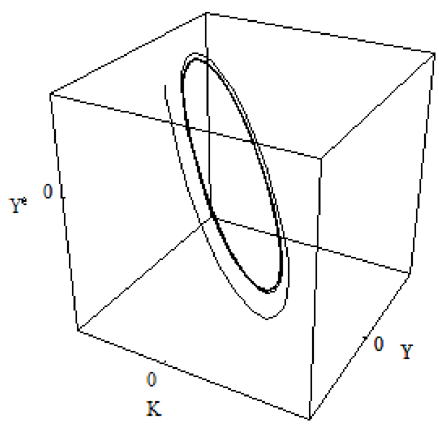

Figure 5. Existence of a limit cycle in the delay Kaldor-Kalecki model

3.3 Delay Nonlinear Cournot Model

We will examine a dynamic Cournot duopoly game when a firm has an information lag in the receipt of information about its competitor's output. We assume that each firm adaptively adjusts its output to the desired level of output:

$$\begin{cases} \dot{x}_1(t) = k_1 \left\{ R_1(x_2(t-\theta_1)) - x_1(t) \right\}, \\ \\ \dot{x}_2(t) = k_2 \left\{ R_2(x_1(t-\theta_2)) - x_2(t) \right\}, \end{cases} \tag{17}$$

where x_i, k_i, θ_i and $R_i(x_j)$ are output, a positive adjustment coefficient, a time lag and the best reply function of firm i for $i, j = 1, 2$ and $i \neq j$. Special duopoly models such as the classical Cournot model with a linear price function and a nonlinear Cournot model with a unit-elastic price function will be considered later to specify the best reply functions.

To consider a linearization of the system, we suppose that the information lags are sufficiently small and an advance θ_1 time in the first equation of (17) and an advance θ_2 time in the second one:

$$\begin{aligned} \dot{x}_1(t+\theta_1) &= k_1 \left\{ R_1(x_2(t)) - x_1(t+\theta_1) \right\}, \\ \\ \dot{x}_2(t+\theta_2) &= k_2 \left\{ R_2(x_1(t)) - x_2(t+\theta_2) \right\}. \end{aligned}$$

Define the difference between the left-hand side and the right-hand side by

$$F_1(\theta_1) = \dot{x}_1(t+\theta_1) - k_1 \left\{ R_1(x_2(t)) - x_1(t+\theta_1) \right\}$$

and

$$F_2(\theta_2) = \dot{x}_2(t+\theta_2) - k_2 \left\{ R_2(x_1(t)) - x_2(t+\theta_1) \right\}.$$

Differentiating each function with its lag at $\theta_i = 0$ and arranging terms yield

$$\theta_1 \ddot{x}_1(t) = -k_1\theta_1\dot{x}_1(t) - \dot{x}_1(t) + k_1 \left\{ R_1(x_2(t)) - x_1(t) \right\}$$

and

$$\theta_2 \ddot{x}_2(t) = -k_2\theta_2\dot{x}_2(t) - \dot{x}_2(t) + k_2 \left\{ R_2(x_1(t)) - x_2(t) \right\}.$$

Introducing the new variables $y_1(t) = \dot{x}_1(t)$ and $y_2(t) = \dot{x}_2(t)$, we can transform the 2D delay differential equation system (17) into the following 4D system of ordinary differential equations:

$$\begin{cases} \dot{x}_1(t) = y_1(t), \\ \\ \dot{x}_2(t) = y_2(t), \\ \\ \dot{y}_1(t) = \dfrac{k_1}{\theta_1} \left\{ R_1(x_2(t)) - x_1(t) \right\} - \left(k_1 + \dfrac{1}{\theta_1} \right) y_1(t), \\ \\ \dot{y}_2(t) = \dfrac{k_2}{\theta_2} \left\{ R_2(x_1(t)) - x_2(t) \right\} - \left(k_2 + \dfrac{1}{\theta_2} \right) y_2(t). \end{cases} \tag{18}$$

