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Abstract

We study the effects of production delays on the local as well as on global dynamics of nonlinear Cobweb model in a continuous-time framework. After reviewing a single delay model, we proceed to two models with two delays. When the two delays are used to form an expected price or feedback for price adjustment, we have a winding stability switching curve and in consequence, obtain repetition of stability losses and gains via Hopf bifurcation. When the two delays are involved in two interrelated markets, we find that the stability switchings occur on straight lines and complicated dynamics can arise in unstable markets.

Keywords: Continuos-time Cobweb model, Nonlinear price dynamics, Production delay, Stability switching, Hopf bifurcation

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1 Introduction

It is now well-known that the cobweb model or cobweb theory has been developed in various directions since the pioneering work of Kaldor (1934). It explains why and how certain types of markets give rise to fluctuations in prices and quantities. Since it mainly focuses on the agricultural markets in which producers determine their outputs before observing market prices and a delay between planting and harvesting is inevitable, its key issues are an expectation formation of price and a production delay. In early stage, the models are essentially linear and constructed in discrete time scales in which production delay is incorporated from the beginning. Thus the main question is on how the expectation formations such as naive, adaptive and rational expectations are responsible for the emergence of fluctuations. During the last two decades, an increasing attention has been given to nonlinear dynamics. The nonlinear and discrete-time cobweb models can generate a wide spectrum of dynamic behavior involving chaos. See Dieci and Westerhof (2010) and Hommes (1994), to name a few.

It is, however, less-known that a continuous time cobweb model with fixed time delay is also developed with the same problem consciousness as early as in the 1930s. In particular, Haldane (1933) found the similarity between the effects caused by the rise in the birth rate in biology and the ones by a rise in commodity price in economics and built a simple economic model to examine the fluctuations in price and the rate of production, coaxing the idea from theoretical biology. Independently from Haldane, Larson (1964) presents a linear continuous time model in which a hog cycle is described as a harmonic motion. It is assumed that realized production has 12 month delay from planned production and the rate of production change is proportional to the deviation of price from equilibrium. Mackey (1989) gives a nonlinear price adjustment model with production delay and rigorously derives a stability switching condition for which the stability of equilibrium is lost. Furthermore, it is shown that a Hopf bifurcation takes place and thus the stable equilibrium bifurcates to a limit cycle after the loss of stability. Recently Gori *et al.* (2014) propose a delay cobweb model with the profit-maximizing behavior to characterize production cycles. Although the delay models have been an object of study for a long time, these are subject to a single delay and little is known about multiple delay models. The purpose of this study is, based on Mackey's formulation, to investigate how multiple delays affect cobweb price dynamics, applying the recent mathematical developments to characterize the stability of two delay differential equations conducted by Gu, *et al.* (2005) and Lin and Wang (2012). Two main results demonstrated in this paper are the following:

(i) Simple dynamics emerges but stability losses and gains are repeatedly taken place in a single market with two time delays.

(ii) No stability gain occurs but complex dynamic can arise when two markets with two delays are unstable.

This paper is organized as follows. In Section 2 a continuous-time nonlinear price adjustment model is presented as a basic model. In Section 3, a single production delay is introduced to review how the delay affects dynamics. In Section 4, the model with two production delays are constructed and the stability switching curve is analytically and numerically derived. In Section 5, two markets model with two delays are considered to develop the conditions under which the two markets are stable or unstable. It is shown that various dynamics arises when the two markets are unstable. In the final section, concluding remarks are given.

2 Basic Cobweb Model

As in Mackey (1989), we consider price dynamics in a continuous-time framework in which relative variations in market price $p(t)$ is adjusted to be proportional to excess demand,

$$\frac{\dot{p}(t)}{p(t)} = K[D(p(t)) - S(p^e(t))] \tag{1}$$

where $K > 0$ is the adjustment coefficient, p^e is the expected price, $D(p)$ and $S(p^e)$ are the demand and supply functions of commodity to be considered. Following the tradition, it is assumed that demand negatively depends on price while supply positively depends on the expected price. For the sake of analytical simplicity it is also assumed that consumers and producers make their decisions based only on the price information appeared in the good market. This assumption is taken away in Section 5. The expected price is formed based on the past observed prices,

$$p^e(t) = F[p(t-\tau_1),\ p(t-\tau_2), ...p(t-\tau_n)] \tag{2}$$

where $\tau_i > 0$ for $n = 1, 2, ...n$, and $p(t-\tau_i)$ is the delayed price or the price realized at time $t - \tau_i$. Again for the sake of simplicity, demand and supply functions are assumed to be linear,

$$D(t) = d_1 - d_2 p(t) \text{ with } d_1 > 0 \text{ and } d_2 > 0 \tag{3}$$

and

$$S(t) = s_1 + s_2 p^e(t) \text{ with } s_1 > 0 \text{ and } s_2 > 0. \tag{4}$$

The equilibrium price and quantity satisfy the conditions of $p^* = p^e(t) = p(t)$ and $q^* = D(p^*) = S(p^*)$ and are obtained as

$$p^* = \frac{d_1 - s_1}{d_2 + s_2} \text{ and } q^* = \frac{d_1 s_2 + d_2 s_1}{d_2 + s_2}$$

where for positivity of the equilibrium price, $d_1 > s_1$ is assumed.

Substituting (3) and (4) into (1), taking $p^e(t) = p(t)$ and then multiplying both sides of the resultant equation by $p(t)$ yield a nonlinear price adjustment equation,

$$\dot{p}(t) = Kp(t)\left[d_1 - s_1 - (d_2 + s_2)p(t))\right]. \tag{5}$$

The equilibrium price p^* is also a stationary point.[1] To examine stability of the equilibrium price, we denote the right hand side of (5) by $G_1(p(t))$ and linearize it around $p = p^*$,

$$\dot{p}_\delta(t) = \left.\frac{dG_1(p(t))}{dp(t)}\right|_{p=p^*} p_\delta(t)$$

or

$$\dot{p}_\delta(t) = -k(d_2 + s_2)p_\delta(t)$$

where $p_\delta(t) = p(t) - p^*$ and $k = Kp^*$. Its solution is

$$p(t) = p^* + (p(0) - p^*)e^{-k(d_2+s_2)t}.$$

Since $k(d_2+s_2) > 0$, the equilibrium price is always locally stable with monotonic convergence.

3 Cobweb Model with a Single Delay

A production time delay is introduced into the basic model (5). Concerning the expectation formation, we start with the simplest form of $F(\cdot)$ where the expected price at time t is the market price realized at time $t - \tau$ with $\tau > 0$.

Assumption 1. $p^e(t) = p(t - \tau) > 0$.

Accordingly, the supply function is modified as

$$S(t) = s_1 + s_2 p(t - \tau). \tag{6}$$

Substituting (6) into (5) presents a delay price adjustment equation,

$$\dot{p}(t) = Kp(t)\left[d_1 - s_1 - d_2 p(t) - s_2 p(t - \tau)\right] \tag{7}$$

that is the first-order nonlinear delay differential equation. It can be confirmed that p^* is also a unique positive stationary state of (7). If $G_2(p(t), p(t-\tau))$ denotes the right hand side of equation (7), then a linearized equation in a neighborhood of the stationary point $\boldsymbol{p}_2^* = (p^*, p^*)$ is

$$\dot{p}_\delta(t) = \left.\frac{\partial G_2}{\partial p(t)}\right|_{\boldsymbol{p}_2^*} p_\delta(t) + \left.\frac{\partial G_2}{\partial p(t-\tau)}\right|_{\boldsymbol{p}_2^*} p_\delta(t-\tau)$$

or

$$\dot{p}_\delta(t) = -kd_2 p_\delta(t) - ks_2 p_\delta(t - \tau).$$

[1] Although dynamic equation (5) has also zero as stationary point, we ignore it in this study.

