Controlling Chaotic Dynamics in N-firm Nonlinear Cournot Games*

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Abstract

The state trajectories of a special class of discrete dynamic economic systems will be considered. The control of their long–term behavior is a major research issue of dynamic markets. An N-firm production game known as oligopoly will be examined with isoelastic price function and linear costs. After the reaction functions of the firms are determined, dynamic systems with adaptive expectations and with adaptive adjustments will be introduced and the equivalence of their local asymptotic behavior will be verified. Then two special cases will be investigated in details, with two and three groups of identical firms, in which the dynamics is two and three dimensional, respectively. Stability conditions will be derived and the global behavior of the equilibria will be illustrated including chaos control.

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1 Introduction

The state sequence of discrete dynamic systems will be considered as time series, with a deterministic rule to obtain the consecutive state variables. Many important classes of dynamic systems are examined frequently in the literature of mathematical economics. Among the large variety of dynamic economic systems the oligopoly models have a very special place, since the long-term behavior of the state trajectories has many different possibilities including global asymptotic stability, limit cycles with increasing number of nodes and even chaos. In this paper a special oligopoly model will be investigated and conditions will be derived to avoid chaotic behavior. Following Cournot (1838) many researchers worked on developing more realistic models and on examining their properties. The existence and uniqueness of the equilibrium was the main focus in earlier studies, and later the researchers turned their focus to the dynamic extensions of these models. A comprehensive summary of earlier results is given in Okuguchi (1976), and their multi-product extensions with different model variants are discussed in Okuguchi and Szidarovszky (1999). Most studies considered concave oligopolies with monotonic reaction functions, and only a little attention was given to the isoelastic case, which can be derived by assuming a Cobb-Douglas type utility function of the market. In this case the best response is not monotonic anymore, making equilibrium and stability analysis more complicated. Agiza (1999) has examined the two-dimensional Kopel map in which the reaction functions are unimodal, derived stability conditions and studied bifurcation and chaos by computing the maximum Lyapunov exponents. Agiza and Elsadany (2003) have investigated discrete-time Cournot duopolies with heterogeneous players. Richter and Stolk (2004) have introduced a new method of controlling coexisting chaotic attractors in Cournot triopolies by means of steering the systems dynamics from one attractor to another. See Puu and Sushko (2002), Puu (2003) and Bischi et al. (2010) for comprehensive summary of recent developments in the theory of nonlinear oligopolies.

This work is an extension and generalization of an earlier paper of Matsumoto (2006), where chaos control for nonlinear duopolies was examined. In line with the economic literature, controlled systems are dynamics with adaptive expectations and uncontrolled systems are those with naive expectations. We will give detailed equilibrium analysis, show the complexity of the dynamics of such models and further show that complex dynamics involving chaos could be stabilized.

The paper develops as follows. In Section 2 the general model will be introduced, the reaction functions and the Cournot equilibrium will be determined. In Section 3 we will show that the local stability properties of systems with adaptive expectations and with inertia control (i.e., adaptive adjustments) are identical. In Section 4 two special cases, two and three groups of firms, will be analyzed both theoretically and numerically in which the dynamics are two and three dimensional, respectively. The stability regions, where chaos is controlled, will be shown and their dependence on the number of firms will be illustrated. Section 5 will conclude the paper.

2 Nonlinear Oligopoly Models

It is assumed that a homogeneous market is supplied by N firms. For the sake of mathematical simplicity only one product is considered. Let x_i denote the production output of firm i, then $y_i = \sum_{j \neq i} x_j$ is the output of the rest of the industry and $Q = \sum_{i=1}^{N} x_i$ is the total output of the industry. We assume isoelastic price function $p = 1/Q$ and linear cost functions $C_i(x_i) = c_i x_i$ as in the duopoly model of Puu (2003). Since the firms make decisions about their production levels simultaneously, the firms do not know the output of the rivals when their decisions are made. They can have only an expectation (prediction) of the output of the rest of the industry, y_i^e. So the expected profit of firm i can be given as

$$\Pi_i^e = \frac{x_i}{x_i + y_i^e} - c_i x_i, \ \ (i = 1, 2, ..., N).$$

Notice that this function is strictly concave in x_i.

2.1 Reaction Functions

The strict concavity of Π_i^e implies that with any given value of y_i^e the profit maximizing output level of firm i can be computed as

$$x_i = f_i(y_i^e) = \begin{cases} \sqrt{\dfrac{y_i^e}{c_i}} - y_i^e & \text{if } 0 < y_i^e \leq \dfrac{1}{c_i}, \\ 0 & \text{if } \dfrac{1}{c_i} < y_i^e. \end{cases} \tag{1}$$

This function is continuous, and strictly concave in the first segment.

Then the best response dynamic process is

$$x_i(t+1) = f_i(y_i^e(t+1)) \text{ for } i = 1, 2, ...N.$$

Dynamic characteristics are sensitive to the expectation formation. In this study we first consider naive expectation in which the firms assume that the output of the rest of the industry remains the same:

$$y_i^e(t+1) = \sum_{j \neq i} x_j(t),$$

and call it an *naive system.* It is well known that the naive system is a special case of the best reply dynamics with adaptive expectations. In the Appendix, local stability conditions for dynamic systems with adaptive expectations are derived. It is, as will be seen, useful to determine the stability of not only naive systems but also that of the controlled systems.

2.2 Cournot Equilibrium

Without losing generality we may assume that at the equilibrium all firms have positive outputs, otherwise we can ignore the firms with zero equilibrium output values and decrease the value of N. Assuming a positive equilibrium, then, from the definition of the naive expectation and the reaction function of firm i,

$$y_i^e = \sum_{j \neq i} x_j \text{ and } x_i = \sqrt{\frac{y_i^e}{c_i}} - y_i^e.$$

By introducing the notation $y = x_i + y_i^e$, from the second equation we have

$$y_i^e = c_i y^2,$$

that is,

$$x_i = y - c_i y^2.$$

Adding this equation for all values of i and denoting the sum of the marginal costs by $C = \sum_{i=1}^N c_i$ gives

$$y = Ny - Cy^2.$$

Therefore there is a trivial equilibrium with $y = x_1 = ... = x_N = 0$, and a nontrivial positive equilibrium with

$$y^c = \frac{N-1}{C},$$

where the Cournot output of firm i becomes

$$x_i^c = \frac{(N-1)(C - c_i(N-1))}{C^2}. \tag{2}$$

The superscript "c" is attached to variables to indicate that they are computed at the Cournot equilibrium. Our concern is on the nontrivial point, $(x_i^c)_{i=1.2,...N}$, and thus no further considerations will be given to the trivial point. For non-negative Cournot outputs, the following inequality has to be satisfied:

$$C - c_i(N-1) \geq 0 \text{ or } c_i \leq \frac{C}{N-1}. \tag{3}$$

This always holds for $N = 2$, and necessarily holds for $N > 2$ if the marginal costs, c_i, are sufficiently close to each other. In the rest of this paper, we assume that this condition is satisfied. Notice that under this condition

$$y_i^e = c_i \left(y^c\right)^2 = c_i \left(\frac{N-1}{C}\right)^2 \leq c_i \frac{1}{c_i^2} \leq \frac{1}{c_1},$$

So at the equilibrium the first case of (1) applies.

