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Discussion Paper No.197

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January 2013

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Tokyo, Japan

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Abstract

A neoclassical growth model is examined with a special mound-shaped production function. Continuous time scales are assumed and a complete steady state and stability analysis is presented. Then fixed delay is assumed and it is shown how the asymptotic stability of the steady state is lost if the delay reaches a certain threshold, where Hopf bifurcation occurs. In the case of continuously distriubuted delays, we show that with small average delays stability is preserved, then lost at a threshold and is regained if the average delay becomes sufficiently large. The occurence of Hopf bifurcation is shown at both critical values.

*The authors highly appreciate financial supports from the Japan Society for the Promotion of Science (Gtant-in-Aid for Scientific Research (C) 24530202) and Chuo University (Grant for Special Research and Joint Research Grant 0981).

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1 Introduction

The examination of economic growth models is one of the most frequently discussed issues in mathematical economics. Day (1982, 1983) has investigated a neoclassical growth, and a productivity and population growth model and showed the emergence of complex behavior even under simple economic structure. His models were based on discrete time scales and a mound-shaped production function that represented the negative effect of pollution resulting from increasing capital. It was demonstrated by numerical computations that these models could generate cyclic and even chaotic behavior. Following Day's pioneering works, a lot of effort has been given to the understanding of complex economic dynamics. Day (1994), Puu (2003) and Bischi et al. (2010) present the earlier contributions of this field. A large number of studies assumed discrete time scales. Li and Yorke (1975) have introduced the "period-three condition" to detect chaos, which has many applications in first-order nonlinear difference equations. The papers collected by Rosser (2004) offer many applications. Only a few studies are devoted to the case of continuous time scales, since there is no general criterion to detect chaos and the system must have at least three dimensions.

In this paper we will examine an extension of the neoclassical aggregate growth model, which can be traced back to the early works of Solow (1956) and Swan (1956). It is a very interesting question to see the dynamic behavior of the growth models with continuous time scales. Matsumoto and Szidarovszky (2011) have introduced a neoclassical growth model with a mound-shaped production function and investigated its dynamic behavior with both discrete and continuous time scales. The production function was assumed to be a Cobb-Douglas type function of the form $F(x) = Ax^a(1-x)^b$, where x is the capital per unit labor. This paper will consider another type of mound-shaped production function and will examine the stability of the steady state with and without time delays. Two kinds of delay will be discussed, fixed and continuously distributed (continuous hereafter) delays.

This paper develops as follows. First the mathematical model is formulated without time delays, and complete steady state and stability analysis is presented. Then models with fixed delays and then with continuous delays are introduced and complete stability analysis is given. The last section concludes the paper.

2 The Mathematical Model

Matsumoto and Szidarovszky (2011) has introduced a special growth model of the form

$$\dot{x}(t) = sF(x(t)) - \alpha x(t)$$

where x is the capital per labor, s and α are positive parameters where $s \in (0, 1)$ is the average propensity to save and $\alpha = n + s\mu$ with μ being the depreciation ratio of capital and n the growth rate of labor. Assume now the mound-shaped production function

$$F(x) = \varepsilon x^{\gamma} e^{-\delta x},$$

so the modified mathematical model becomes

$$\dot{x}(t) = -\alpha x(t) + \beta x(t)^{\gamma} e^{-\delta x(t)}, \tag{1}$$

where α, γ, δ and $\beta = s\varepsilon$ are positive parameters. The number of steady states and their locations depend on the specific values of the model parameters. We will consider three different cases: $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$. Let now $f(x)$ denote the right hand side of equation (1).[1]

Case I

Assume first that $\gamma < 1$. The steady states are the solutions of $f(x) = 0$. Notice that $f(0) = 0$, so zero is a steady state. $f(x)$ converges to $-\infty$ as $x \to \infty$. Since

$$f'(x) = -\alpha + \beta\gamma x^{\gamma-1} e^{-\delta x} - \beta\delta x^{\gamma} e^{-\delta x},$$

