Price and Quantity Competition in Differentiated Oligopoly Revisited*

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Abstract

This study examines the competition in Bertrand and Cournot markets from both statics and dynamic points of view. It formalizes optimal behavior in the n-firm framework with product differentiation. Our first findings is that differentiated Bertrand and Cournot equilibria can be destabilized when the number of the firms is strictly greater than three. This finding extends the well-known stability result shown by Theocharis (1960) in which the stability of a non-differentiated Cournot equilibrium is confirmed only in duopolies framework. A complete analysis is then given in comparing Bertrand and Cournot outputs, prices and profits. The focus is placed upon the effects caused by the increasing number of firms. Our second finding exhibits that the number of the firms really matters in the comparison. In particular, it demonstrates that the comparison results obtained in the duopoly framework do not necessarily hold in the general n-firm framework. This finding extends the results shown by Singh and Vives (1984) examining the duality of these two competitions in the duopoly markets and complements the analysis developed by Häcker (2000) that makes comparison in the case of n-firm markets.

*The authors highly appreciate financial supports from the Japan Society for the Promotion of Science (Grant-in-Aid Scientific Research (C) 21530172), Chuo University (Joint Research Grant 0981) and the Japan Economic Research Foundation. Part of this paper was done when the first author visited the Department of Systems and Industrial Engineering of the University of Arizona. He appreciated its hospitality over his stay. The usual disclaimer applies.

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1 Introduction

A considerable number of studies has been made so far on the nature of Cournot and Bertrand competitions in which the firms are adjusting their quantities and prices, respectively. In differentiated Bertrand and Cournot markets using the linear duopoly framework, Singh and Vives (1984) show, among others, the followings clear-cut results[1]:

(i-SV) Prices are higher and quantities lower under Cournot competition than under Bertrand competition regardless of whether the goods are substitutes or complements;

(ii-SV) Cournot competition is more profitable than Bertrand competition if the goods are substitutes;

(iii-SV) Bertrand competition is more profitable than Cournot competition if the goods are complements.

In n-firm differentiated oligopoly markets, Häckner (2000) points out that some of Singh and Vives' results are sensitive to the duopoly case: although (iii-SV) is robust in the n-firm framework, (i-SV) and (ii-SV) can be reversed in the n-firm case with $n > 2$. In particular, it is shown that prices can be higher under Bertrand competition than under Cournot competition when the goods are complements and Bertrand competition can be more profitable when the goods are substitutes.

In this study, adopting the n-firm framework, we shall look more carefully into the results developed by Häckner (2000) from both static and dynamic points of view. It has been well-known since Theocharis (1960) that if the number of firms is more than three, then the Cournot equilibrium becomes unstable even in a linear structure where the demand and the cost functions are linear. This controversial result is shown when the goods are homogenous (i.e., non-differentiated). Considering the two types of competitions with n firms, we may raise a natural question whether the similar result holds or not when the goods are differentiated, namely, whether the differentiated Cournot and Bertrand equilibria become destabilized by the increasing number of firms. We answer the question in the affirmative. The Cournot equilibrium is possibly destabilized when the goods are substitutes and remains stable when the goods are complements. On the other hand, the Bertrand equilibrium is possibly destabilized when the goods are complements and is stable when the goods are substitutes.

In the linear structure, it is not difficult to convert an inverse demand function into a direct demand function. To make the direct demand economically meaningful, it is usually, but implicitly, assumed that its non-induced or price independent demand is positive. This assumption is explicitly considered in Singh and Vives (1984). However, its role is not examined in Häckner (2000). Taking this assumption into account in the n-firm framework, we may improve Häckner's analysis and demonstrate that all of the above-mentioned Singh-Vives results can be reversed:

[1] SV stands for Singh and Vives. MS to be appeared in the later part of the Introduction means Matsumoto and Szidarovszky.

(i-MS) Bertrand price can be higher than Cournot price when the goods are complements whereas Cournot output can be larger than Bertrand output when the goods are substitutes;

(ii-MS) Bertrand profit can be higher than Cournot profit when the goods are substitutes;

(iii-MS) Cournot profit can be higher than Bertrand profit when the goods are complements.

The rest of the paper is organized as follows. In Section 2 we present an n-firm linear oligopoly model and determine the firm's optimal behavior under Cournot and Bertrand competitions. In Section 3, employing a combination of analytical and numerical methods, we compare the optimal price, output and profit under Cournot competition with those under Bertrand competition. Concluding remarks are given in Section 4.

2 n-Firm Oligopoly Models

We will assume consumer's utility maximization in Section 2.1 to obtain a special demand function. In Section 2.2, the firm's profit maximization will be considered under quantity (Cournot) competition, and in Section 2.3 we will derive the optimal prices, outputs and profits under price (Bertrand) competition.

2.1 Consumers

As in Singh and Vives (1984) and Häckner (2000), it is assumed that there is a continuum of consumers of the same type and the utility function of the representative consumer is given as

$$U(\boldsymbol{q}, I) = \sum_{i=1}^{n} \alpha_i q_i - \frac{1}{2}\left(\sum_{i=1}^{n} q_i^2 + 2\gamma \sum_{i=1}^{n}\sum_{j>i}^{n} q_i q_j\right) - I, \tag{1}$$

where $\boldsymbol{q} = (q_i)$ is the quantity vector, $I = \sum_{i=1}^{n} p_i q_i$ with p_i being the price of good k, α_i measures the quality of good i and $\gamma \in [-1, 1]$ measures the degree of relation between the goods: $\gamma > 0$, $\gamma < 0$ or $\gamma = 0$ imply that the goods are substitutes, complements or independent. Moreover, the goods are perfect substitutes if $\gamma = 1$ and perfect complements if $\gamma = -1$. In this study, we confine our analysis to the case in which the goods are imperfect substitutes or complements and are not independent, by assuming that $|\gamma| < 1$ and $\gamma \neq 0$.

The linear inverse demand function (or the price function) of good k is obtained from the first-order condition of the interior optimal consumption of good k and is given by

$$p_k = \alpha_k - q_k - \gamma \sum_{i \neq k}^{n} q_i \text{ for } k = 1, 2, ..., n, \tag{2}$$

where $n \geq 2$ is assumed. That is, the price vector is a linear function of the output vector:

$$\boldsymbol{p} = \boldsymbol{\alpha} - \boldsymbol{B}\boldsymbol{q}, \tag{3}$$

where $\boldsymbol{p} = (p_i)$, $\boldsymbol{\alpha} = (\alpha_i)$ and $\boldsymbol{B} = (B_{ij})$ with $B_{ii} = 1$ and $B_{ij} = \gamma$ for $i \neq j$. Since $\boldsymbol{B}$ is invertible[2], solving (3) for $\boldsymbol{q}$ yields the direct demand

$$\boldsymbol{q} = \boldsymbol{B}^{-1}(\boldsymbol{\alpha} - \boldsymbol{p}) \tag{4}$$

where the diagonal and the off-diagonal elements of $\boldsymbol{B}^{-1}$ are, respectively,

$$\frac{1+(n-2)\gamma}{(1-\gamma)(1+(n-1)\gamma)} \text{ and } -\frac{\gamma}{(1-\gamma)(1+(n-1)\gamma)}.$$

Hence the direct demand of good k, the k^{th}-component of $\boldsymbol{q}$, is linear in the prices and is given as

$$q_k = \frac{(1+(n-2)\gamma)(\alpha_k - p_k) - \gamma \sum_{i \neq k}^{n} (\alpha_i - p_i)}{(1-\gamma)(1+(n-1)\gamma)} \text{ for } k = 1, 2, ..., n. \tag{5}$$

Since Singh and Vives (1984) have already examined the duopoly case (i.e., $n = 2$), we will mainly consider a more general case of $n > 2$ henceforth. For the sake of the later analysis, let us define the admissible region of (γ, n) by $D_{(+)}$ or $D_{(-)}$ according to whether the goods are substitutes or complements:

$$D_{(+)} = \{(\gamma, n) \mid 0 < \gamma < 1 \text{ and } 2 < n\}$$

and

$$D_{(-)} = \{(\gamma, n) \mid -1 < \gamma < 0 \text{ and } 2 < n\}.$$

2.2 Quantity-adjusting firms

In Cournot competition, firm k chooses a quantity q_k of good k to maximize its profit $\pi_k = (p_k - c_k)q_k$ subject to its price function (2), taking the other firms' quantities given. We assume a linear cost function for each firm, so that the marginal cost c_k is constant and non-negative. To avoid negative optimal production, we also assume that the *net quality* of good k, $\alpha_k - c_k$, is positive.

Assumption 1. $c_k \geq 0$ and $\alpha_k - c_k > 0$ for all k.

Assuming interior maximum and solving its first-order condition yield the best reply of firm k,

$$q_k = \frac{\alpha_k - c_k}{2} - \frac{\gamma}{2} \sum_{i \neq k}^{n} q_i \text{ for } k = 1, 2, ..., n. \tag{6}$$

It can be easily checked that the second-order condition is certainly satisfied. The Cournot equilibrium output and price for firm k are obtained by solving the following simultaneous equations:

$$q_k + \frac{\gamma}{2} \sum_{i \neq k}^{n} q_i = \frac{\alpha_k - c_k}{2} \text{ for } k = 1, 2, ..., n,$$

[2] The n by n matrix $\boldsymbol{B}$ is invertible if $\det \boldsymbol{B} = (1-\gamma)^{n-1}(1+(n-1)\gamma) \neq 0$. It is invertible when $\gamma > 0$. In the case of $\gamma < 0$, the inequality constraint $1 + (n-1)\gamma > 0$ will be assumed in Assumption 2 below and it will guarantee the invertiblity of $\boldsymbol{B}$.

or in vector form,

$$\boldsymbol{B}^C \boldsymbol{q} = \mathbf{A}^C,$$

where $\mathbf{A}^C = (\alpha_i - c_i)/2$ and $\boldsymbol{B}^C = (B^C_{ij})$ with $B^C_{ii} = 1$ and $B^C_{ij} = \gamma/2$ for $i \neq j$. Since $\boldsymbol{B}^C$ is invertible, the Cournot output vector is given by

$$\boldsymbol{q}^C = \left(\boldsymbol{B}^C\right)^{-1} \mathbf{A}^C,$$

where the diagonal and off-diagonal elements of $\left(\boldsymbol{B}^C\right)^{-1}$ are, respectively,

$$\frac{2(2+(n-2)\gamma)}{(2-\gamma)(2+(n-1)\gamma)} \text{ and } -\frac{2\gamma}{(2-\gamma)(2+(n-1)\gamma)}.$$