Introducing the new variable $x(t) = p_\delta(t)$ and the new parameters, $\alpha = kd_2$ and $\beta = ks_2$, we obtain the following form of the linearized equation

$$\dot{x}(t) + \alpha x(t) + \beta x(t - \tau) = 0 \tag{8}$$

where $x(t) = 0$ is the only stationary point. Assuming an exponential solution

$$x(t) = e^{\lambda t} u$$

and substituting it into (8) give the corresponding characteristic equation

$$\lambda + \alpha + \beta e^{-\lambda\tau} = 0. \tag{9}$$

Without delay $\tau = 0$, the stationary point is locally asymptotically stable. If stability of the trivial solution $x(t) = 0$ of (8) switches to instability at $\tau = \bar{\tau}$, then (9) must have a pair of pure conjugate imaginary roots. It is then assumed, without loss of generality, that $\lambda = i\omega$ with $\omega > 0$ is a root. Substituting it into (9) breaks down the characteristic equation to the real and imaginary parts

$$\alpha + \beta \cos \tau\omega = 0 \tag{10}$$

and

$$\omega - \beta \sin \tau\omega = 0. \tag{11}$$

Moving the constant terms to the right hand sides and adding the squares of the resulted equations give

$$\omega^2 = k^2(s_2 + d_2)(s_2 - d_2).$$

If $s_2 \leq d_2$, then there is no $\omega > 0$, implying that the delay is *harmless.*[2]

Theorem 1 *If $s_2 \leq d_2$, then the positive steady state of (5) is locally asymptotically stable for any positive values of τ.*

On the other hand, if $s_2 > d_2$, then we can define $\bar{\omega} > 0$ as

$$\bar{\omega} = k\sqrt{(s_2 + d_2)(s_2 - d_2)}.$$

It is substituted into (10) to obtain threshold values of τ,[3]

$$\bar{\tau} = \frac{1}{\bar{\omega}} \left[\cos^{-1} \left(-\frac{d_2}{s_2} \right) + 2n\pi \right] \quad (n = 0, 1, 2, ...). \tag{12}$$

[2] The stability condition in a continuous time model is the same as the one in a discrete-time model. Assuming $\tau = 1$, we can construct a discrete-time cobweb model,

$$p(t) = -\frac{s_2}{d_2} p(t-1) + (d_1 - s_1)$$

where $s_2 \leq d_2$ is the stability condition including a cyclic solution.

[3] It is possible to substitute it into (11) to obtain the same value in a different form,

$$\bar{\tau} = \frac{1}{\bar{\omega}} \left[\pi - \sin^{-1} \left(-\frac{\bar{\omega}}{\beta} \right) \right].$$

In order to determine the direction of the stability switch, we can think of the roots of (9) as a continuous function of the delay τ. Then differentiating (9) with respect to τ and arranging terms yield

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1 - \tau\beta e^{-\lambda\tau}}{\lambda\beta e^{-\lambda\tau}}.$$

Thus

$$\begin{aligned}
\left(\frac{d(\operatorname{Re}\lambda)}{d\tau}\right)^{-1}_{\lambda=i\omega} &= \operatorname{Re}\left[\frac{1 - \tau\beta e^{-\lambda\tau}}{\lambda\beta e^{-\lambda\tau}}\right]_{\lambda=i\omega} \\
&= \operatorname{Re}\left[-\frac{1}{\lambda(\lambda+\alpha)}\right]_{\lambda=i\omega} \\
&= \operatorname{Re}\left[\frac{\omega + i\alpha}{\omega(\omega^2+\alpha^2)}\right] \\
&= \frac{1}{\omega^2+\alpha^2}
\end{aligned}$$

Hence

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\lambda=i\omega} > 0. \tag{13}$$

This inequality implies that all roots that cross the imaginary axis at $i\omega$ cross from left to right as τ increases. So at point $\bar{\tau}$ with $n = 0$ stability is lost and cannot be regained later.

Further substituting $\bar{\omega}$ into (12) determines the critical value of the delay

$$\bar{\tau}(d_2, s_2) = \frac{\cos^{-1}\left(-\dfrac{d_2}{s_2}\right)}{Kp^*\sqrt{(s_2+d_2)(s_2-d_2)}} \tag{14}$$

with

$$\frac{\partial\bar{\tau}}{\partial d_2} = \frac{s_2\sqrt{1-\left(\frac{d_2}{s_2}\right)^2}\left[(s_2^2-d_2^2) + s_2^2\sqrt{1-\left(\frac{d_2}{s_2}\right)^2}\cos^{-1}\left(-\frac{d_2}{s_2}\right)\right]}{K(d_1-s_1)(s_2-d_2)(s_2^2-d_2^2)^{3/2}} > 0$$

and

$$\frac{\partial\bar{\tau}}{\partial s_2} = -\frac{d_2\sqrt{1-\left(\frac{d_2}{s_2}\right)^2}\left[(s_2^2-d_2^2) + s_2^2\sqrt{1-\left(\frac{d_2}{s_2}\right)^2}\cos^{-1}\left(-\frac{d_2}{s_2}\right)\right]}{K(d_1-s_1)(s_2-d_2)(s_2^2-d_2^2)^{3/2}} < 0.$$

In Figure 1(A), the stability switching curve is depicted as a hyperbolic curve on which the real parts of the eigenvalues are zero. The equilibrium price is stable in the shaded region below the curve and unstable in the white region above.

Since $\partial\bar{\tau}/\partial d_2 > 0$ and $\partial\bar{\tau}/\partial s_2 < 0$, increasing the value of d_2 and decreasing the value of s_2 shift the stability switching curve upward, implying that those parameter changes enlarge the stability region and thus have stabilizing effects. Figure 1(B) illustrates the bifurcation diagram with respect to τ where the parameters are specified as follows.

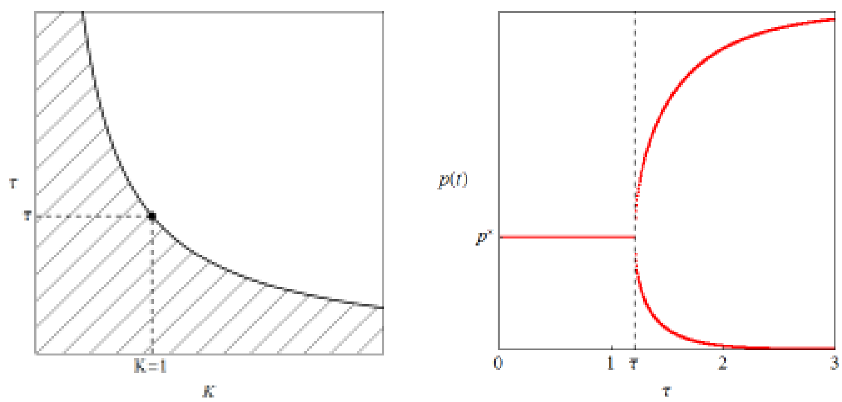
Assumption 2. $K = 1,\ d_1 = 4,\ s_1 = 1,\ d_2 = 1$ and $s_2 = 2$.