The Jacobian matrix is

$$\mathbf{J_L} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1}{\theta_1} & \dfrac{k_1}{\theta_1}\gamma_1 & -\left(k_1 + \dfrac{1}{\theta_1}\right) & 0 \\ \dfrac{k_2}{\theta_2}\gamma_2 & -\dfrac{k_2}{\theta_2} & 0 & -\left(k_2 + \dfrac{1}{\theta_2}\right) \end{pmatrix},$$

where γ_i is the derivative of $R_i(x_j)$ evaluated at the stationary point. The characteristic equation of $\mathbf{J_L}$ can be written as

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{19}$$

where

$$\begin{aligned} a_0 &= \theta_1\theta_2, \\ a_1 &= \theta_1 + \theta_2 + (k_1 + k_2)\theta_1\theta_2, \\ a_2 &= 1 + k_1k_2\theta_1\theta_2 + (k_1 + k_2)(\theta_1 + \theta_2), \\ a_3 &= k_1 + k_2 + k_1k_2(\theta_1 + \theta_2), \\ a_4 &= k_1k_2(1 - \gamma_1\gamma_2). \end{aligned} \tag{20}$$

The above procedure is suitable for a situation in which the information lag is fixed and sufficiently small. If the lags are uncertain, we can model time lags in a continuously distributed manner. If firm 1's expectation of the competitor's output is denoted by $x_2^e(t)$ and firm 2's expectation of the competitor's output is denoted by $x_1^e(t)$ and both expectations are based on the entire history of the outputs from zero up to t with exponentially decreasing weights, then the delay differential equation system (17) can be written as the 2D system of integro-differential equations:

$$\begin{cases} \dot{x}_1(t) = k_1 \left\{ R_1(x_2^e(t)) - x_1(t) \right\}, \\ \dot{x}_2(t) = k_2 \left\{ R_2(x_1^e(t)) - x_2(t) \right\}, \end{cases} \tag{21}$$

with

$$x_1^e(t) = \int_0^t \frac{1}{\theta_1} e^{-\frac{t-s}{\theta_1}} x_1(s)ds$$

$$x_2^e(t) = \int_0^t \frac{1}{\theta_2} e^{-\frac{t-s}{\theta_2}} x_2(s)ds.$$

This system is equivalent to the following 4D system of ordinary differential equations:

$$\begin{cases} \dot{x}_1(t) = k_1 \left\{ R_1(x_2^e(t)) - x_1(t) \right\}, \\ \dot{x}_2(t) = k_2 \left\{ R_2(x_1^e(t)) - x_2(t) \right\}, \\ \dot{x}_1^e(t) = \dfrac{1}{\theta_1}(x_1(t) - x_1^e(t)), \\ \dot{x}_2^e(t) = \dfrac{1}{\theta_2}(x_2(t) - x_2^e(t)). \end{cases}$$

The Jacobian of this system can be written as

$$\begin{pmatrix} -k_1 & 0 & 0 & k_1\gamma_1 \\ 0 & -k_2 & k_2\gamma_2 & 0 \\ \dfrac{1}{\theta_1} & 0 & -\dfrac{1}{\theta_1} & 0 \\ 0 & \dfrac{1}{\theta_2} & 0 & -\dfrac{1}{\theta_2} \end{pmatrix}.$$

Simple calculation shows that the characteristic equation of this matrix can be written as a quartic equation in λ:

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

with the same coefficients as defined in (20). The identical characteristic equation means that (17) and (18) exhibit the same dynamics in a neighborhood of the stationary point as Theorem 1 claims.