$f'(x)$ converges to ∞ as x tends to zero with positive values. Hence $f(x)$ increases for small values of $x > 0$. The steady state equation $f(x) = 0$ can be written as

$$x^{\gamma}(-\alpha x^{1-\gamma} + \beta e^{-\delta x}) = 0, \tag{2}$$

so the positive steady state is the unique solution of equation

$$\alpha x^{1-\gamma} = \beta e^{-\delta x}. \tag{3}$$

The left hand side is zero at $x = 0$ and strictly increasing, furthermore, converges to ∞ as x tends to infinity. The right hand side is $\beta > 0$ at $x = 0$, strictly decreases and converges to zero as $x \to \infty$. Hence there is a unique solution $\bar{x} > 0$ of (3), and $f(x) > 0$ if $x < \bar{x}$ and $f(x) < 0$ as $x > \bar{x}$. These relations imply that if $x(0) < \bar{x}$, then $x(t)$ increases and converges to $\bar{x}$, and if $x(0) > \bar{x}$, then $x(t)$ decreases and converges to $\bar{x}$. If $x(0) = \bar{x}$, then $x(t)$ remains $\bar{x}$ for all $x > 0$. Thus $\bar{x}$ is globally asymptotically stable.

Case II

Assume next that $\gamma = 1$. Then the steady state equation has the form

$$x(-\alpha + \beta e^{-\delta x}) = 0, \tag{4}$$

so zero is a steady state and there is a unique root of the second factor,

$$\bar{x} = \frac{1}{\delta} \ln \frac{\beta}{\alpha}. \tag{5}$$

[1] If $x(0) = 0$, then the identically zero function is a solution which case is not interesting from the economic point of view and is eliminated from further considerations.

If $\beta \le \alpha$, then the value of $f(x)$ is negative for all $x > 0$. Therefore $x(t)$ is decreasing and converges to zero with arbitrary $x(0) > 0$. If $\beta > \alpha$, then $\bar{x} > 0$, furthermore $f(x) > 0$ as $x < \bar{x}$, and $f(x) < 0$ as $x > \bar{x}$. If $x(0) < \bar{x}$, then $x(t)$ increases and if $x(0) > \bar{x}$, then x decreases and converges to $\bar{x}$, and if $x(0) = \bar{x}$, then $x(t)$ remains $\bar{x}$ for all $t > 0$. Hence $\bar{x}$ is globally asymptotically stable.

Case III

Consider finally the case of $\gamma > 1$. The steady state equation has now the form

$$x(-\alpha + \beta x^{\gamma-1} e^{-\delta x}) = 0, \tag{6}$$

so zero is a steady state again, and any other steady state is the solution of equation

$$g(x) = -\alpha + \beta x^{\gamma-1} e^{-\delta x} = 0. \tag{7}$$

Notice that

$$g(0) = \lim_{x \to \infty} g(x) = -\alpha$$

and

$$\begin{aligned} g'(x) &= \beta(\gamma-1)x^{\gamma-2}e^{-\delta x} - \beta\delta x^{\gamma-1}e^{-\delta x} \\ &= \beta x^{\gamma-2}e^{-\delta x}(\gamma - 1 - \delta x). \end{aligned}$$

Therefore $g(x)$ has its global maximum at

$$\hat{x} = \frac{\gamma - 1}{\delta}, \tag{8}$$

increases for $x < \hat{x}$ and decreases for $x > \hat{x}$. Now we have three sub-cases.

(i) if $g(\hat{x}) < 0$, then there is no positive steady state and with arbitrary $x(0) > 0$, $x(t)$ decreases and converges to zero.

(ii) if $g(\hat{x}) = 0$, then $\bar{x} = \hat{x}$ is the unique positive steady state and $f(x) < 0$ for all $0 < x \ne \bar{x}$. If $x(0) < \bar{x}$, then $x(t)$ decreases and converges to 0, and if $x(0) > \bar{x}$, then $x(t)$ decreases again and now converges to $\bar{x}$. If $x(0) = \bar{x}$, then $x(t) = \bar{x}$ for all $t > 0$.

(iii) If $g(\hat{x}) > 0$, then equation (7) has two positive solutions, $\bar{x}_1 < \hat{x}$ and $\bar{x}_2 > \hat{x}$. Relation (6) implies that $f(x) < 0$ as $x < \bar{x}_1$ or $x > \bar{x}_2$, and $f(x) > 0$ if $\bar{x}_1 < x < \bar{x}_2$. Therefore if $x(0) < \bar{x}_1$, then $x(t)$ decreases and converges to zero, if $\bar{x}_1 < x(0) < \bar{x}_2$, then $x(t)$ increases and converges to $\bar{x}_2$, if $x(0) > \bar{x}_2$, then $x(t)$ decreases and converges to $\bar{x}_2$. That is, $\bar{x}_1$ is locally unstable and $\bar{x}_2$ is locally asymptotically stable. If $x(0) = \bar{x}_1$ or $x(0) = \bar{x}_2$, then $x(t)$ remains at that steady state level for all $t > 0$.