The diagram is constructed in the following way. With a fixed value of τ, we run the delay system (7) for $0 \leq t \leq 1000$. To take away the initial disturbance, we discard the data of $p(t)$ for $t \leq 950$ and plot the local maximum and local minimum of $p(t)$ for $950 \leq t \leq 1000$ against the value of τ. The value of τ is increased with an increment 0.01 and then repeat the same procedure until τ arrives at 3. Under these circumstance, when $K = 1$, the threshold value of τ denoted as $\bar{\tau}$ is obtained as

$$\bar{\tau}(d_2, s_2) = \frac{2\pi}{3\sqrt{3}} \simeq 1.209.$$

It is seen in Figure 1(A) that the equilibrium is stable for $\tau < \bar{\tau}$ and unstable for $\tau > \bar{\tau}$. The two branches of the diagram given in Figure 1(B) indicate two issues; one is that a trajectory has one maximum and one minimum, implying the birth of a limit cycle which is confirmed by Hopf bifurcation theorem with (13) and the other is that a cycle becomes larger as the length of the delay increases. Theorem 2 summarizes the results:

Theorem 2 *Given $s_2 > d_2$, the positive steady state of (7) is locally asymptotically stable if $0 < \tau < \bar{\tau}$, loses stability at $\tau = \bar{\tau}$ and bifurcates to a limit cycle if $\tau > \bar{\tau}$ where $\bar{\tau} = \bar{\tau}(1, 2) \simeq 1.209$.*



(A) Stability region (B) Bifurcation diagram

Figure 1. Numerical results on the one delay model (7)

4 Cobweb Model with Two Delays

The price expectation is formed naive-wisely under Assumption 1. Convergence to the stationary point occurs for $\tau < \bar{\tau}$. The producers with delayed price information can eventually arrive at the stationary state. On the other hand, since cyclic oscillation arises for $\tau > \bar{\tau}$, the producers sooner or later realize that their expectations are systematically wrong. As a natural consequence, even though they are assumed to be boundedly rational, the producers may change somehow the way they form expectations. One possible way is to utilize more price information obtained in the past. To make it simpler, the producers are assumed to use different prices at times, $t - \tau_1$ and $t - \tau_2$. There are at least two different ways to employ these two prices. One is to make the price expectation as the weighted average of these prices and the other is to use the difference between these two prices as delay feedback. We first make the following assumption which is a direct extension of Assumption 2 when the price expectation formation is the weighted average of the past two prices:

Assumption 3. $p^e(t) = \theta p(t - \tau_1) + (1 - \theta)p(t - \tau_2)$ with $0 < \theta < 1$.

Accordingly the supply function is modified as

$$S\left(p^e(t)\right) = s_1 + s_2\left[\theta p(t - \tau_1) + (1 - \theta)p(t - \tau_2)\right] \tag{15}$$

and then the price adjustment is governed by a two delay differential equation,

$$\dot{p}(t) = Kp(t)\left[(d_1 - s_1) - d_2 p(t) - s_2\theta p(t - \tau_1) - s_2(1 - \theta)p(t - \tau_2)\right]. \tag{16}$$

It is clear that the equilibrium price is a unique positive stationary point of (16). To examine local dynamics, we let $G_3(p(t),\ p(t - \tau_1),\ p(t - \tau_2))$ be the right hand side of (16). The linear approximation in the neighborhood of the stationary point $\boldsymbol{p}_3^* = (p^*, p^*, p^*)$ is

$$\dot{p}_\delta(t) = \left.\frac{\partial G_3}{\partial p(t)}\right|_{\boldsymbol{p}_3^*} p_\delta(t) + \left.\frac{\partial G_3}{\partial p(t - \tau_1)}\right|_{\boldsymbol{p}_3^*} p_\delta(t - \tau_1) + \left.\frac{\partial G_3}{\partial p(t - \tau_2)}\right|_{\boldsymbol{p}_3^*} p_\delta(t - \tau_2)$$

or

$$\dot{p}_\delta(t) = -kd_2 p_\delta(t) - ks_2\theta p_\delta(t - \tau_1) - ks_2(1 - \theta)p_\delta(t - \tau_2).$$

As in the same way as before, the last form can be reduced to

$$\dot{x}(t) + \alpha x(t) + \beta x(t - \tau_1) + \gamma x(t - \tau_2) = 0 \tag{17}$$

with

$$\alpha = kd_2,\ \beta = ks_2\theta \text{ and } \gamma = ks_2(1 - \theta).$$

We now turn our attention to the delay feedback with which the price adjustment equation (1) is modified as

$$\frac{\dot{p}(t)}{p(t)} = k_1\left[(d_1 - s_1) - d_2 p(t) - s_2 p(t - \tau_1)\right] + k_2\left[p(t - \tau_1) - p(t - \tau_2)\right]$$

where k_1 is an adjustment coefficient and k_2 is a coefficient of the feedback. This can be rewritten as a differential equation with two delays

$$\dot{p}(t) = k_1 p(t)\left[(d_1 - s_1) - d_2 p(t) - s_2\left(1 - \frac{k_2}{s_2 k_1}\right)p(t-\tau_1) - s_2\frac{k_2}{s_2 k_1}p(t-\tau_2)\right]. \tag{18}$$

If we assume $k_2 < s_2 k_1$ and denote

$$1 - \frac{k_2}{s_2 k_1} = \theta \text{ and } \frac{k_2}{s_2 k_1} = 1 - \theta,$$

then the differential equation with the delay feedback (18) is identical with equation (16), the differential equation with the average price. Hence these equations generate essentially the same dynamics, although their economic interpretations are definitely different. In the following we focus on equation (16) only because the number of the parameters is smaller, however, the same results can be obtained by examining equation (18).

The corresponding characteristic equation of (17) is obtained by substituting an exponential solution $x(t) = e^{\lambda t}u$

$$\lambda + a + \beta e^{-\lambda\tau_1} + \gamma e^{-\lambda\tau_2} = 0. \tag{19}$$

Stability of equation (17) depends on the locations of the eigenvalues of (19) that is investigated by applying the method developed by Gu *et al.* (2005). Dividing both sides of (19) by $\lambda + a$ and introducing the new functions,

$$a_1(\lambda) = \frac{\beta}{\lambda + a} \text{ and } a_2(\lambda) = \frac{\gamma}{\lambda + \alpha} \tag{20}$$

simplify the left hand side of (19),

$$a(\lambda) = 1 + a_1(\lambda)e^{-\lambda\tau_1} + a_2(\lambda)e^{-\lambda\tau_2}. \tag{21}$$

Substituting a possible solution $\lambda = i\omega$ with $\omega > 0$ into the two equations of (20) results in

$$a_1(i\omega) = \frac{a\beta}{\alpha^2 + \omega^2} - i\frac{\beta\omega}{\alpha^2 + \omega^2} \tag{22}$$

and

$$a_2(i\omega) = \frac{a\gamma}{\alpha^2 + \omega^2} - i\frac{\gamma\omega}{\alpha^2 + \omega^2}. \tag{23}$$

Their absolute values are

$$|a_1(i\omega)| = \frac{\beta}{\sqrt{\alpha^2 + \omega^2}} \tag{24}$$

and

$$|a_2(i\omega)| = \frac{\gamma}{\sqrt{\alpha^2 + \omega^2}}. \tag{25}$$

We can consider the three terms in (21) as three vectors in the complex plane with the magnitudes 1, $|a_1(\lambda)|$ and $|a_2(\lambda)|$. The solution of $a(\lambda) = 0$ means that these vectors form a triangle if we put them head to tail. That is, solving $a(\lambda) = 0$ algebraically is equivalent to constructing a triangle geometrically with the following three conditions,

$$1 \leq |a_1(i\omega)| + |a_2(i\omega)|$$

$$|a_1(i\omega)| \leq 1 + |a_2(i\omega)|$$

and

$$|a_2(i\omega)| \leq 1 + |a_1(i\omega)|.$$

Substituting the absolute values of (24) and (25) convert these three conditions to the following conditions,

$$(\beta - \gamma)^2 - \alpha^2 \leq \omega^2 \leq (\beta + \gamma)^2 - \alpha^2$$

that can be rewritten as

$$k^2 \left[s_2^2 (2\theta - 1)^2 - d_2^2 \right] \leq \omega^2 \leq k^2 (s_2^2 - d_2^2). \tag{26}$$

It is clear that the second inequality does not hold if $d_2 \geq s_2$. Hence there is no $\omega > 0$, implying that the delays are *harmless* in the two delay dynamic model.

Theorem 3 *If $s_2 \leq d_2$, then the positive steady state of (16) is locally asymptotically stable for any positive values of τ_1 and τ_2.*

On the other hand, if $s_2 > d_2$, then conditions (26) hold for ω in the interval $[\omega_s, \ \omega_e]$ where

$$\omega_s = \begin{cases} 0 & \text{if } \theta \leq \theta_0 \\ k\sqrt{s_2^2(2\theta - 1)^2 - d_2^2} & \text{if } \theta > \theta_0 \end{cases}$$

with

$$\theta_0 = \frac{1}{2}\left(1 + \frac{d_2}{s_2}\right) \text{ that solves } s_2^2(2\theta - 1)^2 - d_2^2 = 0$$

and

$$\omega_e = k\sqrt{s_2^2 - d_2^2}$$

where the subscripts "s" and "e" mean the starting point and the end point, respectively.