Adaptive expectations are generalizations of naive expectation where the expected output of the rest of the industry is computed as

$$y_i^e(t+1) = (1 - \alpha_i) y_i^e(t) + \alpha_i \sum_{j \neq i} x_j(t).$$

Notice that by selecting $\alpha_i = 1$ this formula reduces to naive expectation. In the Appendix the local stability conditions for system with adaptive expectations are derived. Combining inequalities (3) and (A-1) from the Appendix with $\alpha_i = 1$ gives

$$\frac{C}{4(N-1)} < c_i \leq \frac{C}{N-1} \text{ for } i = 1.2, ...N,$$

and the eigenvalue equation (A-2) of the Appendix reduces to the following equation:

$$\sum_{i=1}^N \frac{\gamma_i}{\gamma_i + \lambda} = 1, \text{ where } \gamma_i \equiv \frac{\partial f_i(y_i^e)}{\partial y_i^e} \text{ at the equilibrium.}$$

It can be written as the quadratic and cubic equations,

$$\lambda^2 - \gamma_1\gamma_2 = 0 \text{ for } N = 2,$$

and

$$-\lambda^3 + (\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3)\lambda + 2\gamma_1\gamma_2\gamma_3 = 0 \text{ for } N = 3.$$

3 General Stability Conditions

In this section, we first consider the dynamic process in which firms cautiously adjust their outputs, that is, the output in the next period is a weighted average of the current output and the naively-determined optimal output:

$$x_i(t+1) = \alpha_i f_i(y_i^e(t+1)) + (1-\alpha_i)x_i(t) \text{ for } i = 1, 2, ..., N,$$

or, equivalently,

$$x_i(t+1) = \alpha_i \left(\sqrt{\frac{\sum_{j\neq i} x_j(t)}{c_i}} - \sum_{j\neq i} x_j(t) \right) + (1-\alpha_i)x_i(t). \qquad (4)$$

This is commonly known as the adaptive adjustment process. Here we assume that firm i moves into the direction toward its profit maximizing output, and reaches it only for $\alpha_i = 1$. Since this adjustment process describes the best reply dynamics with inertia, we call it the *inertia control system*. Here α_i is the inertia or control parameter of firm i and assumed to be positive and not greater than unity. It is easy to see that the fixed point of the inertia control system is the same as that of the naive system. The Jacobian of the system has the form

$$\mathbf{H}^c = \begin{pmatrix} 1-\alpha_1 & \gamma_1\alpha_1 & \cdot & \gamma_1\alpha_1 \\ \gamma_2\alpha_2 & 1-\alpha_2 & \cdot & \gamma_2\alpha_2 \\ \cdot & \cdot & \cdot & \cdot \\ \gamma_N\alpha_N & \gamma_N\alpha_N & \cdot & 1-\alpha_N \end{pmatrix}.$$

In the Appendix, we show that the nonzero eigenvalues of the Jacobian of the system with adaptive expectations are the eigenvalues of matrix

$$\mathbf{H} = \begin{pmatrix} 1-\alpha_1 & \gamma_2\alpha_1 & \cdot & \gamma_N\alpha_1 \\ \gamma_1\alpha_2 & 1-\alpha_2 & \cdot & \gamma_N\alpha_2 \\ \cdot & \cdot & \cdot & \cdot \\ \gamma_1\alpha_N & \gamma_2\alpha_N & \cdot & 1-\alpha_N \end{pmatrix}.$$

Next we will prove that the characteristic polynomials of matrixes $\mathbf{H}$ and $\mathbf{H}^c$ are identical, so as far as local stability is concerned, the asymptotic behavior of the two systems is identical. Assume first that all $\gamma_i \neq 0$. Define

$$\mathbf{R} = diag(\gamma_1, \gamma_2, ..., \gamma_N).$$

Then clearly

$$\begin{aligned}
\mathbf{R}^{-1}\mathbf{H}^c\mathbf{R} &= \begin{pmatrix} \frac{1}{\gamma_1} & 0 & \cdot & 0 \\ 0 & \frac{1}{\gamma_2} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & \frac{1}{\gamma_N} \end{pmatrix} \mathbf{H}^c \begin{pmatrix} \gamma_1 & 0 & \cdot & 0 \\ 0 & \gamma_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & \gamma_N \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-\alpha_1}{\gamma_1} & \alpha_1 & \cdot & \alpha_1 \\ \alpha_2 & \frac{1-\alpha_2}{\gamma_2} & \cdot & \alpha_2 \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_N & \alpha_N & \cdot & \frac{1-\alpha_N}{\gamma_N} \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 & \cdot & 0 \\ 0 & \gamma_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & \gamma_N \end{pmatrix} \\
&= \mathbf{H}.
\end{aligned}$$

Since $\mathbf{H}^c$ and $\mathbf{H}$ are similar, their characteristic polynomials are identical. If any or more $\gamma_i = 0$, then the continuity of the characteristic polynomial coefficients in the matrix elements implies the result. Hence the local stability conditions of dynamics with adaptive expectations and with inertia control are identical. So in the rest of this paper we will consider only inertia control.

In the case of concave oligopolies (see for example, Szidarovszky and Chiarella (2001), Bischi et al. (2010)) it is proved that $-1 < \gamma_i \leq 0$, all eigenvalues are real and they are inside the unit circle if and only if for all i,

$$\alpha_i < \frac{2}{1+\gamma_i}$$

and

$$\sum_{i=1}^{N} \frac{\alpha_i \gamma_i}{2 - \alpha_i(1+\gamma_i)} > -1.$$

These conditions are clearly satisfied if the speeds α_i of adjustments are sufficiently small. However in the case of isoelastic inverse demand functions there is the possibility of complex eigenvalues, so no such simple general stability conditions can be derived. In the next section two special cases will be examined both theoretically and numerically.

4 Stability Conditions in Special Models

In this section we will focus on two cases: one case where the industry consists of two groups of firms and the other case with three groups of firms. Our aim is to see whether the unstable naive system can be stabilized by the inertia control method.