3 Model with Fixed Delay

The fixed delay $T > 0$ is assumed in the second term of the right hand side of equation (1), so we have the following equation:

$$\dot{x}(t) = -\alpha x(t) + h(x(t-T)) \tag{9}$$

where

$$h(x) = \beta x^{\gamma} e^{-\delta x}. \tag{10}$$

The local asymptotic behavior of the trajectory can be examined by linearization. Let $\bar{x}$ be a positive steady state. Then the linearized equation has the form

$$\dot{x}_{\delta}(t) = -\alpha x_{\delta}(t) + h'(\bar{x}) x_{\delta}(t-T),$$

where $x_{\delta}(t)$ is the deviation of $x(t)$ from the steady state level. Looking for the solution in the usual form $x_{\delta}(t) = e^{\lambda t} u$, we have

$$\lambda e^{\lambda t} u = -\alpha e^{\lambda t} u + h'(\bar{x}) e^{\lambda(t-T)} u$$

which gives the characteristic equation

$$\lambda + \alpha = h'(\bar{x}) e^{-\lambda T}$$

or

$$(\lambda + \alpha) e^{\lambda T} = h'(\bar{x}). \tag{11}$$

Lemma 1 *Assume that* $|h'(\bar{x})| < \alpha$. *Then* $\bar{x}$ *is locally asymptotically stable.*

Proof. Assume that $\mathrm{Re}\lambda \geq 0$. Then

$$|\lambda + \alpha| \geq \alpha$$

and since

$$\begin{aligned} e^{\lambda T} &= e^{T(\mathrm{Re}\,\lambda)} e^{iT(\mathrm{Im}\,\lambda)} \\ &= e^{T(\mathrm{Re}\,\lambda)} (\cos[T(\mathrm{Im}\,\lambda)] + i \sin[T(\mathrm{Im}\,\lambda)]), \end{aligned}$$

clearly

$$\left| e^{\lambda T} \right| \geq 1.$$

Therefore

$$\left| (\lambda + \alpha) e^{\lambda T} \right| \geq \alpha \text{ and } |h'(\bar{x})| < \alpha$$

implying that λ cannot be an eigenvalue. ■

Notice that

$$h'(x) = \beta x^{\gamma - 1} e^{-\delta x} (\gamma - \delta x) \tag{12}$$

and at the steady state

$$\beta x^{\gamma} e^{-\delta x} = \alpha x$$

implying that

$$\beta x^{\gamma-1} e^{-\delta x} = \alpha.$$

Therefore

$$h'(\bar{x}) = \alpha(\gamma - \delta\bar{x}), \tag{13}$$

so the characteristic equation (11) can be rewritten as

$$(\lambda + \alpha)e^{\lambda T} = \alpha(\gamma - \delta\bar{x}).$$

We also mention that the condition of Lemma 1 can be rewritten as

$$|\gamma - \delta\bar{x}| < 1$$

or equivalently

$$\frac{\gamma - 1}{\delta} < \bar{x} < \frac{\gamma + 1}{\delta}. \tag{14}$$

In the special case of $\gamma = 1$, this condition has the form

$$0 < \frac{1}{\delta} \ln \frac{\beta}{\alpha} < \frac{2}{\delta}$$

which is equivalent to relation

$$\alpha < \beta < \alpha e^2.$$

In order to give a complete stability analysis, we have to find the possible stability switches. At any stability switch, $\lambda = i\omega$ with $\omega > 0$ and substituting it into equation (11) yields

$$i\omega + \alpha = h'(\bar{x})\left(\cos\omega T - i\sin\omega T\right). \tag{15}$$