We will next find all the pairs of (τ_1, τ_2) satisfying $a(i\omega) = 0$. Let $|1|$ be the base of the triangle and then denote an angle between $|1|$ and $|a_1(i\omega)|$ by θ_1 and an angle between $|1|$ and $|a_2(i\omega)|$ by θ_2. By the law of cosine, the angle's magnitudes are expressed in terms of the model parameters,

$$\theta_1(\omega) = \cos^{-1}\left(\frac{\omega^2 + \alpha^2 + \beta^2 - \gamma^2}{2\beta\sqrt{\alpha^2 + \omega^2}}\right)$$

and

$$\theta_2(\omega) = \cos^{-1}\left(\frac{\omega^2 + \alpha^2 - \beta^2 + \gamma^2}{2\gamma\sqrt{\alpha^2 + \omega^2}}\right).$$

From (22) and (23), we obtain their arguments,

$$\arg[a_1(i\omega)] = -\tan^{-1}\left(\frac{\omega}{\alpha}\right)$$

and

$$\arg[a_2(i\omega)] = -\tan^{-1}\left(\frac{\omega}{\alpha}\right).$$

Since the triangle can be located above and under the real axis, the following two equations hold for τ_1 and τ_2:

$$\left\{\arg\left[a_1(i\omega)e^{-i\omega\tau_1}\right] + 2m\pi\right\} \pm \theta_1(\omega) = \pi$$

and

$$\left\{\arg\left[a_2(i\omega)e^{-i\omega\tau_2}\right] + 2n\pi\right\} \mp \theta_2(\omega) = \pi$$

which yield the threshold values of the delays

$$\tau_1^{\pm}(\omega, m) = \frac{1}{\omega}\left[-\tan^{-1}\left(\frac{\omega}{\alpha}\right) + (2m-1)\pi \pm \theta_1(\omega)\right] \tag{27}$$

and

$$\tau_2^{\mp}(\omega, n) = \frac{1}{\omega}\left[-\tan^{-1}\left(\frac{\omega}{\alpha}\right) + (2n-1)\pi \mp \theta_2(\omega)\right]. \tag{28}$$

Here m and n are nonnegative integers such that $\tau_1 > 0$ and $\tau_2 > 0$. Thus for any m, n and $\omega \in [\omega_s, \omega_e]$, we can define the pairs of (τ_1, τ_2) constructing the stability switching curve as follows:

Theorem 4 *Given $s_2 > d_2$, the stability switching curve is described by $C_1(m,n) \cup C_2(m,n)$ with $m, n = 0, 1, 2...$, where*

$$C_1(m,n) = \{\tau_1^+(\omega, m), \tau_2^-(\omega, n)\}$$

and

$$C_2(m,n) = \{\tau_1^-(\omega, m), \tau_2^+(\omega, n)\}.$$

The segments $C_1(m, n+1)$ and $C_2(m,n)$ have the same starting point whereas the segment $C_1(m,n)$ and $C_2(m,n)$ have the same end point.

Proof. It can be verified that

$$\frac{\omega_s^2 + \alpha^2 + \beta^2 - \gamma^2}{2\beta\sqrt{\alpha^2 + \omega_s^2}} = 1, \quad \frac{\omega_e^2 + \alpha^2 + \beta^2 - \gamma^2}{2\beta\sqrt{\alpha^2 + \omega_e^2}} = 1$$

and

$$\frac{\omega_s^2 + \alpha^2 - \beta^2 + \gamma^2}{2\gamma\sqrt{\alpha^2 + \omega_s^2}} = -1 \text{ and } \frac{\omega^2 + \alpha^2 - \beta^2 + \gamma^2}{2\gamma\sqrt{\alpha^2 + \omega^2}} = 1$$

which leads to

$$\theta_1(\omega_s) = 0,\ \ \theta_1(\omega_e) = 0$$

and

$$\theta_2(\omega_s) = \pi,\ \ \theta_2(\omega_e) = 0.$$

Let the starting points of $C_1(m, n)$ and $C_2(m, n)$ be denoted by

$$C_1^s(m,n) = \{\tau_1^+(\omega_s, m), \tau_2^-(\omega_s, n)\}$$

and

$$C_2^s(m,n) = \{\tau_1^-(\omega_s, m), \tau_2^+(\omega_s, n)\}.$$

Substituting $\theta_1(\omega_s) = 0$ and $\theta_2(\omega_s) = \pi$ into (27) and (28) yields

$$\tau_1^+(\omega_s, m) = \tau_1^-(\omega_s, m) \text{ and } \tau_2^-(\omega_s, n+1) = \tau_2^+(\omega_s, n)$$

which imply that $C_1^s(m, n+1) = C_2^s(m, n)$. In the same way, let the end points of $C_1(m, n)$ and $C_2(m, n)$ be denoted by

$$C_1^e(m,n) = \{\tau_1^+(\omega_e, m), \tau_2^-(\omega_e, n)\}$$

and

$$C_2^e(m,n) = \{\tau_1^-(\omega_e, 0), \tau_2^+(\omega_e, n)\}.$$

Then substituting $\theta_1(\omega_e) = 0$ and $\theta_2(\omega_e) = 0$ into (27) and (28) yields

$$\tau_1^+(\omega_e, m) = \tau_1^-(\omega_e, m) \text{ and } \tau_2^-(\omega_e, n) = \tau_2^+(\omega_e, n)$$

which imply that $C_1^e(m, n) = C_2^e(m, n)$. This completes the proof. ■

Under Assumption 2 with $\theta = 0.8$ and $m = 1$, the stability switching curve is illustrated in Figure 2 in which the red segments show $C_1(1, n)$ and the blue segments show $C_2(1, n)$ for $n = 1, 2, 3$. Notice the segment shifts upward as the value of n increases and to the right as the value of m increases.[4] The lowest red and blue segments are $C_1(1, 1)$ and $C_2(1, 1)$ where they are connected to each other at point $C_1^e(1, 1) = C_2^e(1, 1)$. The middle red and blue segments are $C_1(1, 2)$ and $C_2(1, 2)$ where $C_1(1, 2)$ is connected to $C_2(1, 1)$ at $C_1^s(1, 2) = C_2^s(1, 1)$. As n increases the two segments are connected in the same way to construct the continuous stability switching curve, $C_1(1, n) \cup C_2(1, n)$. This curve divides the first quadrant into two parts as shown in Figure 2. One contains the origin and its every point can be reached from the origin via continuous curve not crossing the stability switching curve. At any point in this region, the real parts of the eigenvalues are negative, so the system is locally asymptotically stable. On the other hand, at the points in the complement of this region except the stability switching curve, the system is unstable. Observing Figure 2, we find the following three issues:

[4] Since the stability switching curve with $m = 0$ is located in the second quadrant of the (τ_1, τ_2) plane and the curve with $n = 0$ is in the fourth quadrant, they are not depicted in Figure 2.

(i) For $\tau_1 \leq \tau_1^h \simeq 1.188$, the system is locally asymptotically stable irrespective of the values of τ_2, implying that delay τ_2 is harmless;

(ii) For $\tau_1 > \tau_1^h$, stability loss and gain repeatedly occur when τ_2 increases from zero;

(iii) Depending on the value of τ_2, two different dynamic phenomena are seen when the value of τ_1 increases. One is when stability is lost and can not be regained as in the one delay model, and the other case is when stability regain can occur.