4.1 Two Groups of Firms

Assume there are two groups of firms in a sense that firms of the same group produce the same output with the same marginal cost and have the same speed of adjustment. So the N firms are divided into two groups. Without loss of generality, we can assume that the first N_a firms are in the first group and the remaining N_b firms in the second group, where $N_a + N_b = N$ and $N_a = \frac{1}{w}N$

with $w > 1$. We denote the outputs produced by the firms of the two groups by x and y, so

$$x_1 = \cdots = x_{N_a} = x \text{ and } x_{N_a+1} = \cdots = x_N = y,$$

the two kinds of marginal costs are denoted by a and b, so

$$c_1 = \cdots = c_{N_a} = a \text{ and } c_{N_a+1} = \cdots = c_N = b,$$

and the two types of speeds of adjustment by α and β, that is,

$$\alpha_1 = \cdots = \alpha_{N_a} = \alpha \text{ and } \alpha_{N_a+1} = \cdots = \alpha_N = \beta.$$

We further assume, without any losses of generality, that the firms in the first group are more efficient than the ones in the second in a sense that

$$a < b.$$

Accordingly, the marginal cost ratio of b over a, which we denote by h, is greater than unity,

$$h = \frac{b}{a} > 1.$$

The sum of the marginal costs and the derivatives of the reaction functions evaluated at the Cournot equilibrium are

$$C = N_a a + N_b b$$

and

$$\gamma_a = \frac{C}{2a(N-1)} - 1 \text{ and } \gamma_b = \frac{C}{2b(N-1)} - 1.$$

From (2), the equilibrium outputs of the firms are

$$x^c = \frac{(N-1)(C - a(N-1))}{C^2},$$

$$y^c = \frac{(N-1)(C - b(N-1))}{C^2},$$

where

$$\begin{aligned} C - a(N-1) &= a(N_b(h-1)+1) > 0, \\ C - b(N-1) &= a(N_a - 1)\left(\frac{N}{N-w} - h\right). \end{aligned}$$

The first inequality is always true because $h > 1$, so x^c is always positive. The second equation indicates that y^c is nonnegative if $N_a > 1$ and the marginal cost ratio is bounded from above:

$$h \leq h_N \equiv \frac{N}{N-w}. \tag{5}$$

Notice that $N_a > 1$ is equivalent to $N > w$ implying the positive denominator in the upper bound h_N of the marginal cost ratio. It can be seen that h_N decreases in N and is approaching unity as N converges to infinity with fixed values of w. Since the ratio h is assumed to be greater than unity, this implies that it becomes more difficult to have a positive Cournot equilibrium as the number of the firms in the industry increases. Similarly, condition (A-1) for each group can be written as

$$\begin{cases} h < \dfrac{4(N-1) - \alpha N_a}{\alpha N_b} \text{ for the first group,} \\ \\ h > \dfrac{\beta N_a}{4(N-1) - \beta N_b} \text{ for the second group.} \end{cases}$$

However neither inequality is effective, since for $N \geq 3$ and $0 < (\alpha, \beta) \leq 1$,

$$\frac{\beta N_a}{4(N-1) - \beta N_b} < 1 \leq \frac{N}{N-w} < \frac{4(N-1) - \alpha N_a}{\alpha N_b},$$

which indicates that if h fulfills (5), then it also satisfies (A-1).

Notice that in this case, equation (A-2) assumes the form

$$1 + N_a \frac{\alpha \gamma_a}{1 - \alpha(1+\gamma_a) - \lambda} + N_b \frac{\beta \gamma_b}{1 - \beta(1+\gamma_b) - \lambda} = 0.$$

Since N_a and N_b firms have identical parameters, in which case $\alpha_i = \alpha$, $\gamma_i = \gamma_a$ $(1 \leq i \leq N_a)$ and $\alpha_i = \beta$, $\gamma_i = \gamma_b$ $(N_a + 1 \leq i \leq N_a + N_b)$. Notice that with notation $R_a = 1 + \gamma_a$ and $R_b = 1 + \gamma_b$, the above equation can be written as

$$(1 - \alpha R_a - \lambda)(1 - \beta R_b - \lambda) + N_a \alpha \gamma_a (1 - \beta R_b - \lambda) + N_b \beta \gamma_b (1 - \alpha R_a - \lambda) = 0.$$

This is a quadratic equation in λ,

$$\lambda^2 + p\lambda + q = 0 \tag{6}$$

with coefficients

$$p = \alpha(R_a(1 - N_a) + N_a) + \beta(R_b(1 - N_b) + N_b) - 2,$$

and

$$\begin{aligned} q \ = \ & -\alpha(R_a(1 - N_a) + N_a) - \beta(R_b(1 - N_b) + N_b) + \\ & \alpha\beta(R_a R_b(1 - N) + R_a N_b + R_b N_a) + 1. \end{aligned}$$

It is well-known (see for example, Bischi et al. (2010)) that the roots of the quadratic equation are inside the unit circle if and only if

$$q < 1,$$

$$q + p + 1 > 0,$$

$$q - p + 1 > 0.$$

The left hand side of the second condition can be rewritten as

$$q + p + 1 = \frac{(N_a + hN_b)^2 \alpha\beta}{4h(N-1)},$$

which is always positive. This indicates that the characteristic equation (6) does not have a root equal to unity. To simplify the other two stability conditions, we make a specializing assumption that the speeds of adjustment are the same for the two groups.

Assumption $\alpha = \beta$.

Solving $q - 1 = 0$ for h yields the *Hopf boundary*

$$h_H = \frac{w^2(N-1) + (w-1)(2-\alpha)N + \sqrt{D_H}}{(w-1)(2(N-w) + (w-1)\alpha N)} \tag{7}$$

with the discriminant

$$D_H = w^2(N-1)\left\{(N-1)w^2 + 4(w-1)((1-\alpha)N-1)\right\}.$$

On the Hopf boundary the eigenvalues are complex and their absolute values are equal to unity. Thus this boundary is valid for $p^2 < 4q$. It is rather difficult to simplify this condition, however it can be checked numerically in all particular situations. Notice that D_H is linear in α. At $\alpha = 0$,

$$D_H = w^2(N-1)^2\left\{w^2 + 4(w-1)\right\} > 0,$$

and at $\alpha = 1$,

$$\begin{aligned} D_H &= w^2(N-1)\left\{(N-1)w^2 - 4(w-1)\right\} \\ &= w^2(N-1)\left\{(N-2)w^2 + (w-2)^2\right\} \geq 0 \text{ for } N \geq 2. \end{aligned}$$

Therefore $D_H \geq 0$ for all $\alpha \in (0, 1]$.

By gradually varying the parameter values α and h, our simulation study indicates that if pair (α, h) crosses the Hopf boundary, then the Cournot point bifurcates to periodic cycles, quasi-periodic cycles and then to chaotic fluctuations. The same phenomenon has been observed earlier by Puu (2003) in his nonlinear duopoly model. As shown below, the Hopf bifurcation of the n-firm oligopoly, however, occurs only in the infeasible region in which the non-negativity condition (4) is violated. In consequence we do not pay much attention to the Hopf bifurcation.