Hence the Cournot equilibrium output of firm k is

$$q^C_k = \frac{\alpha_k - c_k}{2-\gamma} - \frac{\gamma}{(2-\gamma)(2+(n-1)\gamma)} \sum_{i=1}^{n} (\alpha_i - c_i) \tag{7}$$

and the Cournot equilibrium price of firm k is

$$p^C_k = \frac{\alpha_k + c_k - \gamma c_k}{2-\gamma} - \frac{\gamma}{(2-\gamma)(2+(n-1)\gamma)} \sum_{i=1}^{n} (\alpha_i - c_i). \tag{8}$$

Subtracting (7) from (8) yields $p^C_k - c_k = q^C_k$ and then by substituting it into the profit function, the Cournot profit is obtained:

$$\pi^C_k = \left(q^C_k\right)^2. \tag{9}$$

The relation $p^C_k - c_k = q^C_k$ also implies that the Cournot price is positive if the Cournot output is positive. Equation (7) implies that the Cournot output is always positive when $\gamma < 0$, and non-negative with $\gamma > 0$ if

$$z^C(\gamma, n) \geq \beta_k, \tag{10}$$

where

$$z^C(\gamma, n) = \frac{2+(n-1)\gamma}{n\gamma} \tag{11}$$

and β_k is the ratio of the average net quality over the individual net quality of firm k,

$$\beta_k = \frac{\frac{1}{n}\sum_{i=1}^{n}(\alpha_i - c_i)}{\alpha_k - c_k}. \tag{12}$$

When $\beta_k < 1$, the individual net quality of firm k is larger than the average net quality. Firm k is called *higher-qualified* in this case. On the other hand, when $\beta_k > 1$, the individual net quality is less than the average net quality. Firm k is then called *lower-qualified.*

We now inquire into the stability of the Cournot output. Best response dynamics is assumed with static expectations, when each firm assumes believes that the other firms remain unchanged with their outputs from the previous period. Then the best response equation (6) gives rise to the time invariant linear dynamic system

$$q_k(t+1) = \frac{\alpha_k - c_k}{2} - \frac{\gamma}{2}\sum_{i \neq k}^{n} q_i(t), \quad k = 1, 2, ..., n. \tag{13}$$

Substituting $q_k(t+1)$ into (2), the price function of firm k, yields the price dynamic equation associated with the output dynamics:

$$p_k(t+1) = \alpha_k - q_k(t+1) - \gamma \sum_{i \neq k}^{n} q_i(t+1), \quad k = 1, 2, ..., n. \tag{14}$$

Equations (13) and (14) imply that the price dynamics is essentially the same as the quantity dynamics. In other words, the Cournot price is stable (resp. unstable) if the Cournot output is stable (resp. unstable). Therefore it is enough for our purpose to draw our attention only to the stability of the Cournot output.

The coefficient matrix of system (13) is its Jacobian:

$$\boldsymbol{J}_C = \begin{pmatrix} 0 & -\frac{\gamma}{2} & \cdot & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & 0 & \cdot & -\frac{\gamma}{2} \\ \cdot & \cdot & \cdot & \cdot \\ -\frac{\gamma}{2} & -\frac{\gamma}{2} & \cdot & 0 \end{pmatrix}.$$

The corresponding characteristic equation reads

$$|\boldsymbol{J}_C - \lambda \boldsymbol{I}| = (-1)^n \left(\lambda - \frac{\gamma}{2}\right)^{n-1} \left(\lambda + \frac{(n-1)\gamma}{2}\right) = 0,$$

which indicates that there are $n-1$ identical eigenvalues and one different eigenvalue. Without a loss of generality, the first $n-1$ eigenvalues are assumed to be identical,

$$\lambda_1^C = \lambda_2^C = ... = \lambda_{n-1}^C = \frac{\gamma}{2} \text{ and } \lambda_n^C = -\frac{(n-1)\gamma}{2}.$$

Since $|\gamma| < 1$ is assumed, the first $n-1$ eigenvalues are less than unity in absolute value. It depends on the absolute value of λ_n^C whether the Cournot output is stable or not. It follows that $\left|\lambda_2^C\right| = \left|-\frac{\gamma}{2}\right| < 1$ for $n = 2$ (i.e., duopoly) and $\left|\lambda_3^C\right| = |\gamma| < 1$ for $n = 3$ (triopoly). Solving $\left|\lambda_n^C\right| < 1$ for $n > 3$ presents the stability conditions of the Cournot output[3]:

$$n < 1 + \frac{2}{\gamma} \text{ if } \gamma > 0 \text{ and } n < 1 - \frac{2}{\gamma} \text{ if } \gamma < 0.$$

We can now summrize these stability results as follows.

[3] If the Jacobian of the price function satisfies a diagonally dominant condition (i.e., $|dp_k/dq_k| > \sum_{i \neq k} |dp_k/dq_i|$), then we have

$$n < 1 + \frac{1}{\gamma}$$

under which $\left|\lambda_n^C\right| < 1$. However, we proceed our analysis without this strong assumption.

Theorem 1 *Under Cournot competition with $n > 2$, (i) the Cournot output and price are stable for $(\gamma, n) \in R_S^C$ and unstable for $(\gamma, n) \in R_U^C = D_{(+)} \backslash R_S^C$ if the goods are substitutes; (ii) they are stable for $(\gamma, n) \in R_s^C$ and unstable for $(\gamma, n) \in R_u^C = D_{(-)} \backslash R_s^C$ if the goods are complements where the stability regions are, respectively, defined by*

$$R_S^C = \{(\gamma, n) \in D_{(+)} \mid n < 1 + \frac{2}{\gamma}\} \text{ and } R_s^C = \{(\gamma, n) \in D_{(-)} \mid n < 1 - \frac{2}{\gamma}\},$$

and the instability regions, R_U^C and R_u^C, are the complements of the stability regions.

Notice that Cournot outputs are locally unstable if $n > 4$ and $|\gamma| > 1/2$, that is when the number of firms is more than four and the products are sufficiently differentiated.

2.3 Price-adjusting firms

In Bertrand competition, firm k chooses the price of good k to maximize the profit $\pi_k = (p_k - c_k)q_k$ subject to its direct demand (5), taking the other firms' prices given. Solving the first-order condition yields the best reply of firm k,

$$p_k = \frac{\alpha_k + c_k}{2} - \frac{\gamma}{2[1 + (n-2)\gamma]} \sum_{i \neq k}^{n} (\alpha_i - p_i), \text{ for } k = 1, 2, ..., n. \tag{15}$$

The second-order condition for an interior optimum solution is

$$\frac{\partial^2 \pi_k}{\partial p_k^2} = -\frac{2\,(1 + (n-2)\gamma)}{(1-\gamma)(1 + (n-1)\gamma)} < 0, \tag{16}$$

where the direction of inequality depends on the parameter configuration.[4] For $(\gamma, n) \in D_{(+)}$, we see that (16) is always satisfied. On the other hand, for $(\gamma, n) \in D_{(-)}$, we need additional condition to fulfill the second-order condition. Since

$$1 + (n-1)\gamma < 1 + (n-2)\gamma,$$

for $\gamma < 0$, the required condition is either $0 < 1 + (n-1)\gamma$ or $1 + (n-2)\gamma < 0$. As in Häckner (2000), we make the following assumption:

Assumption 2. $1 + (n-1)\gamma > 0$ when $\gamma < 0$.

The Bertrand equilibrium prices are obtained by solving the simultaneous equations

$$p_k - \frac{\gamma}{2[1 + (n-2)\gamma]} \sum_{i \neq k}^{n} p_i = \frac{\alpha_k + c_k}{2} - \frac{\gamma}{2[1 + (n-2)\gamma]} \sum_{i \neq k}^{n} \alpha_i$$

for $k = 1, 2, ..., n$ with unknown p_k. In vector form,

$$\boldsymbol{B}^B \boldsymbol{p} = \boldsymbol{A}^B,$$

[4] Note that inequality (16) is always fulfilled for $n = 2$.

with $\boldsymbol{A}^B = (\frac{\alpha_k + c_k}{2} - \frac{\gamma}{2[1+(n-2)\gamma]} \sum_{i \neq k}^{n} \alpha_i)$, $\boldsymbol{B}^B = (B_{ij}^B)$ where $B_{ii}^B = 1$ and $B_{ij}^B = -\frac{\gamma}{2[1+(n-2)\gamma]}$ for $i \neq j$. Since $\boldsymbol{B}^B$ is invertible, the solution is

$$\boldsymbol{p} = \left(\boldsymbol{B}^B\right)^{-1} \boldsymbol{A}^B,$$

where the diagonal and off-diagonal elements of $\left(\boldsymbol{B}^B\right)^{-1}$ are, respectively,

$$\frac{2(1+(n-2)\gamma)(2+(n-2)\gamma)}{(2+(n-3)\gamma)(2+(2n-3)\gamma)} \text{ and } \frac{2\gamma(1+(n-2)\gamma)}{(2+(n-3)\gamma)(2+(2n-3)\gamma)}.$$

Hence, the Bertrand equilibrium price and output of firm k are given by

$$p_k^B = \frac{(2+(n-3)\gamma)[(1+(n-1)\gamma)(\alpha_k+c_k)-\gamma c_k]-\gamma(1+(n-2)\gamma)\sum_{i=1}^{n}(\alpha_i - c_i)}{(2+(2n-3)\gamma)(2+(n-3)\gamma)} \tag{17}$$

and

$$q_k^B = \frac{1+(n-2)\gamma}{(1-\gamma)(1+(n-1)\gamma)}(p_k^B - c_k) \tag{18}$$

with

$$p_k^B - c_k = \frac{(2+(n-3)\gamma)(1+(n-1)\gamma)(\alpha_k - c_k) - \gamma(1+(n-2)\gamma)\sum_{i=1}^{n}(\alpha_i - c_i)}{(2+(2n-3)\gamma)(2+(n-3)\gamma)}. \tag{19}$$

Due to (18), the Bertrand profit of firm k becomes

$$\pi_k^B = \frac{(1-\gamma)(1+(n-1)\gamma)}{1+(n-2)\gamma}(q_k^B)^2. \tag{20}$$

Equation (18) implies that the Bertrand output is positive if $p_k^B - c_k$ is positive. Under Assumption 2, equation (19) implies that $p_k^B - c_k$ is always positive if $\gamma < 0$, and is nonnegative with $\gamma > 0$ if

$$z^B(\gamma, n) \geq \beta_k \tag{21}$$

where

$$z^B(\gamma, n) = \frac{(2+(n-3)\gamma)(1+(n-1)\gamma)}{(1+(n-2)\gamma)n\gamma} \tag{22}$$

and β_k is defined by (12).