If $\gamma_1\gamma_2 < 1$, then all coefficients of the characteristic equation are positive, and the Routh-Hurwitz theorem implies that the roots have negative real parts if and only if

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0 \text{ and } \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} > 0.$$

The first condition is satisfied because the second-order determinant is always positive,

$$(k_1 + k_2)(1 + k_1\theta_2)(1 + k_2\theta_2)\theta_1^2 + \theta_2(1 + (k_1 + k_2)\theta_1) + \theta_1(1 + (k_1 + k_2)\theta_2) > 0.$$

The second condition depends on the value of $\gamma_1\gamma_2$. Expanding the third-order determinant, and solving the inequality gives a lower bound for $\gamma_1\gamma_2$, and by combining it with the upper bound $\gamma_1\gamma_2 < 1$, we get the following condition for the local asymptotic stability of the stationary state:

$$1 > \gamma_1\gamma_2 > -\frac{(k_1 + k_2)(1 + k_1\theta_1)(1 + k_2\theta_1)(\theta_1 + \theta_2)(1 + k_1\theta_2)(1 + k_2\theta_2)}{k_1k_2(\theta_1 + \theta_2 + \theta_1\theta_2(k_1 + k_2))^2}. \tag{22}$$

In the case of the linear Cournot model, the price function is given by

$$p = a - b(x_1 + x_2)$$

and so the profit function of firm i is defined as

$$\pi_i = (a - b(x_1 + x_2))\, x_i - c_i x_i,$$

where c_i is the constant marginal cost. The best reply function and its derivative are

$$R_i(x_j) = \frac{a - c_i - bx_j}{2b} \text{ and } \gamma_i = -\frac{1}{2}.$$

Since $1 > \gamma_1\gamma_2 = 1/4 > 0$, (22) is satisfied. Hence the delay linear Cournot model is always stable for any values of information lags, θ_i.

In the case of the unit-elastic demand, the price function is given by

$$p = \frac{1}{x_1 + x_2}$$

and the profit function of firm i is defined as

$$\pi_i = \frac{x_i}{x_1 + x_2} - c_i x_i.$$

Assuming an interior solution, the profit maximization yields a bell-shaped best reply function,

$$R_i(x_j) = \sqrt{\frac{x_j}{c_i}} - x_j.$$

Cournot outputs are determined by an intersection of the best reply curves,

$$x_1^C = \frac{c_2}{(c_1 + c_2)^2} \text{ and } x_2^C = \frac{c_1}{(c_1 + c_2)^2}.$$

The derivatives of the best response functions evaluated at the Cournot point are derived as

$$\gamma_1 = -\frac{c_1 - c_2}{2c_1} \text{ and } \gamma_2 = \frac{c_1 - c_2}{2c_2}.$$

If there are no time lags, the dynamic system is represented by (17) with $\theta_i = 0$. The asymptotic properties of the trajectories $x_1(t)$ and $x_2(t)$ depend on the location of the eigenvalues of the Jacobian matrix of the system. The eigenvalues are obtained by solving the associated characteristic equation,

$$\lambda^2 + (k_1 + k_2)\lambda + k_1 k_2(1 - \gamma_1\gamma_2) = 0.$$

Here $k_1 + k_2 > 0$ by the definition of the adjustment coefficient and $\gamma_1\gamma_2 < 1$, since

$$\gamma_1\gamma_2 = -\frac{(1-c)^2}{4c} \text{ with } c = \frac{c_2}{c_1}.$$

The roots of the characteristic equation have negative real parts. Hence the nonlinear Cournot model with no information lags is always asymptotically stable.[1]

Now we examine the asymptotic behavior of the delay nonlinear Cournot model. The value of $\gamma_1\gamma_2$ can be any negative number between $-\infty$ and zero by the appropriate choice of the cost ratio c. Notice that the stability condition (22) is violated if $\gamma_1\gamma_2$ is negative with large absolute value. In particular, Figure 6 illustrates the dynamic behavior of the trajectories when the stability condition is violated, in which the parameters are specified as $k_1 = k_2 = 0.8$, $\theta_1 = \theta_2 = 2$, $c_1 = 1$ and $c_2 = 0.045$.[2] It can be seen that a trajectory starting at the dot point converges to a limit cycle surrounding a locally unstable Cournot point.