Solving $1 - p + q = 0$ for h yields the *flip boundary*

$$h_F = \frac{(4-\alpha)(w-1)\alpha N^2 + 2w^2(N-1)(N\alpha - 4) + 2\sqrt{D_F}}{(w-1)(4(N-w) + (w-1)\alpha N)\alpha N} \tag{8}$$

with

$$D_F = w^2(N-1)(\alpha N - 4)\left\{2(2-\alpha)(w-1)\alpha N^2 + w^2(N-1)(\alpha N - 4)\right\}.$$

On the flip boundary, at least one of the eigenvalues is equal to -1. Crossing this boundary the equilibrium point goes through a period doubling cascade to chaos. Note that the sign of this discriminant depends on the particular value of α. It can be confirmed that $D_F = 0$ has three real, positive roots:

$$\alpha_S = \frac{-w^2 + N(w^2 + 4(w-1)) - \sqrt{\Delta}}{4N(w-1)},$$

$$\alpha_L = \frac{-w^2 + N(w^2 + 4(w-1)) + \sqrt{\Delta}}{4N(w-1)},$$

$$\alpha_M = \frac{4}{N},$$

with $\Delta = \left[w^2 - N\left(w^2 + 4(w-1)\right)\right]^2 - 32(N-1)(w-1)w^2$, since it can be shown that $\Delta \geq 0$. Here S, L and M stand for "smaller", "larger" and "middle," as $\alpha_S < \alpha_M < \alpha_L$ holds. Therefore, $D_F > 0$ for $\alpha < \alpha_S$ and $\alpha_M < \alpha < \alpha_L$, and $D_F \leq 0$ otherwise. The adaptive coefficient α is assumed to be positive and not greater than unity, and the number of firms N must be greater than 2. Since $\alpha_L \geq 2$ for $N \geq 2$, α_L is outside the unit domain of α. When $N \geq 4$,

$$D_F \geq 0 \text{ for either } \alpha_M \leq \alpha \leq 1 \text{ or } 0 < \alpha \leq \alpha_S,$$

and

$$D_F < 0 \text{ for } \alpha_S < \alpha < \alpha_M.$$

When $2 \leq N < 4$, α_M becomes greater than unity and thus it is outside the domain. The sign of D_F is positive or negative according to α is less or greater than α_S.

Judging from the above considerations, the stability domain for non-negative Cournot equilibrium is surrounded by the non-negativity boundary h_N, the Hopf boundary h_H, and the flip boundary h_F. The last two boundaries, however, still have complicated expressions. So, instead of analytic study we will consider an important special case and perform numerical simulation. For further simplification, we set $w = 2$ (that is, the industry consists of two groups with equal size) and increase the number of the firms by two from $N = 4$ to $N = 10$ to see how the shape of the stability domain changes as the number of firms in the industry increases. We note that the present model with $N = 2$ reduces to the Puu (duopoly) model in which the Hopf bifurcation is shown to occurs for $3 + 2\sqrt{2} < h \leq 25/4$.

The *feasible region* in terms of the adjustment speeds and the marginal cost ratios for which the Cournot equilibrium is non-negative is

$$P_N = \{(\alpha, h) \mid 0 < \alpha \leq 1 \text{ and } 1 < h \leq h_N\} \text{ for } N = 4, 6, 8, 10,$$

where h_N is the upper bound of the marginal cost ratio defined in (5). Since the upper bound is determined by N and w, the feasible region is presented by a rectangular, $P_N = (0, 1] \times (1, h_N]$ which decreases in N (i.e., $P_{N+1} \subset P_N$). Then the *stability region* is given as

$$S_N = \{(\alpha, h) \mid 0 < \alpha \leq 1 \text{ and } Max\{1, h_{F_N}\} < h \leq h_{H_N}\},$$

where h_{F_N} and h_{H_N} denote the flip boundary (7) and the Hopf boundary (6). The *feasible and stable region* is the intersection of these two regions, $P_N \cap S_N$.

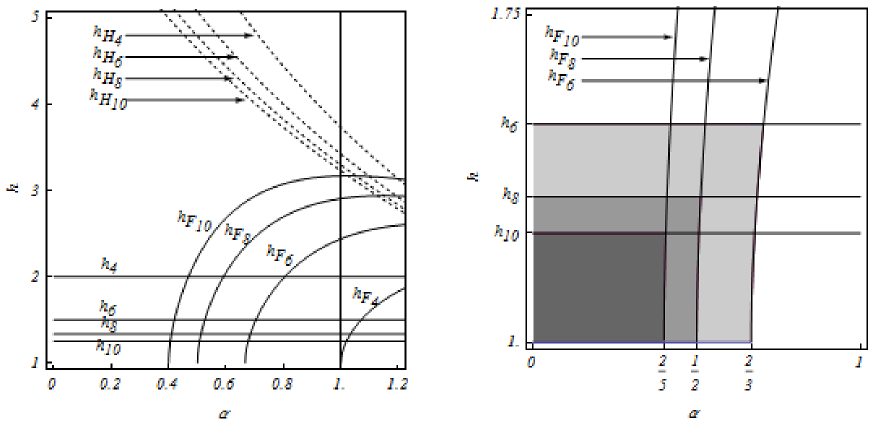
Figure 1(A) illustrates these boundaries and the upper bound of the marginal cost ratio, h_N, for each value of N $(= 4, 6, 8, 10)$. The Hopf boundary is negative-sloping and shifts downward as N increases. In addition it is bounded from below and its lower limit is defined by

$$\lim_{N \to \infty} h_{H_N} = \frac{6 - \alpha + 4\sqrt{2 - \alpha}}{2 + \alpha}.$$

It can be checked that the condition $p^2 < 4q$ holds if $h > 2.19$ for $N = 4$, $h > 2.62$ for $N = 6$, $h > 2.94$ for $N = 8$ and $h > 3.17$ for $N = 10$. It is also observed in Figure 1(A) that the Hopf boundaries h_{H_N} are located far above the $h = h_N$ locus for each N. Thus the Hopf bifurcation does not occur in the feasible region P_N. The Flip boundary is bounded from above with its upper limit

$$\lim_{N \to \infty} h_{F_N} = \frac{12 - \alpha + 4\sqrt{2(4 - \alpha)}}{4 + \alpha}.$$

It is positive-sloping for the appropriate values of α and shifts leftward as N increases. Although the speed of adjustment α is positive and not greater than unity, it is measured to the right up to the point 1.2 on the horizontal line in Figure 1(A) to confirm that the flip boundary with $N = 4$, the most inner flip boundary, takes unity for $\alpha = 1$. Thus $P_4 \cap S_4 = P_4$, so all positive Cournot points are stable for $N = 4$. However, the flip boundary with $N \geq 6$ divides the corresponding feasible region P_N into two sub regions, stability region with $h_{F_N} < h$ and instability region with $h_{F_N} > h$. Figure 1(B) enlarges the lower-left part of Figure 1(A) and depicts three stability regions: the largest rectangular with light-gray is the stability region with $N = 6$, the smallest rectangular with dark-gray is the one with $N = 10$ and the remaining middle rectangular is the one with $N = 8$. Since the $h = h_N$ locus and the flip boundary shift downward and leftward, respectively, the feasible and stable region becomes smaller as the number of N increases.