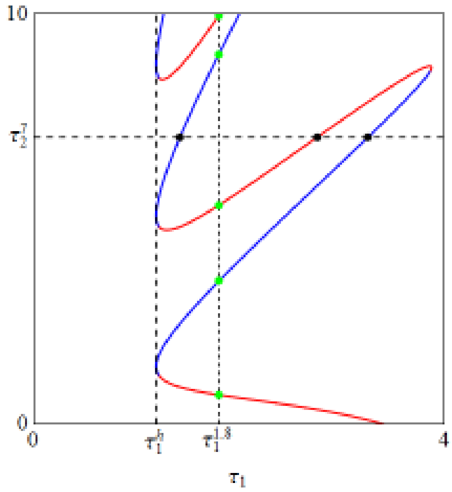
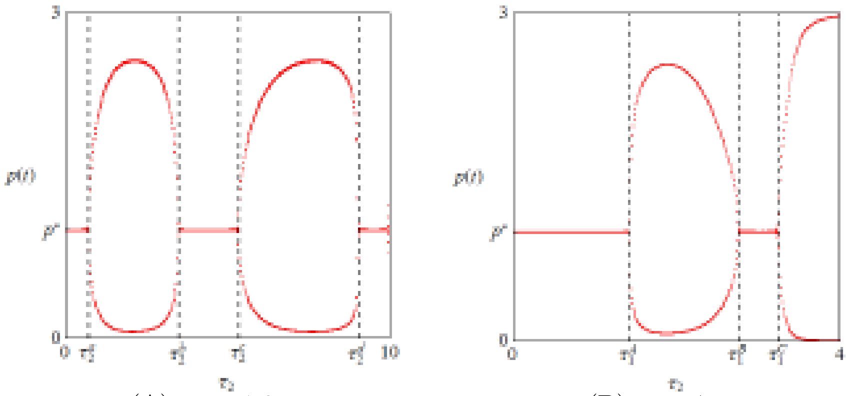


Figure 2. Stability switching curve

We now examine the effect caused by changing τ_2, keeping the value of τ_1 at some positive value. In Figure 3(A), the bifurcation diagram of $p(t)$ with respect to τ_2 is illustrated. The value of τ_1 is fixed at $\tau_1^{1.8} = 1.8$ and the value of τ_2 is increased along the vertical dotted line at $\tau_1^{1.8}$ in Figure 2. For each value of τ_2, the dynamic system runs for $0 \leq t \leq 1000$ and discard the data for $0 \leq t \leq 950$ to get rid of the transients. The local maximum and minimum obtained from the remaining data are plotted against the value of τ_2. The value of τ_2 is increased with $1/400$ and then the same procedure is repeated until τ_2 arrives at 10. If the resultant bifurcation diagram has only one point against the value of τ_2, then the system is locally stable and that point corresponds to the stationary point. If it has two points, then a limit cycle with one maximum and one minimum emerges. As is seen in Figure 2, the vertical line at $\tau_1 = \tau_1^{1.8}$ crosses the stability switching curves five times. We denote the values of τ_2 of the green intersection points by τ_2^a ($\simeq 0.714$), τ_2^b ($\simeq 3.489$), τ_2^c ($\simeq 5.342$), τ_2^d ($\simeq 9.028$) and τ_2^e ($\simeq 9.928$) in the ascending order. The bifurcation diagram in Figure 3(A) indicates the following dynamics: after stability is lost at τ_2^a, a limit cycle emerges for $\tau_2 \in (\tau_2^a, \tau_2^b)$ and its amplitude first expands, then shrinks to zero at $\tau_2 = \tau_2^b$ when stability is regained. The same process is repeated for larger values of τ_2.

We draw attention to the effect caused by changing the value of τ_1. With the similar procedure, Figure 3(B) illustrates the bifurcation diagram with respect to τ_1 along the dotted horizontal line at $\tau_2 = \tau_2^7 (= 7)$ shown in Figure 2 in which the line crosses the stability switching line three times denoted by three black dots at τ_1^A ($\simeq 1.421$), τ_1^B ($\simeq 2.758$) and τ_1^C ($\simeq 3.252$) in the ascending order. A limit cycle emerges when stability is lost and stability losses and gains are repeatedly observed.



(A) $\tau_1 = 1.8$ (B) $\tau_2 = 7$

Figure 3. Bifurcation diagrams showing stability losses and gains

5 Two-Market Model with Two Delays

In this section we examine two delay price adjustments in the interrelated markets like hog and corn markets. Let us denote corn and hogs by c and h. To simplify the analysis, the demand and supply functions in the corn market are assumed to be linear

$$\begin{cases} D^c(p_c(t)) = \delta_1 - \delta_2 p_c(t) \\ S^c(p_c^e(t)) = \sigma_1 + \sigma_2 p_c^e(t) \end{cases}$$

and those in the hog market are also linear

$$\begin{cases} D^h(p_h(t)) = d_1 - d_2 p_h(t) \\ S^h(p_h^e(t),\ p^c(t)) = s_1 + s_2 p_h^e(t) - s_3 p_c(t-\tau) \end{cases}$$

where all parameters δ_i, σ_i, d_i, s_i and delay τ are positive. In each market, the demand for the commodity depends on its current price observed at time t. The supply of corn depends only on the expected corn price while the supply of hogs depends on the expected price of hogs and the delay price of corn since hog suppliers are corn demanders and determine their demand decisions observing the price at time when they determine their supplies of hogs or the price at an earlier time. It is natural for hog producers to have $s_3 > 0$ because they decrease the quantity of hogs when the corn price increases. Equilibrium prices satisfying $D^c(p_c^*) = S^c(p_c^*)$ and $D^h(p_h^*) = S^h(p_h^*,\ p_c^*)$ are

$$p_c^* = \frac{\delta_1 - \sigma_1}{\delta_2 + \sigma_2} \text{ and } p_h^* = \frac{d_1 - s_1 + s_3 p_c^*}{d_2 + s_2}$$

where $\delta_1 > \sigma_1$ and $d_1 > s_1$ are assumed to assure positivity of the equilibrium prices. We assume the simplest expectation formations as in Assumption 1.

Assumption 4. $p_c^e(t) = p_c(t - \tau_c)$ and $p_h^e(t) = p_h(t - \tau_h)$ with $\tau_c > 0$ and $\tau_h > 0$ and $\tau = \tau_h$.

The price adjustment system is given by a two dimensional system of delay differential equations,

$$\begin{aligned} \dot{p}_c(t) &= k_c p_c(t) \left[(\delta_1 - \sigma_1) - \delta_2 p_c(t) - \sigma_2 p_c(t - \tau_c) \right], \\ \dot{p}_h(t) &= k_h p_h(t) \left[(d_1 - s_1) - d_2 p_h(t) - s_2 p_h(t - \tau_h) + s_3 p_c(t - \tau_h) \right]. \end{aligned} \tag{29}$$

It can be confirmed that the equilibrium prices are the stationary point of the adjustment system. Linearizing this system in a neighborhood of the stationary point and introducing new variables and new parameters

$$x(t) = p_c(t) - p_c^*,\ y(t) = p_h(t) - p_h^*,\ \alpha = k_c p_c^*,\ \beta = k_h p_h^*,\ \tau_x = \tau_c \text{ and } \tau_y = \tau_h,$$

yield the linearized system,

$$\begin{aligned} \dot{x}(t) &= \alpha \left[-\delta_2 x(t) - \sigma_2 x(t - \tau_x) \right], \\ \dot{y}(t) &= \beta \left[s_3 x(t - \tau_y) - d_2 y(t) - s_2 y(t - \tau_y) \right]. \end{aligned} \tag{30}$$

Supposing exponential solutions, $x(t) = e^{-\lambda t} u$ and $y(t) = e^{-\lambda t} v$ with $u \neq 0$ and $v \neq 0$ and substituting them into (30), we see that nontrivial solutions for u and v exist if and only if

$$\begin{pmatrix} \lambda + \alpha\delta_2 + \alpha\sigma_2 e^{-\lambda\tau_x} & 0 \\ -\beta s_3 e^{-\lambda\tau_y} & \lambda + \beta d_2 + \beta s_2 e^{-\lambda\tau_y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The corresponding characteristic equation is

$$(\lambda + \alpha\delta_2 + \alpha\sigma_2 e^{-\lambda\tau_x})(\lambda + \beta d_2 + \beta s_2 e^{-\lambda\tau_y}) = 0 \tag{31}$$

which implies that the two delays are independent.[5] Notice that the delay corn price $p_c(t - \tau_h)$ does not affect local dynamics. Equation (31) can be divided into two independent equations

$$\lambda + \alpha\delta_2 + \alpha\sigma_2 e^{-\lambda\tau_x} = 0 \tag{32}$$

[5] In the Appendix, we apply a more general method to solve the two delay differential equation for this particular case and obtain the same result.