In examining stability of the Bertrand price, we assume best response dynamics with static expectations on price formation and obtain the following system of time-invariant difference equations:

$$p_k(t+1) = \frac{\alpha_k + c_k}{2} - \frac{\gamma}{2[1+(n-2)\gamma]} \sum_{i \neq k}^{n} [\alpha_i - p_i(t)] \text{ for } k = 1, 2, ..., n. \tag{23}$$

Similarly to the Cournot competition, we can also obtain the output difference equations under Bertrand competition by substituting $p_k(t+1)$ into the direct demand function (5):

$$q_k(t+1) = \frac{(1+(n-2)\gamma)(\alpha_k - p_k(t+1)) - \gamma \sum_{i \neq k}^{n} (\alpha_i - p_i(t+1))}{(1-\gamma)(1+(n-1)\gamma)}. \tag{24}$$

It is clear from (23) and (24) that the output dynamics is synchronized with the price dynamics. The coefficient matrix of this price adjusting system is

$$\boldsymbol{J}_B = \begin{pmatrix} 0 & \dfrac{\gamma}{2[1+(n-2)\gamma]} & \cdot & \dfrac{\gamma}{2[1+(n-2)\gamma]} \\ \dfrac{\gamma}{2[1+(n-2)\gamma]} & 0 & \cdot & \dfrac{\gamma}{2[1+(n-2)\gamma]} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\gamma}{2[1+(n-2)\gamma]} & \dfrac{\gamma}{2[1+(n-2)\gamma]} & \cdot & 0 \end{pmatrix}.$$

This matrix has the same structure as $\boldsymbol{J}_C$ when γ is replaced by $-\gamma/2[1+(n-2)\gamma]$, so the eigenvalues are

$$\lambda_1^B = \lambda_2^B = ... = \lambda_{n-1}^B = -\frac{\gamma}{2[1+(n-2)\gamma]} \text{ and } \lambda_n^B = \frac{(n-1)\gamma}{2[1+(n-2)\gamma]}.$$

When $\gamma > 0$ and $n > 2$, we have $\left|\lambda_k^B\right| < 1$ for $k = 1, 2, ..., n$. That is, the Bertrand price is asymptotically locally stable in $D_{(+)}$.[5] On the other hand, when $\gamma < 0$, $0 < \lambda_k^B < 1$ also holds for $k = 1, 2, ..., n-1$. The value of λ_n^B is clearly negative and $\lambda_n^B > -1$ can be rewritten as

$$n < \frac{5}{3} - \frac{2}{3\gamma}$$

under which the Bertrand price is stable[6]. Since the Bertrand competition synchronizes output dynamics with price dynamics, the stability conditions of the Bertrand price and output are summarized as follows:

Theorem 2 *Under Bertrand competition with* $n > 2$, *(i) the Bertrand price and output are stable if the goods are substitutes; (ii) if the goods are complements, then they are stable for* $(\gamma, n) \in R_s^B$ *and unstable for* $(\gamma, n) \in R_u^B = D^B \backslash R_s^B$ *where* D^B *is the feasible region under Assumption 2,*

$$D^B = \{(\gamma, n) \in D_{(-)} \mid 0 < 1 + (n-1)\gamma\},$$

R_s^B *is the stable region,*

$$R_s^B = \{(\gamma, n) \in D^B \mid n < \frac{5}{3} - \frac{2}{3\gamma}\}$$

and R_u^B *is the unstable region, which is the complement of the stable region.*

Theorems 1 and 2 consider stability of the Cournot output and the Bertrand price as well as stability of the Cournot price and the Bertrand output through the difference equations (14) and (24). Graphical explanations of Theorems 1

[5] Okuguchi (1987) has already shown the same result with a more general demand function.

[6] If the Jacobian of the demand function is diagonally dominant (i.e., $|dq_k/dp_k| > \sum_{i \neq k} |dq_i/dp_k|$), then we have

$$n < \frac{3}{2} - \frac{1}{2\gamma}$$

which is stronger than the price-stability condition and thus leads to the stability of Bertrand price. However we do not assume this strong condition in what follows.

and 2 are given in Figure 1. In the first quadrant where $\gamma > 0$, the admissible region $D_{(+)}$ is divided into two parts by the neutral stability locus of the Cournot output $\lambda_n^C = -1$; the light-gray region R_S^C below the locus and the dark-gray region R_U^C above. The Cournot output is stable in the former and unstable in the latter while the Bertrand price is stable in both regions. In the second quadrant where $\gamma < 0$, the admissible region $D_{(-)}$ of the Bertrand price is reduced to D^B by Assumption 2. The neutral stability locus of the Bertrand price $\lambda_n^B = -1$ cuts across the locus of $1 + (n-1)\gamma = 0$ from left to right at point $(-1/2, 3)$ and divides the region D^B into two parts: the light-gray region R_s^B and the dark-gray region R_u^B. The Bertrand price is stable in the former and unstable in the latter. In comparing the Cournot and the Bertrand optimal behavior of output, price and profit, we should confine our analysis to the parametric region in which both equilibria are feasible, otherwise the comparison has no economic meanings. Two facts are clear. One is that we can ignore the white region of the second quadrant in all further discussions as Assumption 2 is violated there. The other is that the Cournot output is always stable in D^B since $D^B \subset R_s^C$.

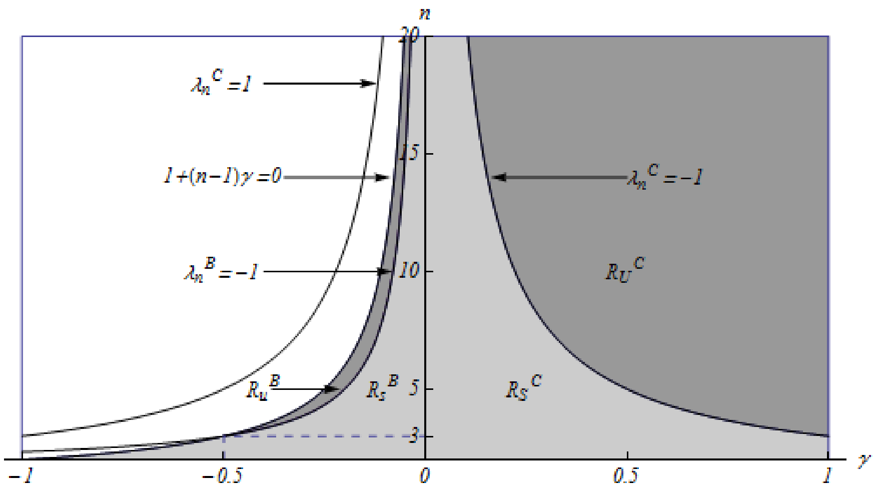


Figure 1. Stable and unstable regions

Theocharis (1960) studies stability of discrete dynamic evolution of the Cournot output under static expectation when the goods are perfect substitutes (i.e., no product differentiation) and demonstrates that the Cournot output is asymptotically stable if and only if the number of firms is equal to two. Theorems 1 and 2 extend Theocharis' classical result and assert that the Cournot output as well as the Bertrand price can be unstable when the number of firms is strictly greater than three and the goods are differentiated. The main point is that Cournot output can be unstable only when the goods are substitutes and the Bertrand price only when the goods are complements.

3 Optimal Strategy Comparison

We call the optimal price, output and profit under Cournot competition *Cournot strategy* and those under Bertrand competition *Bertrand strategy*. In this section we will compare Cournot strategy with Bertrand strategy to examine which strategy is more preferable when the number of the firms becomes more than three.

Assuming $n > 2$ and subtracting (17) from (8) yield a price difference

$$p_k^C - p_k^B = \frac{(\alpha_k - c_k)(n-1)\gamma^2}{(2-\gamma)(2+(2n-3)\gamma)} \frac{1}{z^P(\gamma, n)} \left(z^P(\gamma, n) - \beta_k\right), \tag{25}$$

where

$$z^P(\gamma, n) = \frac{(2+(n-1)\gamma)(2+(n-3)\gamma)}{(n-2)n\gamma^2}.$$

Since the first two factors multiplying the parenthesized term on the right hand side of (25) are positive,

$$sign\left[p_k^C - p_k^B\right] = sign\left[z^P(\gamma, n) - \beta_k\right]. \tag{26}$$

Subtracting (18) from (7) yields the output difference,

$$q_k^C - q_k^B = \frac{(\alpha_k - c_k)(n-1)\gamma^2}{(2-\gamma)(1-\gamma)(2+(2n-3)\gamma)} \frac{1}{z^Q(\gamma, n)} \left(\beta_k - z^Q(\gamma, n)\right) \tag{27}$$

where

$$z^Q(\gamma, n) = \frac{(2+(n-3)\gamma)(1+(n-1)\gamma)(2+(n-1)\gamma)}{n\gamma(4+5(n-2)\gamma+(n^2-5n+5)\gamma^2)}.$$

Consider the right hand side of (27). The first factor is positive while the second one (i.e., the reciprocal of $z^Q(\gamma, n)$) is ambiguous: it is positive when the goods are substitutes and negative when the goods are complements. The sign of the output difference is therefore determined by the simplified expression:

$$sign\left[q_k^C - q_k^B\right] = sign\left[\gamma\left(\beta_k - z^Q(\gamma, n)\right)\right]. \tag{28}$$

Finally, dividing (9) by (20) gives a profit ratio,

$$\frac{\pi_k^C}{\pi_k^B} = \frac{1+(n-2)\gamma}{(1-\gamma)(1+(n-1)\gamma)} \left(\frac{q_k^C}{q_k^B}\right)^2.$$