(A) Various boundaries (B) Enlargment of Figure 1(A)

Figure 1. Stability and feasible regions for $N = 4, 6, 8, 10$

Our extension from the two-firm (i.e., duopoly) model to the two-group model reveals interesting features of the nonlinear oligopoly.[1] To see them, we

[1] See Puu (2003) for the theoretical and numerical analyses of the two-firm (duopoly) and the three-firm (triopoly) models.

perform numerical simulations in two models and compare the results. To this end, in the two-group model, we take the set of parameters $N = 6$, $w = 2$ and choose the two bifurcation parameters: one is the common adjustment coefficient $\alpha = \beta$ and the other is the cost ratio $h = b/a$ between the two groups where $a = 1$ is assumed for simplicity. The dynamic system has now the from (4)

$$x(t+1) = (1-\alpha)x(t) + \alpha\left(\sqrt{\frac{2x(t)+3y(t)}{a}} - (2x(t)+3y(t))\right),$$

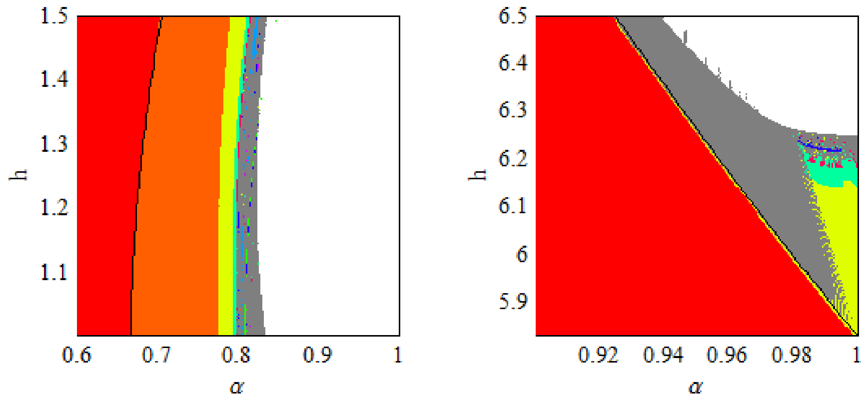
$$y(t+1) = (1-\alpha)y(t) + \alpha\left(\sqrt{\frac{3x(t)+2y(t)}{b}} - (3x(t)+2y(t))\right).$$

On the other hand, in the two-firm model, replacing $N = 6$ with $N = 2$ reduces the this dynamic system to

$$x(t+1) = (1-\alpha)x(t) + \alpha\left(\sqrt{\frac{y(t)}{a}} - y(t)\right),$$

$$y(t+1) = (1-\alpha)y(t) + \alpha\left(\sqrt{\frac{x(t)}{b}} - x(t)\right).$$

Changing α from 0.6 to 1.0 and h (actually b) from 1 to 1.5 generates the two-parameter bifurcation diagram shown in Figure 2 (A) while changing α from 0.9 to 1.0 and h from $3+2\sqrt{2}$ to 6.5 yields the two-parameter bifurcation diagram presented in Figure 2 (B). Different colors in the (α, h) plane indicate different periods of the cycles up to 16. Periodic points with a period larger than 16 and aperiodic points are colored in gray. The solution becomes imaginary if parameters are selected from the white region. The upward sloping black curve in Figure 2(A) and the downward sloping black curve in Figure 2(B) are the flip boundary of the two-group model and the Hopf boundary of the two-firm (i.e., duopoly) model, respectively. As the destabilizing scenario is concerned, the comparison shows two issues. First the stationary state is destabilized through a flip bifurcation in the two-group model and with Hopf bifurcation in the two-firm model. Even though both models can generate complex dynamics involving chaos, the way to chaos is different. Second, the loss of stability occurs for a relatively lower production cost ratio in the two-group model and for a higher ratio in the two-firm model. Furthermore, Figure 2(A) indicates that the value of the adjustment speed seems to be a main source of the flip bifurcation scenario since similar bifurcation takes place as α increases regardless of the value of h. This implies that the adjustment speed can be an effective control variable in the two-group model.



(A) The two-group model (B) The two-firm model

Figure 2. Two-parameter bifurcation diagrams in the (α, h) plane

4.2 Three Groups of Firms

Assume that the industry is now divided into three groups with N_a, N_b and N_c firms where $N_a + N_b + N_c = N$ and $N_a = \frac{1}{w_a}N$ and $N_b = \frac{1}{w_b}N$ with $w_a > 1$ and $w_b > 1$. The outputs of the three groups of firms are denoted by x, y and z,

$$x_1 = \cdots = x_{N_a} = x,\ x_{N_a+1} = \cdots = x_{N_a+N_b} = y \text{ and } x_{N_a+N_b+1} = \cdots = x_N = z,$$

the marginal costs are by a, b and c,

$$c_1 = \cdots = c_{N_a} = a,\ c_{N_a+1} = \cdots = c_{N_a+N_b} = b \text{ and } c_{N_a+N_b+1} = \cdots = c_N = c,$$

and the speeds of adjustment are by α, β and γ,

$$\alpha_1 = \cdots = \alpha_{N_a} = \alpha,\ \alpha_{N_a+1} = \cdots = \alpha_{N_a+N_b} = \beta \text{ and } \alpha_{N_a+N_b+1} = \cdots = \alpha_N = \gamma.$$

As in the previous section, we assume that the firms in the first group are the most efficient in a sense that their marginal cost is the smallest,

$$a < min\{b, c\}.$$

The marginal cost ratios are denoted by h and k,

$$h = \frac{b}{a} > 1 \text{ and } k = \frac{c}{a} > 1.$$

The Cournot outputs are obtained from equation (2) as

$$x^c = \frac{N-1}{C^2}(C - a(N-1)),$$

$$y^c = \frac{N-1}{C^2}(C - b(N-1)),$$

$$z^c = \frac{N-1}{C^2}(C - c(N-1))$$

with $C = N_a a + N_b b + N_c c$. It can also be seen that in the nonnegativity conditions (3),

$$C - a(N-1) = a\left\{N_b(h-1) + N_c(k-1) + 1\right\} > 0,$$

$$C - b(N-1) = a\left\{N_c k - (N_a + N_c - 1)h + N_a\right\}$$

and

$$C - c(N-1) = a\left\{-(N_a + N_b - 1)k + N_b h + N_a\right\}.$$

The first inequality is always true. In order to have positive values of $y^c$and z^c, we have to assume that

$$k > \frac{N_c + N_a - 1}{N_c} h - \frac{N_a}{N_c}$$

and

$$k \leq \frac{N_b}{N_a + N_b - 1} h + \frac{N_a}{N_a + N_b - 1}.$$

Let the right hand sides of the first and the second inequalities be denoted by $f_b(h)$ and $f_c(h)$. There is no guarantee that $f_b(h) > 1$, but $f_c(h) > 1$ always for $h > 1$. Therefore the set of the marginal cost ratios that generate non-negative Cournot equilibrium can be defined by

$$N_N = \left\{(h,k) \mid \max\{1, f_b(h)\} \leq k \leq f_c(h) \text{ and } h > 1\right\}.$$