and

$$\lambda + \beta d_2 + \beta s_2 e^{-\lambda \tau_y} = 0. \tag{33}$$

Before proceeding we check the no delay case (i.e., $\tau_x = \tau_y = 0$) in which, from (31), the characteristic roots are real and negative,

$$\lambda = -\alpha(\delta_2 + \sigma_2) < 0 \text{ and } \lambda = -\beta(d_2 + s_2) < 0.$$

The stationary point is always stable without delays. Thus stability can be preserved for positive delays as far as the delays are sufficiently small. In order to confirm to what extent the stationary point is stable, we determine the threshold values of the delays for which stability is just lost. Equations (32) and (33) have the same form as equation (9) and can be solved in the same way. Substituting $\lambda = i\omega_x$ and $\lambda = i\omega_y$ into equations (32) and (33) yields

$$i\omega_x + \alpha\delta_2 + \alpha\sigma_2 (\cos \tau_x \omega_x - \sin \tau_x \omega_x) = 0$$

and

$$i\omega_y + \beta d_2 + \beta s_2 (\cos \tau_y \omega_y - \sin \tau_y \omega_y) = 0.$$

Dividing each equation into the real and imaginary parts and solving them for ω_j for $j = x, y$, we have

$$\omega_x^* = \alpha\sqrt{\sigma_2^2 - \delta_2^2} \text{ and } \omega_y^* = \beta\sqrt{s_2^2 - d_2^2}. \tag{34}$$

There are several combinations of ω_x and ω_y according to whether $\sigma_2 - \delta_2$ and $s_2 - d_2$ are positive or not. To simplify the analysis, we assume the following:

Assumption 5. $\sigma_2 - \delta_2 > 0$ and $s_2 - d_2 > 0$.

Under Assumption 5 we can determine the threshold values of the delays[6]

$$\tilde{\tau}_{x,m} = \frac{1}{\omega_x^*}\left[\cos^{-1}\left(-\frac{\delta_2}{\sigma_2}\right) + 2m\pi\right]$$

and

$$\tilde{\tau}_{y,n} = \frac{1}{\omega_y^*}\left[\cos^{-1}\left(-\frac{d_2}{s_2}\right) + 2n\pi\right].$$

At $\tau_j = \tilde{\tau}_{j,0}$ for $j = x, y$, the stationary point loses stability. Hence the $\tau_x = \tilde{\tau}_{x,0}$ line and the $\tau_y = \tilde{\tau}_{y,0}$ line form the stability switching curve. Returning to the original notation of the delays and using these lines, we divide the (τ_c, τ_h)

[6] We can have the different form for the same threshold value,

$$\tilde{\tau}_{x,m} = \frac{1}{\omega_x^*}\left[\pi - \sin^{-1}\left(\frac{\omega_x^*}{\alpha\sigma_2}\right) + 2m\pi\right]$$

and

$$\tilde{\tau}_{y,m} = \frac{1}{\omega_y^*}\left[\pi - \sin^{-1}\left(\frac{\omega_y^*}{\beta s_2}\right) + 2n\pi\right].$$

plane into two parts as shown in Figure 4. The stationary point is locally asymptotically stable in the yellow region in which $0 \leq \tau_c \leq \tilde{\tau}_c$ and $0 \leq \tau_h \leq \tilde{\tau}_h$ and unstable otherwise (in the union of the white regions and the blue region) where $\tilde{\tau}_c = \tilde{\tau}_{x,0}$ and $\tilde{\tau}_h = \tilde{\tau}_{y,0}$. More precisely, one of the two equations in (30) is unstable in the white regions and the two equations are unstable in the blue region. To see the effects caused by the delays, we perform simulations under the following numerical specification:

Assumption 5. $\delta_1 = 4,\ \delta_2 = 1,\ \sigma_1 = 1,\ \sigma_2 = 2,\ d_1 = 4,\ d_2 = 1,\ s_1 = 1,\ s_2 = 3,\ s_3 = 2.$

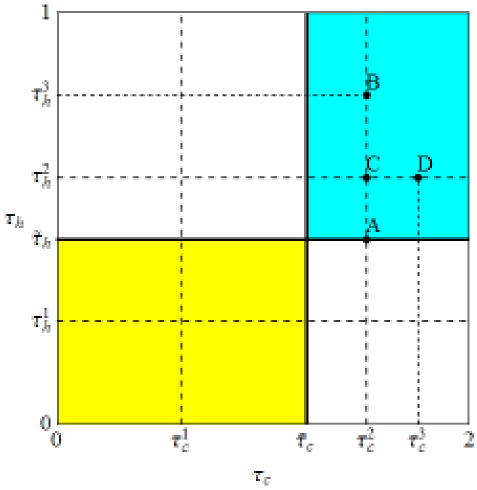
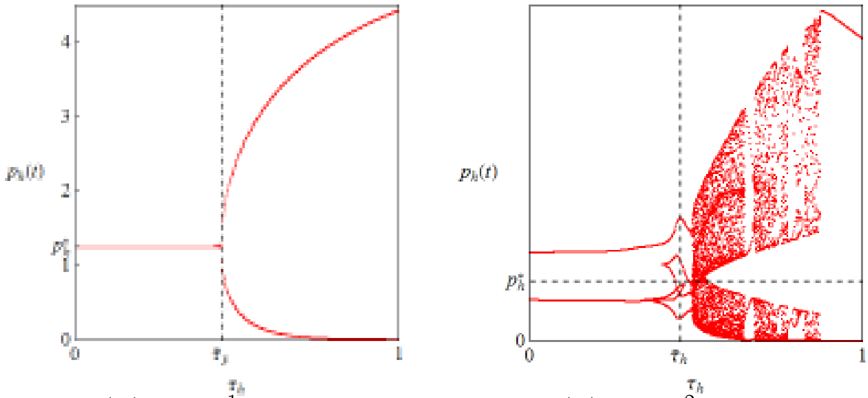


Figure 4. Stability switching curve

Under Assumption 5, the threshold values of the delays are

$$\tilde{\tau}_c = \frac{2\pi}{2\sqrt{3}} \simeq 1.209 \text{ and } \tilde{\tau}_h = \frac{\cos^{-1}(-1/3)}{3\sqrt{2}} \simeq 0.450.$$