Since the first factor of the right hand side is positive and greater than unity, we have $\pi_k^C > \pi_k^B$ if $q_k^C > q_k^B$. In order to find a more general condition determining whether the profit ratio is greater or less than unity, we substitute (18) and (7) into the last expression to have

$$\frac{\pi_k^C}{\pi_k^B} = G(\beta_k)$$

with

$$G(\beta_k) = B(\gamma, n)\left(A(\gamma, n)\frac{z^C(\gamma, n) - \beta_k}{z^B(\gamma, n) - \beta_k}\right)^2, \tag{29}$$

where

$$A(\gamma, n) = \frac{(1-\gamma)(1+(n-1)\gamma)(2+(2n-3)\gamma)(2+(n-3)\gamma)}{(2-\gamma)(2+(n-1)\gamma)(1+(n-2)\gamma)^2} > 0$$

and

$$B(\gamma, n) = \frac{1+(n-2)\gamma}{(1-\gamma)(1+(n-1)\gamma)} > 0.$$

When the net quality of firm k is equal to the average net quality offered by all firms, the profit ratio is

$$G(1) = \frac{(2+(n-3)\gamma)^2(1+(n-1)\gamma)}{(1-\gamma)(1+(n-2)\gamma)(2+(n-1)\gamma)^2}.$$

The difference of the denominator and the numerator of $G(1)$ is

$$(n-1)^2(2+(n-2)\gamma)\gamma^3,$$

which then implies that

$$G(1) > 1 \text{ if } \gamma > 0 \text{ and } G(1) < 1 \text{ if } \gamma < 0. \tag{30}$$

Differentiating $G(\beta_k)$ with respect to β_k gives, after arranging terms,

$$\frac{dG(\beta_k)}{d\beta_k} = \frac{2A(\gamma,n)^2 B(\gamma,n)(z^C(\gamma,n) - z^B(\gamma,n))(z^C(\gamma,n) - \beta_k)}{(z^B(\gamma,n) - \beta_k)^2}. \tag{31}$$

Noticing that $z^C(\gamma,n) - z^B(\gamma,n) > 0$, $z^C(\gamma,n) - \beta_k > 0$ and $z^B(\gamma,n) - \beta_k > 0$ when $\gamma > 0$ and $z^C(\gamma,n) - z^B(\gamma,n) < 0$, $z^C(\gamma,n) - \beta_k < 0$ and $z^B(\gamma,n) - \beta_k < 0$ when $\gamma < 0$, we find that the sign of the derivative of $G(\beta_k)$ is positive when the goods are substitutes and negative when complements:

$$\frac{dG(\beta_k)}{d\beta_k} > 0 \text{ when } \gamma > 0 \text{ and } \frac{dG(\beta_k)}{d\beta_k} < 0 \text{ when } \gamma < 0. \tag{32}$$

3.1 Duopoly Case: $n = 2$

As a benchmark case, we consider duopolies and confirm the Singh-Vives results in our framework. Substituting $n = 2$ into (25), (27) and (29) yields

$$p_k^C - p_k^B = \frac{(\alpha_k - c_k)\gamma^2}{4-\gamma^2}, \tag{33}$$

$$q_k^C - q_k^B = \frac{\gamma^2}{(1-\gamma^2)(4-\gamma^2)} \left\{ 2(\alpha_k - c_k)\gamma \left[\beta_k - \frac{1+\gamma}{2\gamma} \right] \right\} \tag{34}$$

and

$$G(\beta_k) = (1-\gamma^2)\left(\frac{z^C - \beta_k}{z^B - \beta_k} \right)^2, \tag{35}$$

where

$$z^C = \frac{2+\gamma}{2\gamma} \text{ and } z^B = \frac{(2-\gamma)(1+\gamma)}{2\gamma}.$$

Given $|\gamma| < 1$, it is fairly straightforward that $p_k^C > p_k^B$ always and $q_k^C < q_k^B$ when $\gamma < 0$. To determine the sign of the output difference in case of $\gamma > 0$, we return to the direct demand (5) and consider consequences of the assumption $\alpha_i - \gamma\alpha_j > 0$ for $i \neq j$, which is implicitly imposed to guarantee that the independent or non-induced demand for $p_i = 0,\ i = 1, 2$ is positive when the goods are substitutes. This assumption can be rewritten as

$$\alpha_k - \gamma\alpha_j = 2\alpha_k\gamma\left(\frac{1+\gamma}{2\gamma} - z_k\right) > 0$$

with

$$z_k = \frac{1}{2}\frac{\sum_{i=1}^{2}\alpha_i}{\alpha_k}$$

being the ratio of the average quality of the two firms over the individual quality of firm k. Since $\gamma > 0$ and $\alpha_k > 0$, this inequality indicates that an upper bound is imposed on z_k,

$$z_k < \frac{1+\gamma}{2\gamma}.$$

Furthermore, $\alpha_i - \gamma\alpha_k > 0$ can be rewritten as

$$\frac{\alpha_i}{\alpha_k} > \gamma,$$

which is substituted into the definition of z_k to have

$$z_k > \frac{1+\gamma}{2}.$$

If the marginal costs are zero, then it is apparent from the definitions that $\beta_k = z_k$. In the future discussions, we retain $c_i > 0$ and make the following assumption,

$$\frac{c_1}{\alpha_1} = \frac{c_2}{\alpha_2},$$

under which, it is not difficult to show that $\beta_k = z_k$. Then β_k is bounded both from above and from below,

$$\frac{1+\gamma}{2} < \beta_k < \frac{1+\gamma}{2\gamma}$$

and

$$2(\alpha_k - c_k)\gamma\left(\frac{1+\gamma}{2\gamma} - \beta_k\right) > 0.$$

With the last inequality, the output difference (34) is negative, so $q_k^C < q_k^B$ in case of $\gamma > 0$. Hence we have $q_k^C < q_k^B$ always regardless of whether the goods are substitutes or complements.

Substituting $n = 2$ into (31) gives

$$\frac{dG(\beta_k)}{d\beta_k} = \gamma\frac{z^C - \beta_k}{z^B - \beta_k} \gtrless 0 \text{ if } \gamma \gtrless 0.$$

The minimum value of β_k is $(1+\gamma)/2$ when $\gamma > 0$ and $1/2$ when $\gamma < 0$, which is substituted into the profit ratio (35) to obtain

$$G\left(\frac{1+\gamma}{2}\right) = \frac{(2-\gamma^2)^2}{4-\gamma^2} > 1 \text{ and } G\left(\frac{1}{2}\right) = \frac{4-\gamma^2}{(2-\gamma^2)^2} < 1.$$

The value of $G(\beta_k)$ increases in β_k and is greater than unity for the minimum value of β_k when $\gamma > 0$ whereas it decreases and is less than unity for the minimum value of β_k when $\gamma < 0$. Hence we obtain that

$$\pi_k^C > \pi_k^B \text{ if } \gamma > 0 \text{ and } \pi_k^C < \pi_k^B \text{ if } \gamma < 0.$$

We have therefore confirmed the Singh-Vives results, (i-SV), (ii-SV) and (iii-SV), mentioned in the Introduction and now we will proceed to the general n-firm case in order to examine the effects of the increasing number of firms on these results.

3.2 The goods are substitutes, $\gamma > 0$

We first assume that firm k is higher-qualified (i.e., $\beta_k \leq 1$). If $\gamma > 0$, then $z^P(\gamma, n) > z^C(\gamma, n) > z^B(\gamma, n) > z^Q(\gamma, n) > 1$ and thus

$$z^Q(\gamma, n) > \beta_k.$$

With this inequality, equations (11), (21), (25) and (28) imply the following three results: (i) q_k^C and p_k^C are positive; (ii) q_k^B and p_k^B are positive; (iii) $p_k^C > p_k^B$ and $q_k^C < q_k^B$. Before examining the profit ratio, we assume, as in the duopoly case, that the non-induced demand of (5) is positive:

Assumption 3. $(1 + (n-2)\gamma)\,\alpha_k - \gamma \sum_{i \neq k}^n \alpha_i > 0$.

This assumption can be rewritten as

$$\alpha_k n\gamma \left(\frac{1+(n-1)\gamma}{n\gamma} - z_k\right) > 0,$$

where

$$z_k = \frac{1}{n}\frac{\sum_{i=k}^n \alpha_i}{\alpha_k}$$

is the ratio of the average quality over the individual quality of firm k. The above inequality implies that Assumption 3 imposes an upper bound on z_k,

$$z_k < \frac{1+(n-1)\gamma}{n\gamma}.$$

The same assumption for firm j ($\neq k$) can be converted into

$$\alpha_j > \frac{n\gamma}{1+(n-1)\gamma} z_k \alpha_k.$$

This inequality is substituted into the definition of z_k to obtain a lower bound of z_k,

$$z_k > \frac{1+(n-1)\gamma}{n}.$$

That is, Assumption 3 restricts the value of z_k into an interval by imposing upper and lower bounds. To simplify the relation between z_k and β_k, we make one more assumption that the ratio of the unit cost over the quality of firm k is identical with the ratio of the average cost over the average quality in the market:

Assumption 4. $\dfrac{c_k}{\alpha_k} = \dfrac{\sum_{i=1}^n c_i/n}{\sum_{i=1}^n \alpha_i/n}$.