The characteristic equation (A-2) can be written in the form

$$1 + N_a \tfrac{\alpha\gamma_a}{1-\alpha(1+\gamma_a)-\lambda} + N_b \tfrac{\beta\gamma_b}{1-\beta(1+\gamma_b)-\lambda} + N_c \tfrac{\gamma\gamma_c}{1-\gamma(1+\gamma_c)-\lambda} = 0.$$

Introducing the notation $R_a = 1 + \gamma_a$, $R_b = 1 + \gamma_b$ and $R_c = 1 + \gamma_c$, it can be reduced to a cubic equation,

$$-\lambda^3 + p\lambda^2 + q\lambda + r = 0,$$

where the coefficients are

$$p = (-N_a + (N_a - 1)R_a)\alpha + (-N_b + (N_b - 1)R_b)\beta + (-N_c + (N_c - 1)R_c)\gamma + 3,$$

$$\begin{aligned} q \;=\; & -2\left\{(-N_a + (N_a - 1)R_a)\alpha + (-N_b + (N_b - 1)R_b)\beta + (-N_c + (N_c - 1)R_c)\gamma\right\} \\ & + (-N_b R_a - N_a R_b + (N_a + N_b - 1)R_a R_b)\alpha\beta \\ & + (-N_c R_a - N_a R_c + (N_a + N_c - 1)R_a R_c)\alpha\gamma \\ & + (-N_c R_b - N_b R_c + (N_b + N_c - 1)R_b R_c)\beta\gamma - 3 \end{aligned}$$

and

$$\begin{aligned} r \;=\; & (-N_a + (N_a - 1)R_a)\alpha + (-N_b + (N_b - 1)R_b)\beta + (-N_c + (N_c - 1)R_c)\gamma \\ & - (-N_b R_a - N_a R_b + (N_a + N_b - 1)R_a R_b)\alpha\beta \\ & - (-N_c R_a - N_a R_c + (N_a + N_c - 1)R_a R_c)\alpha\gamma \\ & - (-N_c R_b - N_b R_c + (N_b + N_c - 1)R_b R_c)\beta\gamma \\ & + (-N_c R_a R_b - N_b R_a R_c - N_a R_b R_c - (N_a + N_b + N_c - 1))\alpha\beta\gamma. \end{aligned}$$

The Cournot equilibrium is locally asymptotically stable if all eigenvalues are less than unity in absolute value. Okuguchi and Irie (1990) have shown that the most simplified form of the necessary and sufficient conditions for the cubic equation to have roots only inside the unit circle is

$$1 - p + (-q) - r > 0,$$

$$1 + p + (-q) + r > 0,$$

$$1 + q + pr - r^2 > 0.$$

It is easy to show that

$$1 - p + (-q) - r = \frac{(n-1)(N_a + hN_b + kN_c)^3 \alpha\beta\gamma}{8hk(n-1)^3} > 0,$$

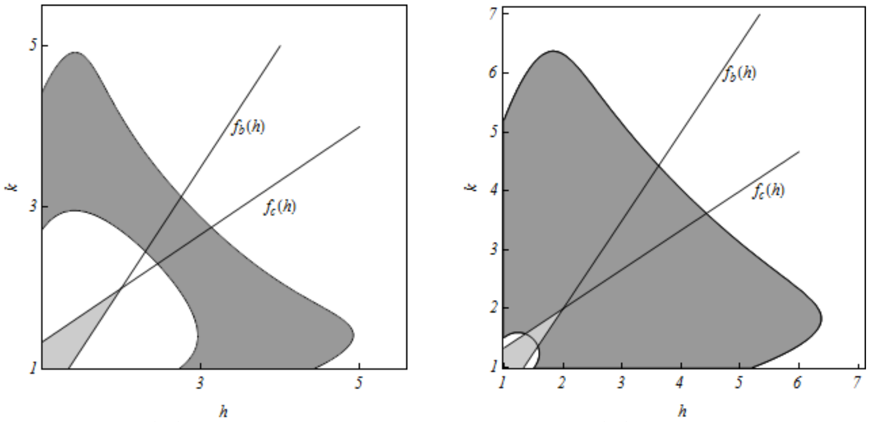
which implies that unity is not a root of the cubic equation.

It seems tedious to examine the remaining two conditions in general, instead we numerically confirm the stability region in the special case with $N = 6$ and $w_1 = w_2 = 3$. The same qualitative conclusions can be obtained for any $N(> 4)$, w_1 and w_2. In this case we have three groups with two firms in each of them. The region for the non-negative Cournot equilibrium is surrounded by the $k = f_a(h)$ and $k = f_b(h)$ loci. Substituting $N_a = N_b = N_c = 2$ into these functions determines the non-negativity region

$$P_6 = \left\{ (h,k) \mid \frac{2}{3}h + \frac{2}{3} \geq k \geq \frac{3}{2}h - 1,\ h > 1 \text{ and } k > 1 \right\}.$$

The nonnegativity and stability regions under naive expectation (with $\alpha = 1$) and those under adaptive adjustment (with $\alpha = 0.7$) are shown in Figures 3A and 3B, in each of which the outer curve is the Hopf boundary, the inner curve is the flip boundary and the two straight lines are the $k = f_a(h)$ and the $k = f_b(h)$ loci. The light gray area is the nonnegativity region and the dark gray area illustrates the stability region. Their intersection is the feasible and stability region. It is seen in Figure 3A that the non-negativity region is completely outside the stability region for naive expectation. This implies that the non-negative Cournot equilibrium is always unstable under naive expectation.[2] It is, in turn, seen in Figure 3B that about half of the non-negativity region is included in the stability region for adaptive adjustment, which implies that an unstable Cournot equilibrium under naive expectations could be stabilized by adaptive adjustments.

[2] This result reminds us the classical result of Theocaris (1960) that the a non-differentiated Cournot equilibrium is stable only in the duopoly case if the expectatitions are naively formed and the price and the cost functions are linear.



(A) $\alpha = 1$ (B) $\alpha = 0.7$

Figure 3. Stability and feasible regions of the naive and adaptive systems

Figure 4 is an enlargement of the south-west corner of Figure 3B. It shows how the flip boundary shifts if the value of α changes. In the current example, solving $1 + p + (-q) + r = 0$ for α yields 0.8, given $k = h = 2$. This implies that the flip boundary with $\alpha = 0.8$ passes through the point $(2, 2)$, the vertex of the triangular part of the stablility region. Thus if $\alpha \geq 0.8$, then the nonnegativity region is completely outside the stability region for adaptive adjustment, so no nonnegative equilibrium becomes stable. On the other hand, solving $1 + p + (-q) + r = 0$ for α yields $\frac{2}{3}$, given $k = h = 1$. This implies that the flip boundary with $\alpha = \frac{2}{3}$ passes through the point $(1, 1)$. If $\alpha \leq \frac{2}{3}$, then the nonnegativity region is entirely inside the stability region, in which case all nonnegative equilibria become stable. If $\frac{2}{3} < \alpha < 0.8$, then only a certain part of the nonnegativity region belongs to the stability region, and this part becomes larger if the value of α decreases. In particular, for $\alpha = 0.7$, the stability region is horizontally-striped and is the triangle with a base of the flip boundary with $\alpha = 0.7$, the most outer circular curve. In the remaining light-gray area of the nonnegativity region, the Cournot output is locally unstable. When the adaptive parameter α decreases, the flip boundary shifts inside accordingly. As a consequence, the stability region enlarges and the unstable region shrinks.