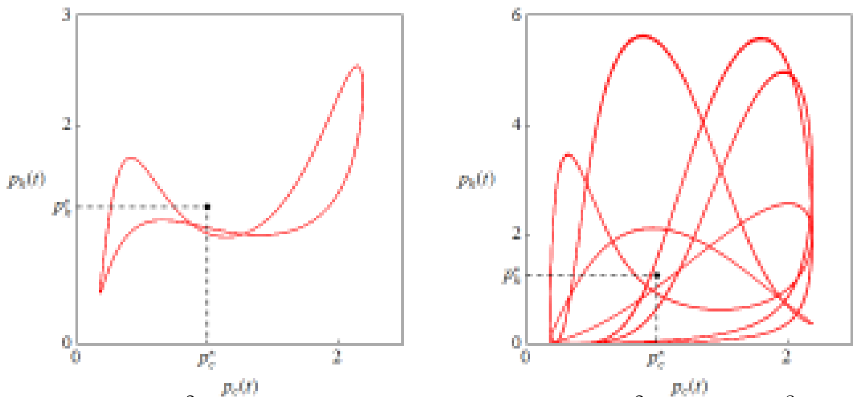
Fixing the value of τ_c at $\tau_c^1 = 0.6$ in the first simulation and $\tau_c^2 = 1.5$ in the second simulation, we investigate the delay effect caused by changing the value of τ_h. The simulations are performed in the same way as before. The value of τ_h is increased from 0 to 1 with an increment of 1/400 and the delay model (29) is run for $0 \leq t \leq 500$ for each value of τ_h. The results obtained are summarized in Figures 5(A) and 5(B) in which the bifurcation diagrams of p_h are illustrated. In the first simulation, the stationary state is locally stable for $\tau_h < \tilde{\tau}_h$ at which the real part of one eigenvalue becomes positive from negative and thus stability is lost. It bifurcates to a limit cycle and no stability regain occurs for further increasing τ_h from $\tilde{\tau}_h$. This result is essentially the same as the one obtained in the single delay model. On the other hand, in the second simulation, we obtain qualitatively different results. The stationary state is unstable even for $\tau_h = 0$ since $\tau_c^2 > \tilde{\tau}_c$. The bifurcation diagram in Figure 5(B) indicates that a limit cycle already emerges for $\tau_h = 0$, it is distorted for τ_h close to $\tilde{\tau}_h$ and further increasing τ_h generates complicated dynamics.



(A) $\tau_c = \tau_c^1 = 0.6$ (B) $\tau_c = \tau_c^2 = 1.5$

Figure 5. Bifurcation diagrams with respect to τ_h

Figure 6 provides two phase diagrams of $p_c(t)$ and $p_h(t)$ for points A and B in Figure 4. It is seen in Figure 5(B) that the vertical dotted line at $\tau_h = \tilde{\tau}_h$ crosses the bifurcation diagram six times. This phenomena is described from a different view point in Figure 6(A) in which the distorted limit cycle has three local maximum and three local minimum for point $A = (\tau_c^2, \tilde{\tau}_h)$. The value of τ_h is increased to $\tau_c^3 = 0.6$ while τ_c is kept at the same value. It is seen in Figure 6(B) in which the limit cycle has seven local maximum and minimum if we observe it carefully and thus its time trajectory exhibits fluctuations with more ups and downs.

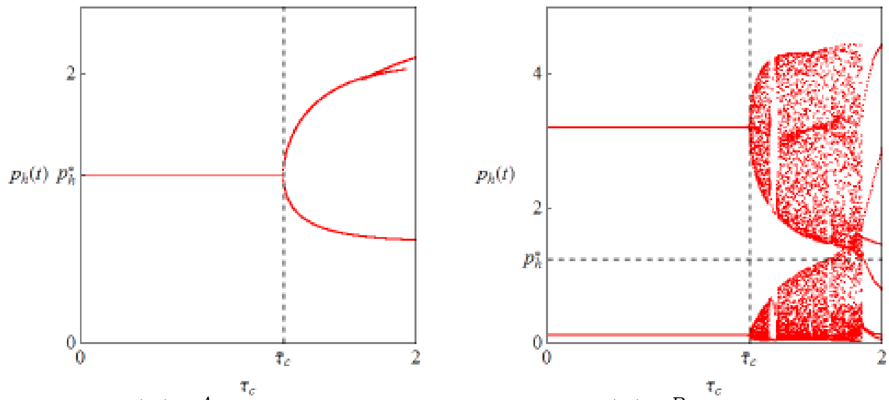


(A) $\tau_c = \tau_c^2$ and $\tau_h = \tilde{\tau}_h$ (B) $\tau_c = \tau_c^2$ and $\tau_h = \tau_h^3$

Figure 6. Phase diagrams at points A and B.

Further we examine the effects caused by changing the value of τ_c. For this purpose, fixing the value of τ_h at $\tau_h^1 = 0.25$ in the first simulation and at

$\tau_h^2 = 0.6$ in the second simulation, we increase the value of τ_c from 0 to 2 with 1/400 increment and and run the delay model (29) for each value of τ_c. When the value of τ_h is chosen in the yellow (i.e., stable) region, the dynamic system generates simple dynamics as shown in Figure 7(A) in which the system is stable for $\tau_c < \tilde{\tau}_c$ and gives rise to a limit cycle after stability is lost for $\tau_c > \tilde{\tau}_c$. As expected, when both equations in (29) are destabilized, more complicated dynamics can arise as shown in Figure 7(B).



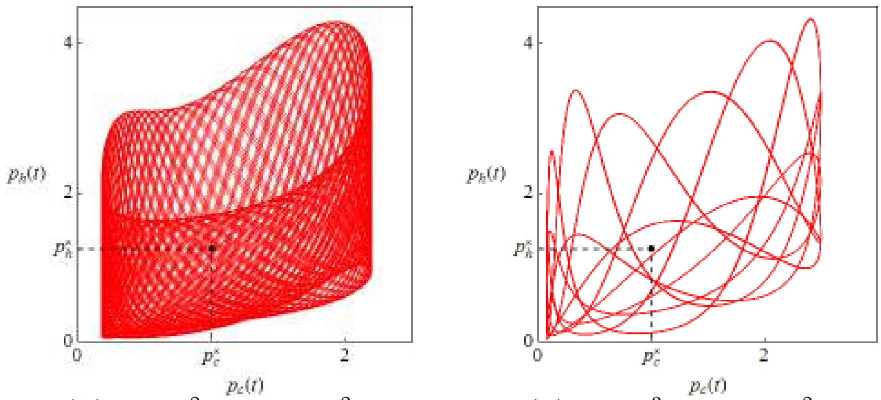
(A) $\tau_h^A = 0.25$ (B) $\tau_h^B = 0.6$

Figure 7. Bifurcation diagram with respect to τ_c

We now shift our emphasis from the bifurcation diagrams to the phase diagrams, especially to see what dynamics arises when both equations of the delay system (29) are locally unstable. In Figure 8(A), we choose point C in Figure 4 and run the system for $0 \leq t \leq 500$. We eliminate the price data for $t \leq 300$ as transitory dynamics and plot the remaining data in the (p_c, p_h) plane. It can be seen that the prices behave in very complicated way. In Figure 8(B), the value of τ_h is increased to τ_h^3. It is also seen that the price behavior is also complicated. What these numerical examples make clear is the following:

(i) If one equation of the two delay system is stable, then the resultant dynamics is essentially the same as in the one delay system.

(ii) If both equations are unstable, then the two delay system can generate various dynamics from a simple limit cycle to complicated dynamics having many ups and drowns.



(A) $\tau_c = \tau_c^2$ and $\tau_h = \tau_h^2$ (B) $\tau_c = \tau_c^3$ and $\tau_h = \tau_h^2$

Figure 8. Phase diagrams at points C and D

6 Concluding Remarks

In this study, we have examined the delay effect on price dynamics in three different models. After reviewing a single delay model in which a limit cycle can emerge via Hopf bifurcation, we proceed to two models with two production delays. When the two delays are used to form an expected price or feedback for price adjustment, we find that the stability switching curve on which stability is changed is winding and as a natural consequence, stability losses and gains are repeated when the length of one delay increases. It is numerically confirmed that only simple dynamics such as a limit cycle can emerge when stability is lost. On the other hand, when the two delays are considered in interrelated markets, they affect price dynamics differently. The stability switching curves become straight lines. When one market is stable and the other market is unstable, the resultant dynamics is simple and essentially the same as the one in the single delay model. On the other hand, when both markets are unstable, a broad spectrum of dynamic behavior can be found. In the cobweb literature, the discrete-time model have been considered and it is known that it can give rise to complicated dynamics when behavior nonlinearities get stronger. This study develops a continuous-time model and indicates that it also reasonably explains various dynamic behavior observed in commodity markets.