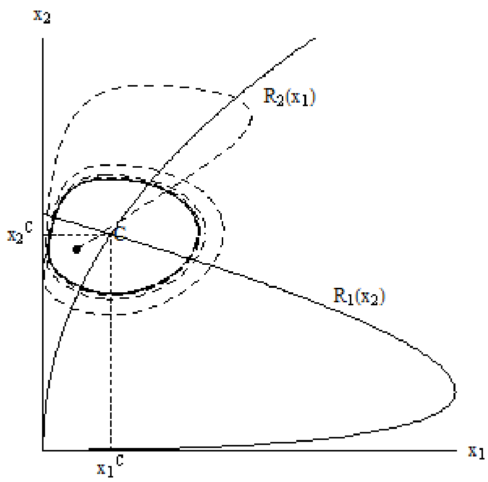

Figure 6. The birth of a Cournot cycle

4 Concluding Remarks

Delay models with fixed lags and models with continuously distributed delays were compared in this paper. By selecting exponential kernel function, we first proved that with small delays the two types of dynamics generate identical local asymptotic properties. However with large delays this interesting equivalence was not true anymore.

Three particular economic models (Goodwin's business cycle model, Kaldorian business cycle model with Kaleckian investment lag and the Cournot oligopoly

[1] The discrete-time version of the nonlinear Cournot model has been extensively studied, and it is demonstrated that simple nonlinear best reply functions can generate a very rich dynamics involving chaos and multistability (Puu (2003), Puu and Sushuko (2002) and Bischi, et al. (2009)). The delay differential Cournot model with product differentiation is considered in Matsumoto and Szidarovszky (2007).

[2] A trajectory seems to cross itself as dynamics generated in a 4D space is projected to a 2D space.

model) illustrated the theoretical results, and computer simulations showed the emerge of more complex dynamics if a large value of time delay was selected.

References

[1] Bellman, R., *Stability Theory of Differential Equations*, Dover Publ. Inc., New York, 1969.

[2] Bischi, G., C. Chiarella, M. Kopel and F. Szidarovszky, *Nonlinear Oligopolies: Stability and Bifurcations*, Springer (in press), 2009.

[3] Chang, W., and D. Smyth, "The Existence and Persistence of Cycles in a Non-linear Model: Kaldor's 1940 model reexamined," Review of Economic Studies, vol.38, 37-44, 1971.

[4] Goldberg, S., *Introduction to Difference Equations*, John Wiley & Sons, New York, 1958.

[5] Goodwin, R., "The Nonlinear Accelerator and the Persistence of Business Cycles," *Econometrica*, vol.19, 1-17, 1951.

[6] Kaldor, N., "A Model of the Trade Cycle," *Economic Journal*, vol.50, 78-92, 1940.

[7] Kalecki, M., "A Macroeconomic Theory of the Business Cycle," *Econometrica*, vol.3, 327-344, 1935.

[8] Krawiec, A., and M. Szydłowski, "The Kaldor-Kalecki business cycle model," *Annals of Operations Research*, vol. 89, 89-100, 1999.

[9] Kuang, Y., *Delay Differential Equations with Applications in Population Dynamics*, Boston, Academic Press, 1993.

[10] Matsumoto, A., "Note on Goodwin's 1951 Nonlinear Accelerator Model with an Investment Lag," *Journal of Economic Dynamics and Control,* forthcoming, 2009.

[11] Matsumoto, A., "Destabilizing Effects on Income Adjustment Process with Fiscal Policy Lag," *Metroeconomica*, vol.59, 713-735, 2008.

[12] Matsumoto, A., and F. Szidarovszky, "Goodwin Business Cycle Model with Continuously Distributed Time Lags," mimeo, 2009.

[13] Matsumoto, A., and F. Szidarovszky, "Delayed Nonlinear Cournot and Bertrand Dynamics with Product Differentiation," *Nonlinear Dynamics, Psychology, and Life Science*, vol.11, 367-395, 2007.