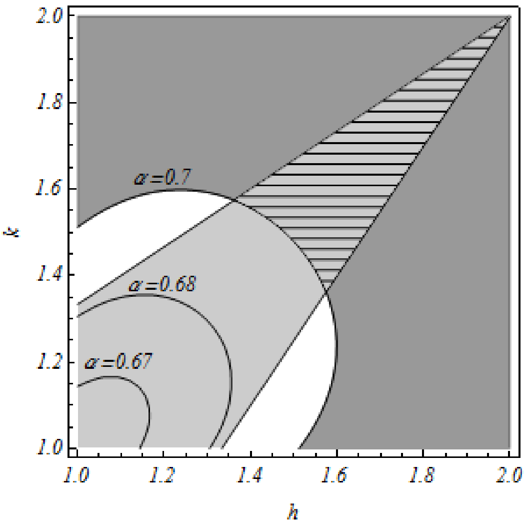


Figure 4. Dependency of the stability region on α

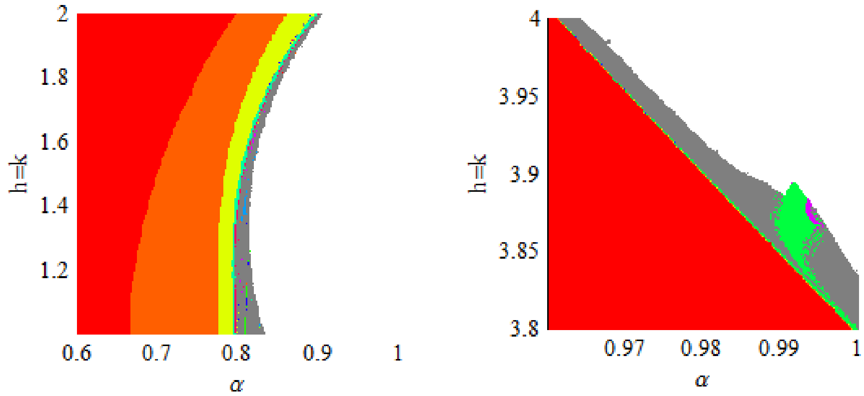
As the comparison of the three-group and the three-firm (triopoly) models, we, again, perform numerical simulations for these two models. The dynamic system in the three-group model is

$$x(t+1) = (1-\alpha)x(t) + \alpha\left(\sqrt{\frac{x(t)+2y(t)+2z(t)}{a}} - (x(t)+2y(t)+2z(t))\right)$$

$$y(t+1) = (1-\beta)y(t) + \beta\left(\sqrt{\frac{2x(t)+y(t)+2z(t)}{b}} - (2x(t)+y(t)+2z(t))\right)$$

$$z(t+1) = (1-\gamma)z(t) + \gamma\left(\sqrt{\frac{2x(t)+2y(t)+z(t)}{c}} - (2x(t)+2y(t)+z(t))\right)$$

where for simplicity, $\alpha = \beta = \gamma$, $h = k$ and $a = 1$ are selected implying $b = c$. The dynamic system in the three-firm model (i.e., triopoly) can be constructed similarly. Selecting α and h (in particular b) as the bifurcation parameters, Figure 5 illustrates the bifurcation diagrams in the (α, h) plane. The numerical investigations clearly reveal qualitatively the same issues as those we saw in Figure 2. Namely, first of all, the destabilizing process goes to chaos through a flip bifurcation with a lower production ratio in the three-group model and through a Hopf bifurcation with a higher ratio in the three-firm model. Second, the adjustment speed can be used to control unstable trajectories. Comparing the bifurcation diagrams of the two-group and the three-group models shows the similarity of the destabilizing process of these models in which period-doubling bifurcation takes place.



(A) The three-group model (B) The three-firm model

Figure 5. Two-parameter bifurcation diagrams in the $(\alpha, h = k)$ plane

In Figure 6, we present the bifurcation diagram in the (h, k) plane to draw attention to the feasibility of the solutions of the three-group model. The value $\alpha = 0.8$ is fixed. As already explored in Figure 4, h and k range from 1 to 2 and the stationary state is unstable for any combination of h and k from this region. Color has the same meanings as before. The feasible region of the three-group model is defined by two upward sloping curves, $k = f_b(h)$ and $k = f_c(h)$. Notice that although the area outside the feasible region is colored in the same way as the feasible region, the stationary point defined in that area becomes negative and thus economically meaningless. Bifurcation makes sense only in the feasible region from an economic point of view. Notice further that a trajectory that oscillates around the stationary point periodically or aperiodically may take negative values and thus becomes economically meaningless. One way is to confine the parameter choice in such a way that the resultant dynamics does not become infeasible. The other way is to reconstruct the dynamic system by taking into account the non-negativity constraint explicitly. However, the former has the difficulty of deriving the confinement conditions in our model as many parameters are involved, and the latter makes the asymptotic behavior of the dynamic system significantly different and more difficult to analyze. Since our main concern is to control the unstable trajectories and our main conclusion is that the adjustment speed is an effective control parameter, which is supposed to hold in those models, we used the model without such modifications at the expense of some economic precision.

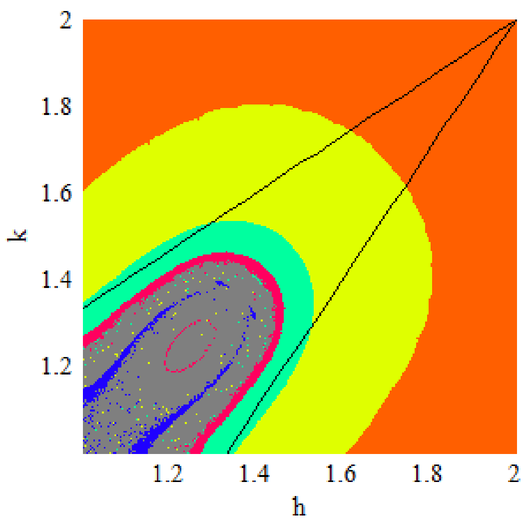


Figure 6. Two-parameter bifurcation diagram in the (h, k) plane

5 Conclusion

The asymptotic behavior of discrete dynamic systems is a fundamental research issue. In this paper a special class of economic models was examined. For the sake of mathematical simplicity we selected N-firm Cournot oligopolies without product differentiation, and with isoelastic price function. The reaction functions and the equilibrium were determined first, and then the asymptotic behavior of the equilibrium was illustrated in two special cases with two or three groups of identical firms. Stability conditions could be derived analytically in the first case, and the dependence of the asymptotic properties of the equilibrium on the number of firms was illustrated by computer simulation in the second case. The results of the nonlinear duopoly and triopoly models show that the Cournot equilibrium can be destabilized through a Hopf bifurcation. We also found the following new dynamic phenomenon. In the multi-group models, the stationary state is destabilized through the Feigenbaum period doubling sequence, and a Hopf bifurcation can occur only in the infeasible regions in which the stationary state is negative. For $N > 4$, the multi-group models are unstable if α is close to one and become stable if α is below a certain threshold, regardless of the production cost ratios. This implies that the main source of instability is the speed of adjustment and thus the stationary state could be stabilized by selecting sufficiently small speed of adjustment. That is, the multi-group models with N firms are unstable under naive expectations but are controllable with the adaptive adjustment process in which the speed of adjustment is the control parameter.