Appendix

In this Appendix, we apply the method developed by Lin and Wang (2012) to solve a two delay differential equation to our model. To this end, we first rewrite the left hand side of equation (31)

$$D(\lambda, \tau_x, \tau_y) = D_x(\lambda, \tau_x) D_y(\lambda, \tau_y) \tag{A-1}$$

where

$$D_x(\lambda, \tau_x) = q_0(\lambda) + q_1(\lambda) e^{-\lambda \tau_x} \tag{A-2}$$

$$D_y(\lambda, \tau_y) = q_0'(\lambda) + q_1'(\lambda) e^{-\lambda \tau_y} \tag{A-3}$$

and $q_j(\lambda)$ and $q_j'(\lambda)$ for $j = 0, 1$ are defined accordingly. Substituting (A-3) and (A-2) into (A-1) and expanding it yield the characteristic equation:

$$D(\lambda, \tau_x, \tau_y) = P_0(\lambda) + P_1(\lambda) e^{-\lambda \tau_x} + P_2(\lambda) e^{-\lambda \tau_y} + P_3(\lambda) e^{-\lambda(\tau_x + \tau_y)}$$

where $P_0(\lambda) = q_0(\lambda) q_0'(\lambda),\ P_1(\lambda) = q_0'(\lambda) q_1(\lambda),\ P_2(\lambda) = q_0(\lambda) q_1'(\lambda)$ and $P_3(\lambda) = q_1(\lambda) q_1'(\lambda)$. Since $\lambda = 0$ is not a solution of $D(\lambda, \tau_x, \tau_y) = 0$, we look for a pair of the delays for which the characteristic equation has purely imaginary roots. Since roots of a real function come in conjugate pairs, we can assume that $\lambda = i\omega$ and $\omega > 0$. Substituting this into $D(\lambda, \tau_x, \tau_y) = 0$, we have two different forms,

$$\left[P_0(i\omega) + P_1(i\omega) e^{-\lambda \tau_x}\right] + \left[P_2(i\omega) + P_3(i\omega) e^{-\lambda \tau_x}\right] e^{-\lambda \tau_y} = 0$$

and

$$\left[P_0(i\omega) + P_2(i\omega) e^{-\lambda \tau_y}\right] + \left[P_1(i\omega) + P_3(i\omega) e^{-\lambda \tau_y}\right] e^{-\lambda \tau_x} = 0.$$

We introduce new functions,

$$A_1(\omega) = \mathrm{Re}\left[P_2 \bar{P}_3 - P_0 \bar{P}_1\right] \text{ and } B_1(\omega) = \mathrm{Im}\left[P_2 P_3 - P_0 \bar{P}_1\right]$$

and

$$A_2(\omega) = \mathrm{Re}\left[P_1 \bar{P}_3 - P_0 \bar{P}_2\right] \text{ and } B_2(\omega) = \mathrm{Im}\left[P_1 \bar{P}_3 - P_0 \bar{P}_2\right]$$

where

$$P_2 \bar{P}_3 - P_0 \bar{P}_1 = q_0 \bar{q}_1 \left[q_1' \bar{q}_1' - q_0' \bar{q}_0'\right]$$

and

$$P_1 \bar{P}_3 - P_0 \bar{P}_2 = q_0' \bar{q}_1' \left[q_1 \bar{q}_1 - q_0 \bar{q}_0\right].$$

Following Lin and Wand (2012), we have

$$|P_0(i\omega)|^2 + |P_1(i\omega)|^2 - |P_2(i\omega)|^2 - |P_3(i\omega)|^2 = 2A_1(\omega)\cos(\omega \tau_z) - 2B_1(\omega)\sin \omega \tau_x$$

and

$$|P_0(i\omega)|^2 - |P_1(i\omega)|^2 + |P_2(i\omega)|^2 - |P_3(i\omega)|^2 = 2A_2(\omega)\cos(\omega \tau_y) - 2B_2(\omega)\sin \omega \tau_y.$$

Using (A-2) and (A-3), we obtain

$$|P_0(i\omega)|^2 + |P_1(i\omega)|^2 - |P_2(i\omega)|^2 - |P_3(i\omega)|^2 = \left(|q_0|^2 + |q_1|^2\right)\left(|q_0'|^2 - |q_1'|^2\right)$$

and

$$|P_0(i\omega)|^2 - |P_1(i\omega)|^2 + |P_2(i\omega)|^2 - |P_3(i\omega)|^2 = \left(|q_0'|^2 + |q_1'|^2\right)\left(|q_0|^2 - |q_1|^2\right).$$

We then finally arrive at the following forms,

$$\begin{aligned}&\left(|q_0|^2 + |q_1|^2\right)\left(|q_0'|^2 - |q_1'|^2\right)\\ &= 2\left\{\mathrm{Re}[q_0\bar{q}_1]\left(|q_1'|^2 - |q_0'|^2\right)\cos\omega\tau_x - \mathrm{Im}[q_0\bar{q}_1]\left(|q_1'|^2 - |q_0'|^2\right)\sin\omega\tau_x\right\}.\end{aligned} \tag{A-4}$$

and

$$\begin{aligned}&\left(|q_0'|^2 + |q_1'|^2\right)\left(|q_0|^2 - |q_1|^2\right)\\ &= 2\left\{\mathrm{Re}[q_0'\bar{q}_1']\left(|q_1|^2 - |q_0|^2\right)\cos\omega\tau_y - \mathrm{Im}[q_0'\bar{q}_1']\left(|q_1|^2 - |q_0|^2\right)\sin\omega\tau_y\right\}\end{aligned} \tag{A-5}$$

It is clear that if $|q_0'|^2 = |q_1'|^2$ and $|q_0|^2 = |q_1|^2$, then equations (A-4) and (A-5) hold for any $\tau_x \geq 0$ and $\tau_y \geq 0$, respectively. Further the conditions $|q_0'|^2 = |q_1'|^2$ and $|q_0|^2 = |q_1|^2$ can be rewritten as

$$\omega_y^* = \beta\sqrt{s_2^2 - d_2^2} \text{ and } \omega_x^* = \alpha\sqrt{\sigma_2^2 - \delta_2^2}. \tag{A-6}$$

Notice that (A-6) is the same as (34). Let τ_x^* and τ_y^* be solutions of $D_x(i\omega_x^*, \tau_x) = 0$ and $D_y(i\omega_y^*, \tau_x) = 0$. These are equivalent to $\tilde{\tau}_{x,0}$ and $\tilde{\tau}_{y,0}$. Hence we obtain

$$D(i\omega_x^*, \tau_x^*, \tau_y) = 0 \text{ holds for any } \tau_y \geq 0$$

and

$$D(i\omega_y^*, \tau_x, \tau_y^*) = 0 \text{ holds for any } \tau_x \geq 0.$$

On the other hand, if $\omega_y \neq \omega_y^*$ or $|q_0'|^2 - |q_1'|^2 \neq 0$, then form (A-4),

$$\begin{aligned}|q_0|^2 + |q_1|^2 &= 2\left(\mathrm{Re}\left[q_0\bar{q}_1\right]\cos\omega\tau_x - \mathrm{Im}\left[q_0\bar{q}_1\right]\sin\omega\tau_x\right)\\ &\leq 2\sqrt{\mathrm{Re}\left[q_0\bar{q}_1\right]^2 + \mathrm{Im}\left[q_0\bar{q}_1\right]^2}\sqrt{\cos^2\omega\tau_x + \sin^2\omega\tau_x}\\ &= 2\left|q_0\bar{q}_1\right| = 2\left|q_0\right|\left|q_1\right|\end{aligned}$$

implying that $(|q_0| - |q_1|)^2 \leq 0$ so $|q_0| = |q_1|$ meaning that in this case $\omega_x = \omega_x^*$ and ω_y is arbitray. A similar argument shows that if $\omega_x \neq \omega_x^*$ or $|q_0|^2 - |q_1|^2 \neq 0$, then $\omega_y = \omega_y^*$ and ω_x is arbitray. So there are no additional stability switching points besides the two lines $\tau_x = \tau_x^*$ and $\tau_y = \tau_y^*$.

In summary the stability switching curves are given by the two line segments as depicted in Figure 4,

$$\tau_x = \tau_x^* \text{ and } \tau_y = \tau_y^*.$$

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