Separating the real and imaginary parts gives two equations,

$$h'(\bar{x})\sin\omega T = -\omega \tag{16}$$

and

$$h'(\bar{x})\cos\omega T = \alpha. \tag{17}$$

Adding the squares of these equations gives

$$h'(\bar{x})^2 = \alpha^2 + \omega^2$$

so

$$\omega = \alpha\sqrt{(\gamma - \delta\bar{x})^2 - 1} \tag{18}$$

In order to have solution we have to assume now that

$$|\gamma - \delta\bar{x}| > 1, \tag{19}$$

that is, (14) is violated with strict inequalities. From (16) we have that if $h'(\bar{x}) > 0$, then $T^{(n)} = T_+^{(n)}$ with

$$T_+^{(n)} = \frac{1}{\omega}\left(2\pi - \sin^{-1}\left(\frac{\omega}{h'(\bar{x})}\right) + 2n\pi\right) \quad \text{for } n \geq 0 \tag{20}$$

and if if $h'(\bar{x}) < 0$, then $T^{(n)} = T_-^{(n)}$ with

$$T_-^{(n)} = \frac{1}{\omega}\left(\pi + \sin^{-1}\left(\frac{\omega}{h'(\bar{x})}\right) + 2n\pi\right) \quad \text{for } n \geq 0 \tag{21}$$

and by (13) and (19),

$$|h'(\bar{x})| > \alpha$$

so $h'(\bar{x})$ cannot be zero.

By selecting T as the bifurcation parameter and implicitly differentiating the characteristic equation with respect to T, we have

$$\begin{aligned} \lambda' &= h'(\bar{x})e^{-\lambda T}(-\lambda' T - \lambda) \\ &= -(\lambda + \alpha)(\lambda' T + \lambda) \end{aligned}$$

implying that

$$\lambda' = \frac{-\lambda(\lambda + \alpha)}{1 + T(\lambda + \alpha)}.$$

If $\lambda = i\omega$, then

$$\lambda' = \frac{\omega^2 - i\alpha\omega}{(1 + T\alpha) + iT\omega}$$

with real part

$$\operatorname{Re}\lambda' = \frac{\omega^2}{(1 + T\alpha)^2 + (T\omega)^2} > 0.$$

Therefore if a steady state is unstable with $T = 0$, then it remains unstable for all $T > 0$, and if a steady state is asymptotically stable at $T = 0$, then this stability is lost at $T = T^{(0)}$ and cannot be regained later.

Taking, $\alpha = 1$, $\beta = 25$, $\gamma = 1$ and $\delta = 1$, we give a numerical example in Figure 1. The critical value $\gamma - \delta\bar{x}$ is denoted by

$$z_c = 1 - \ln\left(\frac{\beta}{\alpha}\right) \simeq -2.22.$$

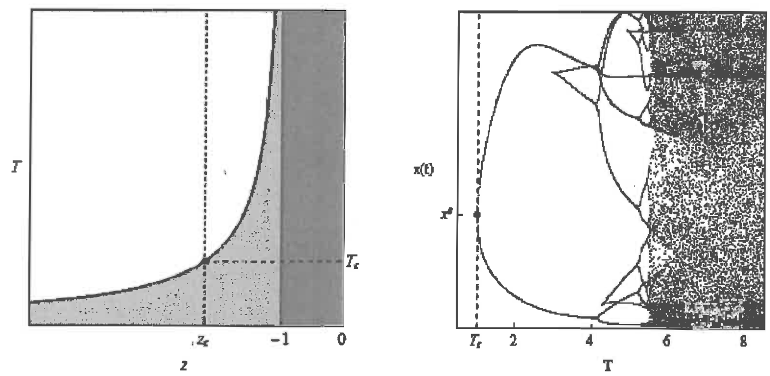
Introducing the notation $z = \gamma - \delta\bar{x}$ transforms the $T_-^{(0)}$ curve to

$$T_-^{(0)} = \frac{1}{\alpha\sqrt{z^2 - 1}}\left(\pi + \sin^{-1}\left(\frac{\sqrt{z^2 - 1}}{z}\right)\right)$$

and then the corresponding critical value of the delay is

$$T_c = \frac{1}{\alpha\sqrt{z_c^2 - 1}}\left(\pi + \sin^{-1}\left(\frac{\sqrt{z_c^2 - 1}}{z_c}\right)\right) \simeq 1.03.$$

In Figure 1(A), the steady state is locally asymptotically stable in the dark-gray region with $z > -1$ due to Lemma 1. It is also locally asymptotically stable in the light-gray region, which is under the critical curve $T = T_-^{(0)}$ and it is unstable in the white region above the curve. Setting $z = z_c$ and increasing T along the vertical dotted line in Figure 1(A), we can see that the steady state loses stability at $T = T_c$. Further increasing T, as observed in Figure 1(B), generates complex dynamics through a *quasi* period-doubling bifurcation in which T increases from $T_c - 0.05$ to 8.5 with an increment of 0.01 and the local maximum and minimum of the corresponding trajectory are plotted against each value of T.