Under Assumptions 3 and 4, the net quality ratio of firm k is equal to the quality ratio (i.e., $\beta_k = z_k$) and thus has the upper and lower bounds,

$$\beta_k^m = \frac{1+(n-1)\gamma}{n} < 1 \text{ and } \beta_k^M = \frac{1+(n-1)\gamma}{n\gamma} > 1$$

and satisfies inequality

$$(\alpha_k - c_k)n\gamma\left(\frac{1+(n-1)\gamma}{n\gamma} - \beta_k\right) > 0.$$

Substituting β_k^m and β_k^M into (29) gives

$$G(\beta_k^m) \gtreqless 1 \text{ and } G(\beta_k^M) > 1.$$

First of all, if $G(\beta_k^m) \geq 1$, then $G(\beta_k) \geq 1$ as $G'(\beta_k) > 0$. Hence $\pi_k^C \geq \pi_k^B$. If, on the other hand, $G(\beta_k^m) < 1$, then there is a threshold value $\bar{\beta}_{ks}(\gamma, n)$ making $G(\bar{\beta}_{ks}(\gamma, n)) = 1$, since $G(\beta_k^M) > 1$ and $G'(\beta_k) > 0$, where the explicit form of $\bar{\beta}_{ks}(\gamma, n)$ is obtained by solving $G(\beta_k) = 1$,[7]

$$\bar{\beta}_{ks}(\gamma, n) = \frac{z^B - A^2Bz^C - A(z^C - z^B)\sqrt{B}}{1 - A^2B}. \tag{36}$$

Here the dependency of each term on γ and n is omitted for the sake of notational simplicity. Given β_k, γ and n, we have the following results on the profit differences:

$$\text{if } \beta_k < \bar{\beta}_k(\gamma, n), \text{ then } G(\beta_k) < 1 \text{ implying } \pi_k^C < \pi_k^B$$

and

$$\text{if } \beta_k \geq \bar{\beta}_k(\gamma, n), \text{ then } G(\beta_k) \geq 1 \text{ implying } \pi_k^C \geq \pi_k^B.$$

It is clear from these discussions that the two conditions $G(\beta_k^m) < 1$ and $\beta_k > \bar{\beta}_{ks}(\gamma, n)$ give rise to $\pi_k^C > \pi_k^B$. We are now ready to consider parametric configurations under which these two inequality conditions are fulfilled. In particular, we take the following three steps to determine the configurations:

[7] Solving $G(\beta_k) = 1$ yields one more solution,

$$\frac{z^B - A^2Bz^C + A(z^C - z^B)\sqrt{B}}{1 - A^2B}.$$

It is, however, becomes greater than unity for $\gamma \in (0, 1)$ and $n \geq 3$ whereas $\beta_k < 1$ is assumed here. This solution is eliminated for further considerations.

Step I. The net quality ratio β_k is assumed to be less than unity and its lower bound β_k^m is also less than unity. Thus given the value of β_k, the locus of $\beta_k^m = \beta_k$ divides the admissible region $D_{(+)}$ into two parts: one is a region with $\beta_k^m \leq \beta_k$ and the other is a region with $\beta_k^m > \beta_k$. The latter region is discarded because Assumption 3 is violated there.

Step II. The former region obtained at Step I is further divided by the locus of $G(\beta_k^m) = 1$ into two parts: one region with $G(\beta_k^m) < 1$ and the other with $G(\beta_k^m) \geq 1$ in which $\pi_k^C \geq \pi_k^B$ follows.

Step III. Finally the former region with $G(\beta_k^m) < 1$ obtained at Step II is further divided by the locus of $\bar{\beta}_{ks}(\gamma, n) = \beta_k$ into two parts: one region with $\beta_k < \bar{\beta}_{ks}(\gamma, n)$ and other region with $\beta_k \geq \bar{\beta}_{ks}(\gamma, n)$. All these divisions make it clear that $\pi_k^C < \pi_k^B$ in the former region and $\pi_k^C \geq \pi_k^B$ in the latter region.

A graphical representation of dividing $D_{(+)}$ is given in Figure 2 with $\beta_k = 1/2$.[8] It is the reproduction of the first quadrant of Figure 1 and thus the downward-sloping hyperbola is the neutral stability locus. The steeper positive sloping curve is the $\beta_k^m = \beta_k$ locus. Assumption 3 is violated in the white region in the right side of this locus. The U-shaped curve is the $G(\beta_k^m) = 1$ locus, above which $G(\beta_k^m) < 1$. The half-real and half-dotted curve is the equal-profit locus $\bar{\beta}_{ks}(\gamma, n) = \beta_k$ and divides the region with $G(\beta_k^m) < 1$ and $\beta_k^m < \beta_k$ into two parts. In the horizontally-striped region we have $\pi_k^C < \pi_k^B$ and the inequality is reversed in the non-striped region.[9]

Notice the two important issues. The first issue is that $\pi_k^C < \pi_k^B$ holds in the horizontally-striped region of Figure 2. As mentioned in (ii-SV) in the Introduction, $\pi_k^C > \pi_k^B$ always in the duopoly framework when $\gamma > 0$. This inequality become reverse in the n-firm framework, however, it has been already pointed out by Häckner (2000) in his Proposition 2(ii). We confirm it and further construct a set of pairs (γ, n) for which it holds under Assumptions 3 and 4. The second issue is that the Cournot output and price are locally unstable whenever $\pi_k^C < \pi_k^B$ since the horizontally-striped region is located within the unstable region, which is the dark-gray domain surrounded by the two loci of $\lambda_n^C = -1$ and $\beta_k^m = \beta_k$. In summary, we arrive at the following conclusions when $\gamma > 0$ and $\beta_k \leq 1$:

Theorem 3 *When β_k, γ and n are given such that firm k is higher-qualified and $\beta_k^m < \beta_k$, then (i) firm k charges a higher price and produces smaller output under Cournot competition than under Bertrand competition; (ii) its Bertrand profit is higher than its Cournot profit when $\beta_k < \bar{\beta}_{ks}(\gamma, n)$ whereas the profit dominance is reversed otherwise; (iii) the Cournot equilibrium is locally unstable when $\pi_k^C < \pi_k^B$.*

[8] Changing the value of β_k keeps the results obtained under $\beta_k = 1/2$ qualitatively the same.

[9] The directions of the inequalities in the divided regions such as $\pi_k^C < \pi_k^B$ and $G(\beta_k^m) < 1$ are numerically confirmed.

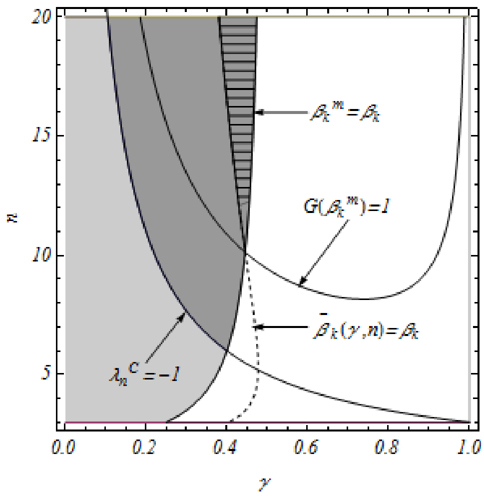


Figure 2. The division of the feasible region $D_{(+)}$ when $\beta_k = 1/2$

Next, firm k is assumed to be lower-qualified (i.e., $\beta_k > 1$). In order to make any comparison meaningful, we have to find out the parametric configurations of γ and n when Assumption 3 is satisfied. Since $\beta_k^m < \beta_k$ as $\beta_k^m < 1$ by definition, β_k should be chosen to be less than its upper bound. The locus of $\beta_k^M = \beta_k$ divides the admissible region $D_{(+)}$ into two parts,

$$R_+ = \{(\gamma, n) \in D_{(+)} \mid \beta_k^M \geq \beta_k\}$$

and

$$R_- = \{(\gamma, n) \in D_{(+)} \mid \beta_k^M < \beta_k\}.$$

For $(\gamma, n) \in R_-$, Assumption 3 is violated. So we eliminate this region from all further considerations and confine our attention to R_+. When $\gamma > 0$, we have the following orderings:

$$z^P(\gamma, n) > z^C(\gamma, n) > z^B(\gamma, n) > 1 \text{ and } z^B(\gamma, n) > \beta_k^M > z^Q(\gamma, n).$$

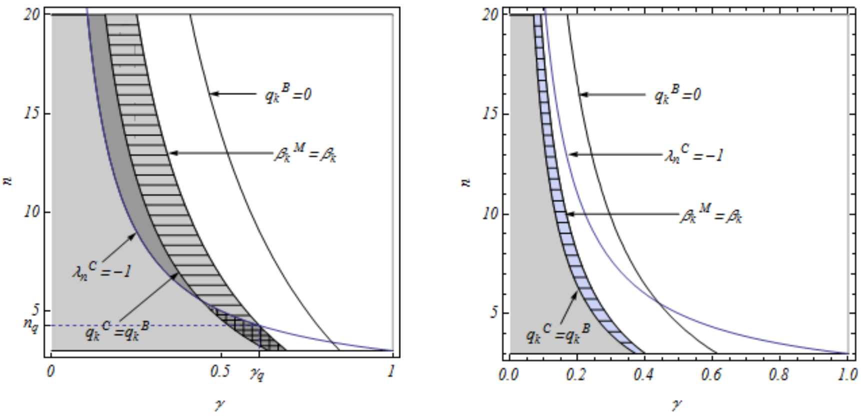
Consequently, $z^P(\gamma, n) > \beta_k$ in R_+ and then equation (25) indicates that $p_k^C > p_k^B$ always in R_+. Since $\beta_k > 1$, $G(1) > 1$ and $G'(\beta_k) > 0$ lead to $G(\beta_k) > 1$, so equation (29) indicates that $\pi_k^C > \pi_k^B$ always in R_+.