Appendix

In this appendix, we derive the stability conditions of the *adaptive system* in which the expectation is adaptively formulated,

$$y_i^e(t+1) = (1-\alpha_i)y_i^e(t) + \alpha_i \sum_{j\neq i} x_j(t).$$

Here $\alpha_i \in (0,1]$ is the speed of adjustment of firm i. The stability conditions are also useful to determine the local dynamic behavior of the naive system as well as that of the inertia system.

We consider the adjustment process with adaptive expectations first, since the one with naive expectations can be obtained by selecting the speed of adjustment equal to unity (i.e., $\alpha_i = 1$). For $i = 1, 2, ..., N$,

$$\begin{cases} x_i(t+1) & = \sqrt{\dfrac{\alpha_i \sum_{j\neq i} x_j(t) + (1-\alpha_i)y_i^e(t)}{c_i}} - \left(\alpha_i \sum_{j\neq i} x_j(t) + (1-\alpha_i)y_i^e(t)\right), \\ y_i^e(t+1) & = \alpha_i \sum_{j\neq i} x_j(t) + (1-\alpha_i)y_i^e(t). \end{cases}$$

The Jacobian at the equilibrium has the form

$$\mathbf{J} = \begin{pmatrix} 0 & \gamma_1\alpha_1 & \cdot & \gamma_1\alpha_1 & \gamma_1(1-\alpha_1) & 0 & \cdot & 0 \\ \gamma_2\alpha_2 & 0 & \cdot & \gamma_2\alpha_2 & 0 & \gamma_2(1-\alpha_2) & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma_N\alpha_N & \cdot & \gamma_N\alpha_N & 0 & 0 & \cdot & 0 & \gamma_N(1-\alpha_N) \\ 0 & \alpha_1 & \cdot & \alpha_1 & 1-\alpha_1 & 0 & \cdot & 0 \\ \alpha_2 & 0 & \cdot & \alpha_2 & 0 & 1-\alpha_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_N & \cdot & \alpha_N & 0 & 0 & \cdot & 0 & 1-\alpha_N \end{pmatrix},$$

where

$$\gamma_i \equiv \frac{\partial f_i(y_i^e)}{\partial y_i^e} = \frac{C}{2c_i(N-1)} - 1.$$

Notice that condition (3) implies that $\gamma_i \geq -\frac{1}{2}$ for all i.

The eigenvalue equation has the form

$$\mathbf{J}\mathbf{x} = \lambda\mathbf{x} \text{ with } \mathbf{x} = (u_1, ...u_N, v_1, ...v_N)^T,$$

or equivalently,

$$\begin{cases} \gamma_i\alpha_i \sum_{j\neq i} u_j + \gamma_i(1-\alpha_i)v_i = \lambda u_i, & 1 \leq i \leq N, \\ \alpha_i \sum_{j\neq i} u_j + (1-\alpha_i)v_i = \lambda v_i, & 1 \leq i \leq N. \end{cases}$$

Subtracting the γ_i-multiple of the second equation from the first one gives

$$\lambda(u_i - \gamma_i v_i) = 0.$$

The value $\lambda = 0$ cannot destroy stability, so we may assume $\lambda \neq 0$. Then $u_i = \gamma_i v_i$, and by substituting it into the second equation, we have

$$\alpha_i \sum_{j\neq i} \gamma_j v_j + (1-\alpha_i)v_i = \lambda v_i, \quad 1 \leq i \leq N.$$

This is the usual eigenvalue problem of the $N \times N$ matrix

$$\mathbf{H} = \begin{pmatrix} 1-\alpha_1 & \gamma_2\alpha_1 & \cdot & \gamma_N\alpha_1 \\ \gamma_1\alpha_2 & 1-\alpha_2 & \cdot & \gamma_N\alpha_2 \\ \cdot & \cdot & \cdot & \cdot \\ \gamma_1\alpha_N & \gamma_2\alpha_N & \cdot & 1-\alpha_N \end{pmatrix}.$$

Notice that

$$\mathbf{H} = \mathbf{D} + \mathbf{a}\mathbf{b}^T,$$

with

$$\mathbf{a}^T = (\alpha_1, \alpha_2, ..., \alpha_N),\ \mathbf{b}^T = (\gamma_1, \gamma_2, ..., \gamma_N),$$

and

$$\mathbf{D} = diag(1-\alpha_1(1+\gamma_1), 1-\alpha_2(1+\gamma_2), ..., 1-\alpha_N(1+\gamma_N)).$$

The characteristic polynomial of $\mathbf{H}$ can be decomposed by using the simple fact that if $\mathbf{x}, \mathbf{y} \in R^N$, then

$$\det\left(\mathbf{I} + \mathbf{x}\mathbf{y}^T\right) = 1 + \mathbf{y}^T\mathbf{x},$$

where $\mathbf{I}$ is the $N \times N$ identity matrix. So we have

$$\begin{aligned} \det(\mathbf{H} - \lambda\mathbf{I}) &= \det(\mathbf{D} + \mathbf{a}\mathbf{b}^T - \lambda\mathbf{I}) \\ &= \det(\mathbf{D} - \lambda\mathbf{I})\det(\mathbf{I} + (\mathbf{D} - \boldsymbol{\lambda}\mathbf{I})^{-1}\mathbf{a}\mathbf{b}^T). \end{aligned}$$

The roots of the first factor are $1-\alpha_i(1+\gamma_i)$ which are inside the unit circle if and only if

$$-1 < 1 - \alpha_i(1+\gamma_i) < 1,$$

which occurs if and only if

$$c_i > \frac{\alpha_i C}{4(N-1)}. \tag{A-1}$$

The other eigenvalues are the roots of equation

$$1 + \sum_{i=1}^{N} \frac{\alpha_i\gamma_i}{1-\alpha_i(1+\gamma_i)-\lambda} = 0. \tag{A-2}$$

The Cournot equilibrium is locally asymptotically stable if all eigenvalues are less than unity in absolute value and is unstable if at least one eigenvalue is outside the unit circle. However, since parameters $\alpha_1, \alpha_2, ..., a_N$ can be selected arbitrarily in interval $(0, 1]$, there is a large flexibility in the location of the eigenvalues.

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