(A) Region division

(B) Bifurcation diagram

Figure 1. Dynamics with $\alpha = 1,\ \beta = 25,\ \gamma = 1$ and $\delta = 1$

4 Model with Continuously Distributed Delay

Assuming continuously distributed delays in the second term of equation (1) gives the following Volterra-type integro-differential equation:

$$\dot{x}(t) = -\alpha x(t) + \int_0^t \omega(t - s, T, m) h(x(s)) ds \qquad (22)$$

where $T > 0$ is a positive parameter, the average delay and $m \geq 0$ is an integer. The kernel function has the form

$$\omega(t-s,T,m) = \begin{cases} \frac{1}{T}e^{-\frac{t-s}{T}} & \text{if } m = 0, \\ \frac{1}{m!}\left(\frac{m}{T}\right)^{m+1}(t-s)^m e^{-\frac{m(t-s)}{T}} & \text{if } m \geq 1. \end{cases}$$

Linearizing equation (22), we have

$$\dot{x}_\delta(t) = -\alpha x_\delta(t) + h'(\bar{x})\int_0^t \omega(t-s,T,m)x_\delta(s)ds$$

where $x_\delta(t)$ is the deviation of $x(t)$ from the steady state level $\bar{x}$. We are looking for the solution in the usual exponential form

$$x_\delta(t) = e^{\lambda t}u,$$

then simple substitution shows that

$$\lambda e^{\lambda t}u = -\alpha e^{\lambda t}u + h'(\bar{x})\int_0^t \omega(t-s,T,m)e^{\lambda s}uds.$$

Notice that by introducing the new variable $S = t - s$ in the integral, we see that

$$\int_0^t \omega(t-s,T,m)e^{\lambda s}ds = \int_0^t \omega(S,T,m)e^{-\lambda S}dSe^{\lambda t},$$

and by letting $t \to \infty$, we have the characteristic equation

$$\lambda + \alpha = h'(\bar{x})\left(1 + \frac{\lambda T}{q}\right)^{-(m+1)}$$

with

$$q = \begin{cases} 1 & \text{if } m = 0, \\ m & \text{if } m \geq 1. \end{cases}$$

This equation can be rewritten as

$$(\lambda + \alpha)\left(1 + \frac{\lambda T}{q}\right)^{(m+1)} = h'(\bar{x}) = \alpha(\gamma - \delta\bar{x}). \tag{23}$$

We then have the following stability result on the dynamic equation (22) with continuously distributed delay which is similar to Lemma 1, the stability result on the dynamic equation (9) with fixed delay:

Lemma 2 *Assume that* $|h'(\bar{x})| < \alpha$. *Then* $\bar{x}$ *is locally asymptotically stable.*

Proof. Assume that $\operatorname{Re}\lambda \geq 0$. Then

$$|\lambda + \alpha| \geq \alpha \text{ and } \left|1 + \frac{\lambda T}{q}\right| \geq 1,$$

therefore

$$\left|(\lambda + \alpha)\left(1 + \frac{\lambda T}{q}\right)^{m+1}\right| \geq \alpha \text{ and } |h'(\bar{x})| < \alpha$$

implying that λ cannot be a solution of equation (23). ■

It has been known that the Routh-Hurwitz stability theorem provides necessary and sufficient conditions for all the roots of the polynomial equation with real coefficients to have negative real parts. It has been also known that it is difficult to locate the eigenvalues with analytic methods in general. However in some special cases as we examine below, analytic results are still possible to obtain.

Case I. $T = 0$

Assume first that $T = 0$, which reduces equation (22) with delays to equation (1) without delays. The asymptotic properties of this equation were already discussed earlier.