The indeterminacy of the relative magnitude between $z^Q(\gamma, n)$ and β_k implies that the equal-product locus of $z^Q(\gamma, n) = \beta_k$ divides R_+ into two parts. In a part with $z^Q(\gamma, n) < \beta_k$, the Cournot output is larger than the Bertrand output, according to equation (28). Notice that the case of $q_k^C > q_k^B$ does not emerge in duopolies and its possibility is not examined in Häckner (2000). The $\beta_k^M = \beta_k$ locus crosses the $\lambda_n^C = -1$ locus at point (γ_q, n_q) with

$$\gamma_q = \frac{3}{\beta_k} - 2 \text{ and } n_q = 1 + \frac{2}{\gamma_q}.$$

Here γ_q is positive for $\beta_k \in (1, 3/2)$ and decreases monotonically form 1 to 0 as β_k increases from 1 to 3/2. Accordingly, n_q increases from 3 to infinity. Graphically this means that the intersection (γ_q, n_q) moves upwards along the neutral stability locus since the upper bound curve shifts leftward in R_+ as β_k increases. It also means that when $\beta_k > 3/2$, the two curves do not intercept and the upper bound curve is within the stable region. Summarizing these observations, we possibly obtain $q_k^C > q_k^B$ for $n \geq 3$ when $\gamma > 0$ and $\beta_k > 1$. Two examples of the division of R_+ are given in Figure 3. In Figure 3(A) where $\beta_k = 1.15$, $q_k^C > q_k^B$ in the horizontally-striped and hatched regions. Furthermore q_k^C is unstable in the horizontally-striped region and stable in the hatched region. In Figure 3(B) where $\beta_k = 3/2$, $q_k^C > q_k^B$ in the horizontally-striped region and $q_k^C < q_k^B$ in the light-gray region while q_k^C is stable in both regions. Since β_k is larger than its upper bound in the white region right to the $\beta_k^M = \beta_k$ locus, we discard it. Comparing Figure 3(A) with Figure 3(B), it can be seen that the whole horizontally-striped region is inside the stable region for $\beta_k > 3/2$ and some part of the region is outside the stable region for $\beta_k < 3/2$. That is, q_k^C can be unstable for a relatively large value of n and $\beta_k < 3/2$ although it is always stable with $\beta_k \geq 3/2$. We summarize these results as follows:

Theorem 4 *When β_k, γ and n are given such that firm k is lower-qualified and $\beta_k < \beta_k^M$, then (i) firm k charges a higher price and earns a larger profit under Cournot competition than under Bertrand competition; (ii) its Cournot output is larger than its Bertrand output for $\beta_k > z^Q(\gamma, n)$ and the relation is reversed for $\beta_k < z^Q(\gamma, n)$; (iii) its Cournot output is locally stable for $\beta_k \geq 3/2$ and the stability may be lost for $\beta_k < 3/2$.*



(A) $\beta_k = 1.15$ (B) $\beta_k = 1.5$

Figure 3. Stable and unstable regions in $D_{(+)}$ when $\beta_k > 1$

3.3 The goods are complements, $\gamma < 0$

When $\gamma < 0$, it should be noticed that the non-induced demand is always positive and Assumption 3 is not necessary. However, β_k still has the lower bound that is defined to be $1/n$ when the net qualities of any other firms are zero. Let us begin with quantity comparison. It is clear that $z^Q(\gamma, n) < 0$ for any γ and n in D^B. Regardless of whether β_k is greater or less than unity, equation (28) implies that

$$q_k^C < q_k^B.$$

Concerning the profit ratio with $\gamma < 0$, $G(1) < 1$ in (30) and $G'(\beta_k) < 0$ in (32) lead to $\pi_k^C < \pi_k^B$ when firm k is not higher-qualified (i.e., $\beta_k \geq 1$). This is the same result as the one obtained in the duopoly framework. We proceed to pursue the inequality reversal of the profit difference when firm k is higher-qualified (i.e., $\beta_k < 1$). Given n, the profit ratio for $\beta_k^m = 1/n$ is reduced to

$$G\left(\frac{1}{n}\right) = A(\gamma, n)^2 B(\gamma, n) \left(\frac{(1+(n-2)\gamma)(2+(n-2)\gamma)}{2+3(n-2)\gamma+(n^2-5n+5)\gamma^2} \right)^2.$$

It is indeterminate in general whether $G(1/n)$ is greater or less than unity. If it is less than or equal to unity, then $G'(\beta_k) < 0$ leads to $G(\beta_k) \leq 1$ for $\beta_k \geq 1/n$. In consequence, we have $\pi_k^C \leq \pi_k^B$. On the other hand, if $G(1/n)$ is greater than unity, then $G'(\beta_k) < 0$ reveals an existence of a threshold value of β_k that solves $G(\beta_k) = 1$. Let this solution be denoted by $\bar{\beta}_{kc}(\gamma, n)$,[10]

$$\bar{\beta}_{kc}(\gamma, n) = \frac{z^B - A^2 B z^C + A(z^C - z^B)\sqrt{B}}{1 - A^2 B}. \tag{37}$$

We thus have the following results on the profit differences:

$$\text{if } \beta_k > \bar{\beta}_{kc}(\gamma, n), \text{ then } G(\beta_k) < 1 \text{ implying } \pi_k^C < \pi_k^B$$

and

$$\text{if } \beta_k < \bar{\beta}_{kc}(\gamma, n), \text{ then } G(\beta_k) > 1 \text{ implying } \pi_k^C > \pi_k^B.$$

The last result does not agree with $\pi_k^C < \pi_k^B$ in (iii-SV), the result obtained in the duopoly framework and indicates a possibility that the inequality reversal may take place in the n-firm framework. In what follows, we will inquire into the parametric configurations that generates $G(1/n) > 1$ and $\beta_k < \bar{\beta}_{kc}(\gamma, n)$.

Since the $G(1/n) = 1$ locus is distorted U-shaped as illustrated in Figure 5 below, it depends on a value of n whether $G(1/n)$ is greater or less than unity,

$$G\left(\frac{1}{n}\right) < 1 \text{ for } \gamma \in (\gamma_0, 0) \text{ when } n = 8$$

and

$$G\left(\frac{1}{n}\right) > 1 \text{ for } \gamma \in (\gamma_1, \gamma_2) \text{ when } n = 9$$

where $\gamma_0 = 1/(1-n)$, $\gamma_i \in (\gamma_0, 0)$ for $i = 1, 2$ are the solutions of equation $G(1/n) = 1$. Hence there is a threshold value $\tilde{n} \in (8, 9)$ such that $G(1/\tilde{n}) = 1$ has a unique solution for $\tilde{\gamma} \in (\tilde{\gamma}_0, 0)$ with $\tilde{\gamma}_0 = 1/(1-\tilde{n})$. It is numerically obtained that $\tilde{\gamma} \simeq -0.133$ and $\tilde{n} \simeq 8.16$. For $n \leq \tilde{n}$, we have $G(1/n) \leq 1$ which implies that $G(\beta_k) \leq 1$ for $\beta_k \geq 1/n$. It then follows that $\pi_k^C \leq \pi_k^B$ for $n \leq \tilde{n}$.

[10] Needless to say, $\bar{\beta}_{ks}(\gamma, n)$ in (36) is also the solution. However, it is negative for (γ, n) such as $1 + (n-1)\gamma > 0$. Therefore we eliminate $\bar{\beta}_{ks}(\gamma, n)$ from further considerations.

Before proceeding further, we return to the definitions of $\bar{\beta}_{kc}(\gamma, n)$ in (37) and $\tilde{n}$ to obtain

$$G(\bar{\beta}_{kc}(\gamma, n)) = G\left(\frac{1}{\tilde{n}}\right) \text{ implying } \bar{\beta}_{kc}(\tilde{\gamma}, \tilde{n}) = \frac{1}{\tilde{n}}.$$

We denote this threshold value of β_k by $\hat{\beta} = \bar{\beta}_{kc}(\tilde{\gamma}, \tilde{n})$ or $\hat{\beta} = 1/\tilde{n}$. Simple calculation shows that $\hat{\beta} \simeq 0.123$. In Figure 4 below, the graphs of $\bar{\beta}_{kc}(\gamma, n)$ with changing values of n are depicted against γ. The left most graph is for $n = \tilde{n}(\simeq 8.16)$ and the next graph is for $n = 9$. As the value of n increases from 9 with two increments, the graph moves rightward accordingly. The right most graph is for $n = 33$. Given n, $\bar{\beta}_{kc}(\gamma, n)$ takes a concave-convex curve and

$$\max_{\gamma_0 < \gamma < 0} \bar{\beta}_{kc}(\gamma, n) \geq \lim_{\gamma \to 0} \bar{\beta}_{kc}(\gamma, n)$$

where the equality holds only for $n = \tilde{n}$. The maximum value of $\bar{\beta}_{kc}(\gamma, n)$ is attained at the vertex of the concave part and decreases with increasing number of n. In other words, $\hat{\beta}$ is the upper bound of $\bar{\beta}_{kc}(\gamma, n)$ for $n > \tilde{n}$ and $0 > \gamma > \tilde{\gamma}_0$.

Given $\beta_k < 1$, let $n_k = 1/\beta_k$. If $n < n_k$, then $\beta_k^m (= 1/n) > \beta_k$. This inequality violates the assumption that the net quality β_k is greater than or equal to its lower bound, β_k^m. Thus the case of $n < n_k$ is not considered as further discuttions. After the above special cases, we may now turn to the case with $n \geq n_k$ and $n > \tilde{n}$, from which the following two sub-cases are identified:

$$n_k \leq \tilde{n} < n \text{ and } \tilde{n} < n_k \leq n.$$

We examine the first case of $n_k \leq \tilde{n} < n$. The latter condition $\tilde{n} < n$ implies that the equation $G(1/n) = 1$ has two distinct solutions γ_1 and γ_2 such that $G(1/n) > 1$ for $\gamma \in (\gamma_1, \gamma_2)$. Since $G'(\beta_k) < 0$, there exists a $\bar{\beta}_{kc}(\gamma, n) > 1/n$ for $\gamma \in (\gamma_1, \gamma_2)$ such that $G(\bar{\beta}_{kc}(\gamma, n)) = 1$. As is explained, $\hat{\beta} = \bar{\beta}_{kc}(\tilde{\gamma}, \tilde{n})$ is greater than $\bar{\beta}_{kc}(\gamma, n)$. The alternative expression of the former condition $n_k \leq \tilde{n}$ is $\hat{\beta} \leq \beta_k$. In consequence, we have $\bar{\beta}_{kc}(\gamma, n) < \beta_k$ implying that $G(\beta_k) < 1$. Therefore our first result on the profit difference is that

$$\pi_k^C < \pi_k^B \text{ if } n_k \leq \tilde{n} < n. \tag{38}$$

In the second case of $\tilde{n} < n_k \leq n$, we can show that $\pi_k^C > \pi_k^B$ is also possible. The former condition $\tilde{n} < n_k$ implies $\beta_k < \hat{\beta} = \bar{\beta}_{kc}(\tilde{\gamma}, \tilde{n})$. The maximum value of $\bar{\beta}_{kc}(\gamma, n)$ with respect to γ decreases when the number of n increases. In consequence, we can find the threshold value $\hat{n}$ such that

$$\max_{\gamma_0 < \gamma < 0} \bar{\beta}_{kc}(\gamma, \hat{n}) = \beta_k \text{ and } \max_{\gamma_0 < \gamma < 0} \bar{\beta}_{kc}(\gamma, n) > \beta_k \text{ for } n < \hat{n}. \tag{39}$$

For $n > \hat{n}$, $\bar{\beta}_{kc}(\gamma, n) < \beta_k$ implying that $G(\bar{\beta}_{kc}(\gamma, n)) = 1 > G(\beta_k)$. Hence our second result on the profit difference is

$$\pi_k^C < \pi_k^B \text{ if } \tilde{n} < n_k \leq \hat{n} < n. \tag{40}$$

On the other hand (39) implies that for $n < \hat{n}$, there are two distinct values γ_k^a and γ_k^b such that $\bar{\beta}(\gamma_k^a, n) = \bar{\beta}(\gamma_k^b, n) = \beta_k$. Hence we have $\bar{\beta}_{kc}(\gamma, n) >$

β_k for $\gamma \in (\gamma_k^a, \gamma_k^b)$ and $n < \hat{n}$. As it turned out, this inequality means that $G(\bar{\beta}_{kc}(\gamma, n)) = 1 < G(\beta_k)$. Hence our third result on the profit difference is

$$\pi_k^C > \pi_k^B \text{ if } \tilde{n} < n_k < n < \hat{n}. \tag{41}$$

Lastly, if $n = \hat{n}$, then $\gamma_k^a = \gamma_k^b$ and $G(\bar{\beta}_{kc}(\gamma_k^i, \hat{n})) = G(\beta_k)$ for $i = a, b$ implying that $\pi_k^C = \pi_k^B$.