Case II. $T > 0$ and $m = 0$

Assume second that $T > 0$ and $m = 0$, with which the kernel function becomes exponentially declining. Then characteristic equation (23) becomes quadratic,

$$(\lambda + \alpha)(1 + \lambda T) = \alpha(\gamma - \delta\bar{x})$$

or

$$\lambda^2 T + \lambda(1 + \alpha T) + \alpha(1 - \gamma + \delta\bar{x}) = 0. \tag{24}$$

If $\gamma \leq 1$, then all coefficients are positive with a positive steady state, so it is locally asymptotically stable. Assume next that $\gamma > 1$. If

$$\bar{x} = \frac{\gamma - 1}{\delta},$$

then the constant term is zero indicating that one eigenvalue is zero and the other is negative. So $\bar{x}$ is marginally stable in the linearized model, so no conclusion can be drawn about its asymptotical behavior in the nonlinear model. If $\bar{x} < (\gamma - 1)/\delta$, then $\bar{x}$ is unstable and if $\bar{x} > (\gamma - 1)/\delta$, then $\bar{x}$ is locally asymptotically stable.

Case III. $T > 0$ and $m = 1$

Assume third that $T > 0$ and $m = 1$, with which the shape of the kernel function takes a bell-shaped form. Then we have a cubic characteristic polynomial:

$$(\lambda + \alpha)(1 + \lambda T)^2 = \alpha(\gamma - \delta\bar{x})$$

or

$$\lambda^3 T^2 + \lambda^2(2T + \alpha T^2) + \lambda(1 + 2\alpha T) + \alpha(1 - \gamma + \delta\bar{x}) = 0. \tag{25}$$

If $\gamma \leq 1$, then all coefficients are positive at a positive steady state. If $\gamma > 1$, then we can consider three cases. If

$$\bar{x} = \frac{\gamma - 1}{\delta},$$

then zero is an eigenvalue and the other two eigenvalues have negative real parts implying that $\bar{x}$ in the linearized system is marginally stable. Therefore no conclusion can be drawn about the stability of $\bar{x}$ in the nonlinear system. If $\bar{x} < (\gamma - 1)/\delta$, then the constant term is negative, so $\bar{x}$ is unstable. If $\bar{x} > (\gamma - 1)/\delta$, then all coefficient of (25) are positive. In this case the Routh-Hurwitz criterion implies that the real parts of the eigenvalues are negative if and only if

$$(2T + \alpha T^2)(1 + 2\alpha T) > T^2\alpha(1 - \gamma + \delta\bar{x})$$

which can be reduced to a quadratic inequality in T:

$$2\alpha^2 T^2 + \alpha T(4 + \gamma - \delta\bar{x}) + 2 > 0. \tag{26}$$

For the sake of simplicity, we re-introduce the notation $z = \gamma - \delta\bar{x}$. If $z \geq 0$, then this inequality holds implying the asymptotical stability of the steady state. So we can assume that $z < 0$. The discriminant of the left hand side of inequality (26) is

$$D = z(z + 8).$$

If $z < -8$, then $D > 0$, so the left hand side of (26) has two roots

$$T^*_{1,2} = \frac{-(4 + z) \pm \sqrt{(4 + z)^2 - 16}}{4\alpha} \tag{27}$$

which are positive and $T^*_1 < T^*_2$. Notice that $T^*_1 T^*_2 = 1/\alpha^2$ and (26) holds if and only if $T < T^*_1$ or $T > T^*_2$ when $\bar{x}$ is locally asymptotically stable. If $T^*_1 < T < T^*_2$, then (26) is violated, so $\bar{x}$ is unstable. If $z = -8$, then $D = 0$ and there are equal roots

$$T^*_1 = T^*_2 = -\frac{4 + z}{4\alpha} = \frac{1}{\alpha},$$

so if $T \neq 1/\alpha$, then $\bar{x}$ is locally asymptotically stable.

If $-8 < z < 0$, then $D < 0$, so (26) holds and $\bar{x}$ is asymptotically stable. The instability region is shown in Figures 2(A) where z is the horizontal axis and T is the vertical axis. If we start with a very small value of T with any given $z < -8$, then $\bar{x}$ is asymptotically stable. If we gradually increase T, then $\bar{x}$ remains asymptotically stable until it reaches the critical value T_1^*, when the steady state becomes unstable. It remains unstable until T_2^* when stability is regained, and $\bar{x}$ remains asymptotically stable for all $T > T_2^*$.