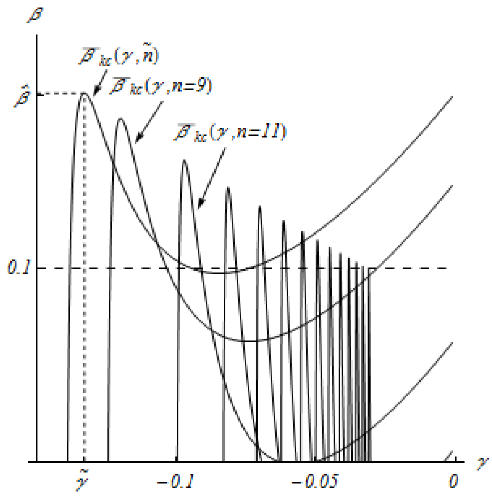


Figure 4. Various $\tilde{\beta}(\gamma, n)$ curves aginst γ, given n

As an illustration, we specify a value of β_k and construct a parametric configuration in which $\pi_k^C < \pi_k^B$. Figure 5 is an enlargement of the second quadrant of Figure 1 in which we take $\beta_k = 0.1$ (i.e., $n_k = 10 > \tilde{n} \simeq 8.16$)[11] and divide the feasible region D^B with the following three steps:

Step I. The white region is a union of the region with $1 + (n-1)\gamma < 0$ and the region with $n < n_k$. It is eliminated from further considerations as the optimal Bertrand solution does not fulfill the second-order condition and /or β_k is less than its lower bound for γ and n in this region.

Step II. The neutral stability locus $\lambda_n^B = -1$ divides the remaining D^B region into two parts: the unstable (dark-gray) region and the stable (light-gray) region. The $G(1/n) = 1$ locus further divides the unstable region into two parts: one with $G(1/n) < 1$ and the other with $G(1/n) > 1$, the least dark-gray region illustrated inside the unstable darker-gray region. In the stable region and the region with $G(1/n) < 1$, we have $\pi_k^C < \pi_k^B$.

Step III. The equal-profit locus of $\bar{\beta}_{kc}(\gamma, n) = \beta_k$ or $\pi_k^C = \pi_k^B$ defined for $n \geq n_k$ divides the lightes-gray region into two parts: one with $\beta_k >$

[11] It is numerically checked that

$$\max_{\gamma} \bar{\beta}_{kc}(\gamma, 31) > 0.1 \text{ and } \max_{\gamma} \bar{\beta}_{kc}(\gamma, 33) < 0.1$$

from which $\hat{n} \in (31, 33)$ follows.

$\bar{\beta}_{kc}(\gamma, n)$ and the other with $\beta_k < \bar{\beta}_{kc}(\gamma, n)$. It crosses the $n = n_k$ locus at points (γ_k^1, n_k) and (γ_k^2, n_k).[12] In the horizontally-striped region, we have $G(\beta_k) > 1$ and $\beta_k < \bar{\beta}_{kc}(\gamma, n)$ so that the Cournot profit is larger than the Bertrand profit while the Bertrand profit is larger in the non-striped gray region.

It is worthwhile to point out two issues. The first issue is that a higher-qualified firm k possibly earns more profits under Cournot competition if its net quality difference is large to the extent that β_k is less than the maximum value of $\bar{\beta}_k(\gamma, n)$, with given n. In the case when the goods are complements, the dominance of Cournot profit over Bertrand profit is not observed in Singh and Vives (1984) and Häckner (2000). The second issue is that the Bertrand price is locally unstable when $\pi_k^C > \pi_k^B$ since the horizontally-striped region is located within the unstable region.

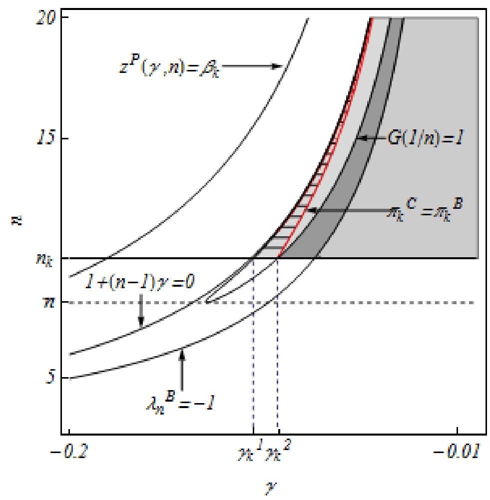


Figure 5. Division of the feasible region D^B when $\beta_k < 1$

Next we turn our attention to the price difference. We have already seen that the Cournot price as well as the Bertrand price is positive when $\gamma < 0$ and that the neutral stability curve $\lambda_n^B = -1$ intercepts the locus of $1 + (n-1)\gamma = 0$ at point $(-1/2, 3)$ so it divides the feasible region D^B into the unstable region R_u^B and the stable region R_s^B as shown in the second quadrant of Figure 1. In addition to this, with given β_k, the equal-price locus of $z^P(\gamma, n) = \beta_k$ also divides D^B into two parts:

$$R_+^B = \{(\gamma, n) \in D^B \mid z^P(\gamma, n) > \beta_k\}$$

[12] Solving $G(1/n_k) = 1$ yields two distinct solutions γ_k^1 and γ_k^2 for which

$$G(\bar{\beta}(\gamma_k^1, n_k)) = G(\bar{\beta}(\gamma_k^2, n_k)) = 1.$$

Since $\beta_k = 1/n_k$, we have

$$\bar{\beta}(\gamma_k^1, n_k) = \bar{\beta}(\gamma_k^2, n_k) = \beta_k,$$

For $n_k = 10$, $\gamma_k^1 \simeq -0.11$ and $\gamma_k^2 \simeq -0.098$.

and

$$R^B_- = \{(\gamma, n) \in D^B \mid z^P(\gamma, n) \leq \beta_k\}.$$

The location of the $z^P(\gamma, n) = \beta_k$ locus is sensitive to the value of β_k. Various combinations of γ and n determining the price difference and possible dynamic behavior can be conveniently classified according to the different values of parameter β_k. We start with a case in which firm k is higher-qualified. The case of lower-qualified firm k will be discussed later.

Case 1. $0 < \beta_k \leq 1$.

In this case, R^B_- is empty because $z^P(\gamma, n) > 1$ for $(\gamma, n) \in D^B$. Consequently $p^C_k > p^B_k$ always in $D^B \subset R^B_+$. In Figure 5 with $\beta_k = 0.1$, the equal-price curve $z^P(\gamma, n) = \beta_k$ is located in the white region that does not belong to D^B. As shown in the second quadrant of Figure 1, p^C_k is always stable while the stability of p^B_k is indeterminate: it is stable in R^B_s and unstable in R^B_u.

Case 2. $1 < \beta_k \leq \frac{8}{3}$.

The equal-price locus shifts downward as β_k increases. For $\beta_k > 1$, it crosses the locus of $1 + (n-1)\gamma = 0$ at the point (γ_1, n_1) with

$$\gamma_1 = \frac{1 - \sqrt{1 - \beta_k + \beta_k^2}}{\beta_k} \text{ and } n_1 = 1 - \frac{1}{\gamma_1}.$$

The equal-price locus divides the unstable region R^B_u into two parts. In Figure 6 where $\beta_k = 2$,[13] $R^B_u \cap R^B_-$ is located above the equal-price locus and vertically-striped, $R^B_s \cap R^B_+$ is the lighter-gray region above the neutral stability locus while $R^B_u \cap R^B_+$ is between these two loci . The intersection moves downwards along the locus of $1 + (n-1)\gamma = 0$ as β_k increases from unity and arrives at the point $(-1/2, 3)$ when $\beta_k = 8/3$. Hence we have the following results concerning the price difference and the stability of the Bertrand price:

(2-*i*) $p^C_k < p^B_k$ and p^B_k is unstable for $(\gamma, n) \in R^B_u \cap R^B_-$,

(2-*ii*) $p^C_k > p^B_k$ and p^B_k is unstable for $(\gamma, n) \in R^B_u \cap R^B_+$,

(2-*iii*) $p^C_k > p^B_k$ and p^B_k is stable for $(\gamma, n) \in R^B_s \cap R^B_+$.

[13] We take $n = 8$ and restrict the interval of γ to $(-0.5, -0.1)$ only for the sake of graphical convenience. Changing the values of n and enlarging the interval do not affect the qualitative aspects of the results.