We will next show that at the critical values T_1^* and T_2^*, Hopf bifurcation occurs giving the possibility of the birth of limit cycles. We select T as the bifurcation parameter. At the critical values (26) is satisfied with equality, so

$$\alpha(1-\gamma+\delta\bar{x}) = \frac{(2T+\alpha T^2)(1+2\alpha T)}{T^2}$$

and the characteristic equation (25) can be rewritten as

$$\begin{aligned}
&\lambda^3 T^2 + \lambda^2(2T+\alpha T^2) + \lambda(1+2\alpha T) + \frac{(2T+\alpha T^2)(1+2\alpha T)}{T^2} \\
= &\left(\lambda + \frac{2+\alpha T}{T}\right)\left(\lambda^2 T^2 + (1+2\alpha T)\right)
\end{aligned}$$

showing that there is a negative eigenvalue

$$\lambda_1 = -\frac{2+\alpha T}{T}$$

and a pair of pure complex eigenvalues

$$\lambda_{1,2} = \pm i\sqrt{\frac{1+2\alpha T}{T^2}} = \pm i\varepsilon.$$

Consider λ as a function of the bifurcation parameter T and differentiate implicitly equation (25) to have

$$\frac{d\lambda}{dT} = \frac{-\lambda^3 2T - \lambda^2(2+2\alpha T) - 2\alpha\lambda}{3\lambda^2 T^2 + 2\lambda(2T+\alpha T^2) + (1+2\alpha T)}.$$

By simple calculation we can see that at $\lambda = \pm i\varepsilon$,

$$\begin{aligned}
\frac{d\lambda}{dT} &= \frac{\pm i\varepsilon^3 2T + \varepsilon^2(2+2\alpha T) \mp 2\alpha i\varepsilon}{-3\varepsilon^2 T^2 \pm 2i\varepsilon(2T+\alpha T^2) + (1+2\alpha T)} \\
&= \frac{\varepsilon^2(1+\alpha T) \pm i(\varepsilon^3 T - \alpha\varepsilon)}{-(1+2\alpha T) \pm i\varepsilon(2T+\alpha T^2)}
\end{aligned}$$

with real part

$$\frac{d(\operatorname{Re}\lambda)}{dT} = \frac{\varepsilon^2(1-\alpha^2 T^2)}{(1+2\alpha T)^2 + (2T+\alpha T^2)^2\varepsilon^2}.$$

Since $T_1^* T_2^* = 1/\alpha^2$, at $T = T_1^*$ the value of $d(\operatorname{Re}\lambda)/dT$ changes from negative to positive showing the loss of stability, and if $T = T_2^*$, then $d(\operatorname{Re}\lambda)/dT$ changes from positive to negative indicating that stability is regained. Since at both critical values $d(\mathrm{Re}\,\lambda)/dT \neq 0$, at both values Hopf bifurcation occurs giving the possibility of the birth of limit cycles.

We perform numerical simulations to confirm the result obtained above. In Figure 2(A), the steady state is locally asymptotically stable in the dark-gray region with $z > -1$ due to Lemma 2. It is also locally asymptotically stable in the light gray region and unstable in the white region when $z < -1$. The appearance and disappearance of a limit cycle can be observed in Figure 2(B) where we take $\alpha = 1$, $\beta = e^{13}$, $\gamma = 1$ and $\delta = 1$ implying $z = -12$,

$$T_1^* = 2 - \sqrt{3} \simeq 0.268 \text{ and } T_2^* = 2 + \sqrt{3} \simeq 3.732.$$

Under these specifications, the Volterra-type integro-differential equation (22) can be written as a 3D system of differential equations,