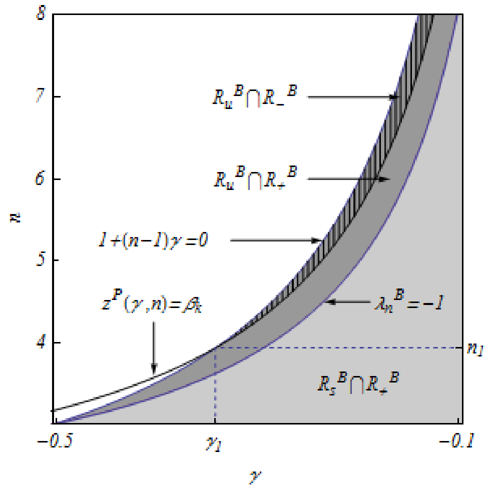


Figure 6, Division of D^B when
$1 < \beta_k < \frac{8}{3}$

Case 3. $\frac{8}{3} < \beta_k \leq 4$.

When β_k increases further from 8/3, then the equal-price locus intercepts the neutral stability locus of $\lambda_n^B = -1$ from below at point (γ_2, n_2) with

$$\gamma_2 = \frac{2(\beta_k - 4)}{5\beta_k - 8} \text{ and } n_2 = \frac{5}{3} - \frac{2}{3\gamma_2}.$$

As shown in Figure 7 where $\beta_k = 3$, the equal-price locus divides the unstable region R_u^B into the vertically-striped gray region above the locus and the darker-gray region below. It also divides the stable region R_s^B into two parts: the hatched region above the locus and the light-gray region below. Since it is not easy to see that the hatched region is bounded by the neutral stability locus and the equal-price locus, the lower-left part of Figure 7 is enlarged and is inserted into Figure 7. We have the following four possibilities concerning the price difference and the stability of the Bertrand price in this case:

(3-i) $p_k^C < p_k^B$ and p_k^B is unstable for $(\gamma, n) \in R_u^B \cap R_-^B$,

(3-ii) $p_k^C > p_k^B$ and p_k^B is unstable for $(\gamma, n) \in R_u^B \cap R_+^B$,

(3-iii) $p_k^C < p_k^B$ and p_k^B is stable for $(\gamma, n) \in R_s^B \cap R_-^B$,

(3-iv) $p_k^C > p_k^B$ and p_k^B is stable for $(\gamma, n) \in R_s^B \cap R_+^B$.

Notice that the value of γ_2 becomes negative for $\beta_k \leq 4$. The intersection moves upward along the neutral stability locus as β_k increases further up to 4.

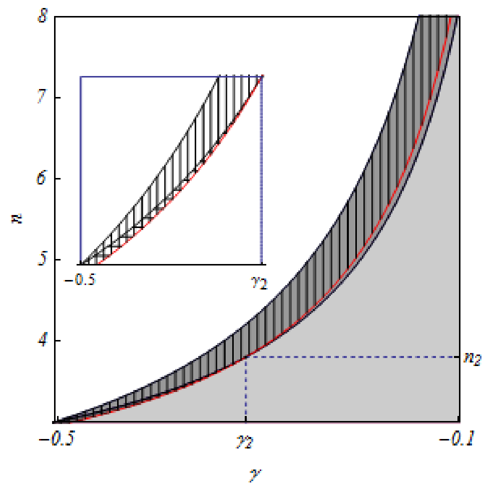


Figure 7. Division of D^B when $\frac{8}{3} < \beta_k < 4$

Case 4. $\beta_k > 4$.

When β_k becomes larger than 4, the equal-price locus is located below the $\lambda_n^B = -1$ locus. It then divides the stable region R_s^B into two parts, $R_s^B \cap R_-^B$ and $R_s^B \cap R_+^B$. The former corresponds to the hatched region and the latter to the light-gray region in Figure 8 in which $\beta_k = 5$. The hatched region that appeared first in Figure 7 becomes larger with increasing value of β_k. The whole region R_u^B is vertically striped, which means that the Bertrand price is larger than the Cournot price and is unstable. We have therefore the following results concerning the price differences and the stability of the Bertrand price in this case:

(4-i) $p_k^C < p_k^B$ and p_k^B is unstable for $(\gamma, n) \in R_u^B$,

(4-ii) $p_k^C < p_k^B$ and p_k^B is stable for $(\gamma, n) \in R_s^B \cap R_-^B$,

(4-iii) $p_k^C > p_k^B$ and p_k^B is stable for $(\gamma, n) \in R_s^B \cap R_+^B$.

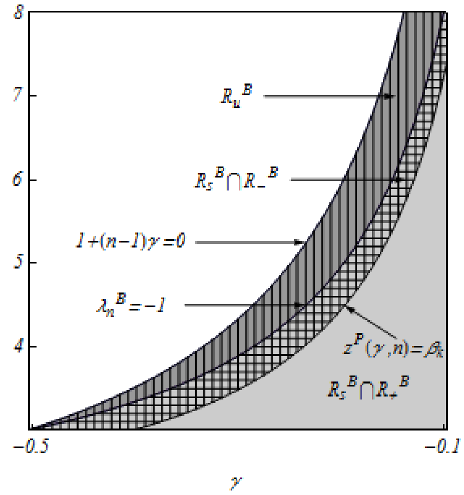


Figure 8. Division of D^B when $\beta_k > 4$

We are now in a position to present our results on the price differences with $\gamma < 0$. Proposition 1(ii) of Häckner (2000) deals with the case where the goods are complements and shows that lower-qualified firms charge higher prices under Bertrand competition than under Cournot competition when quality differences are large. The same result is obtained in our analysis (see (3-*i*), (3-*iii*), (4-*i*) and (4-*ii*)) when $\beta_k > 8/3$. However, (2-*i*) implies that large quality differences are not necessary to obtain $p_k^C < p_k^B$. It is shown there that even if the deviation of β_k from unity is small enough, $p_k^C < p_k^B$ is still possible when the number of firms are relatively large. Furthermore, two new results are obtained in our analysis: it is shown first that the region of (γ, n) with $p_k^C < p_k^B$ becomes larger as β_k increases and second that p_k^B can become unstable.

We summarize the comparison between the Cournot strategy and the Bertrand strategy when the goods are complements in the following theorem:

Theorem 5 *(i) When firm k is higher-qualified, then* $p_k^C > p_k^B$ *and* $q_k^C < q_k^B$ *always whereas* $\pi_k^C > \pi_k^B$ *if* $\tilde{n} < n_k < n < \hat{n}$ *and* $\pi_k^C \leq \pi_k^B$ *otherwise. In addition, the Bertrand price is locally unstable when* $\pi_k^C > \pi_k^B$. *(ii) When firm k is lower-qualified, then* $q_k^C < q_k^B$ *and* $\pi_k^C < \pi_k^B$ *always whereas* $p_k^C < p_k^B$ *is possible.*

Theorems 3, 4 and 5 are summarized in Table 1. When a firm is higher-qualified, the quantity and price comparisons given in the first two rows support the conventional wisdom that the Bertrand competition is more competitive than the Cournot competition in the sense that the Bertrand firm charges a lower price and produces a higher output. Profitability between these competitions are ambiguous. It is not the case when the firm is lower-qualified. The results with "$\lesseqgtr$" are obtained in the n-firm framework. Two of them, however, have already been exhibited by Häckner (2000): when the goods are complements lower-qualifed firms charge higher prices under Bertrand competition than under Cournot competition (i.e., $p_k^C < p_k^B$ when $\gamma < 0$ and $\beta_k > 1$) in his Proposition 1(ii) and when the goods are substitutes, higher-qualified firms earn higher profits under Bertrand competition than under Cournot competition (i.e., $\pi_k^C < \pi_k^B$ when $\gamma > 0$ and $\beta_k < 1$) in his Proposition 2(ii). In this study, we first confirm these results and then classify the parameter region into specified subregions in which these results hold as seen in Figures 2-8 except Figure 4. In addition, we demonstrate two new results: when the goods are complements, then higher-qualified firms earn higher profits under Cournot competition (i.e., $\pi_k^C > \pi_k^B$ when $\gamma < 0$ and $\beta_k < 1$) and when the goods are substitutes, then lower-qualified firms may produce more output under Cournot competition (i.e., $q_k^C > q_k^B$).

	Substitutes ($\gamma > 0$)	Complements ($\gamma < 0$)
Higher-qualified ($\beta_k < 1$)	$p_k^C > p_k^B$ $q_k^C < q_k^B$ $\pi_k^C \lesseqgtr \pi_k^B$	$p_k^C > p_k^B$ $q_k^C < q_k^B$ $\pi_k^C \lesseqgtr \pi_k^B$
Lower-qualified ($\beta_k > 1$)	$p_k^C > p_k^B$ $q_k^C \lesseqgtr q_k^B$ $\pi_k^C > \pi_k^B$	$p_k^C \lesseqgtr p_k^B$ $q_k^C < q_k^B$ $\pi_k^C < \pi_k^B$

Table 1. Comparison of Cournot and Bertrand strategies

4 Concluding Remarks

Singh and Vives (1984) have shown that the duopoly model with linear demand and cost functions have definitive results concerning the nature of Cournot and Bertrand competition as it was mentioned in the Introduction. Häckner (2000) increases the number of firms to n from 2 and exhibits that some of these results are sensitive to the duopoly assumption. In this study, we examine the general n-firm oligopoly model and add two main findings to the existing literature on Cournot and Bertrand competitions. The first finding is concerned with the stability of Cournot and Bertrand equilibria. As stated in Theorems 1 and 2, Cournot equilibrium may be unstable whereas Bertrand equilibrium is always stable when the goods are substitutes. It is further shown that Bertrand equilibrium may be unstable whereas Cournot equilibrium is always stable when the goods are complements. This finding extends the stability result of Theocharis (1960) that a Cournot oligopoly model is unstable if more than three firms are involved and the goods are homogenous (i.e., perfectly substitutes).

The second finding is concerned with the comparison of Cournot and Bertrand strategies. In addition to the inequality reversal of the price and quantity differences, the profit differences shown in the duopoly framework may be reversed in the n-firm framework. Furthermore, as shown in Figures 2 and 5, the horizontally-striped regions are located inside the instability regions. This means that, for example, $\pi_k^C < \pi_k^B$ is possible when $\gamma > 0$ and $\beta_k < 1$, however, π_k^C is locally unstable. The result of $\pi_k^C < \pi_k^B$ does not have much economic implication from the dynamic point of view. Therefore the natural question to be next raised should be concerned with the global dynamic properties of the locally unstable model. Matsumoto and Szidarovszky (2010) have already started their research in this direction.

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