$$\dot{x}(t) = -\alpha x(t) + \beta x^e(t) e^{-\delta x^e(t)}$$

$$\dot{x}^e(t) = \frac{1}{T}\left(y(t) - x^e(t)\right)$$

$$\dot{y}(t) = \frac{1}{T}\left(x(t) - y(t)\right)$$

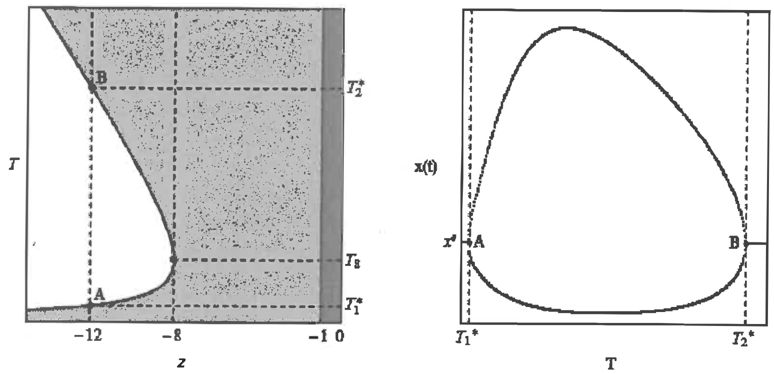
where

$$x^e(t) = \int_0^t \left(\frac{1}{T}\right)^2 (t-s) e^{-\frac{t-s}{T}} x(s) ds$$

and

$$y(t) = \int_0^t \frac{1}{T} e^{-\frac{t-s}{T}} x(s) ds.$$

When T increases from $T_1^* - 0.1$ to $T_2^* + 0.3$ with an increment of 0.01, the steady state loses stability at point A and regains stability at point B. In Figure 2(B), the local maximum and local minimum of a trajectory generated by the 3D system are depicted against each value of T indicating the birth of a limit cycle for $T_1^* < T < T_2^*$.



(A) Region Division (B) Bifurcation diagram

Figure 2. Dynamics with $T > 0$ and $m = 1$

The cases of $m \geq 2$ result in fourth or larger degree polynomial equations, the stability of the steady states can be examined similarly, but the mathematical details become much more complicated. It can be mathematically confirmed that as $m \longrightarrow \infty$, equation (23) converges to the characteristic equation (11) of the model with fixed delay. In particular, if $m \longrightarrow \infty$, then expression

$$\left(1 + \frac{\lambda T}{m}\right)^{(m+1)} = \left(1 + \frac{\lambda T}{m}\right)^m \left(1 + \frac{\lambda T}{m}\right)$$

converges to $e^{\lambda T}$. For a larger value of m, dynamics generated by the differential equation with continuously distributed time delay is similar to dynamics generated by the differential equation with fixed time delay.

5 Conclusions

In this paper, a special neoclassical growth model was introduced and examined. A mound-shaped production function for capital growth was assumed in the dynamic equation. Zero is always a steady state, and depending on model parameters there is either no positive steady state, or one, or two positive steady states. A complete steady state analysis was followed by the derivation of stability conditions. By introducing fixed delay we demonstrated that stability can be lost at a certain value of the delay and the equilibrium remains unstable afterwards. In the case of continuously distributed delays it has been shown how stability can be lost at a certain value of the average delay and by further increasing the average delay it can be regained. At the critical values, Hopf bifurcation occurs giving the possibility of the birth of limit cycles. In our further study more complex kernel function will be considered and their effect on the asymptotic behavior of the steady state will be examined.

References

[1] Bischi, G. I., C. Chiarella, M. Kopel and F. Szidarovszky, *Nonlinear Oligopolies: Stability and Bifurcations,* Berlin, Heidelberg, New York, Springer-Verlag, 2010.

[2] Day, R., "Irregular Growth Cycles," *American Economic Review*, vol. 72, 406-414, 1982.

[3] Day, R.,. "The Emergence of Chaos from Classical Economic Growth," *Quarterly Journal of Economics*, vol.98, 203-213, 1983.

[4] Day, R., *Complex Economic Dynamics: An Introduction to Dynamical Systems and Market Mechanism*, Cambridge, MIT Press, 1994.

[5] Li, T-Y., and J. A. Yorke, "Period Three Implies Chaos," *American Mathematical Monthly*, vol. 82, 985-92, 1975.

[6] Matsumoto, A., and F. Szidarovszky, "Delay Differential Neoclassical Growth Model," *Journal of Economic Behavior and Organization*, vol. 78, 272-289, 2011.

[7] Puu, T., *Attractions, Bifurcations and Chaos: Nonlinear Phenomena in Economics, 2nd Edition*, Berlin, Heidelberg, New York, Springer-Verlag, 2003.

[8] Rosser, J.B. (Ed.), *Complexity in Economics I, II, III, The International Library of Critical Writings in Economic Series,* vol.174, Edward Elgar, Aldergate, 2004.

[9] Solow, R., "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, vol. 70, 65-94, 1956.

[10] Swan, T., "Economic Growth and Capital Accumulation," *Economic Record*, vol. 32, 334-361, 1956.