Heterogeneous Strategies in Nonlinear Duopoly with Product Differentiation*

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Abstract

This study constructs a nonlinear duopoly model with product differentiation. Its purpose is twofold; to compare profitability of quantity strategy and price strategy when heterogeneity of production costs exists and then to show circumstances under which complex dynamics occurs. It is demonstrated that a price strategy may dominate a quantity strategy if the cost difference is large. It is also demonstrated that a long-run average profit of a price-setter taken along a chaotic path is negative but that of a quantity setter is larger than its own equilibrium profit. The results imply that in heterogeneous competition, the price strategy could be favorable when an economy is stable and is unfavorable when destabilized.

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1 Introduction

In the present paper, we consider strategic behavior of economic agents with heterogeneity in a nonlinear system. To this end, we construct a product-differentiated duopoly model in which heterogeneity and nonlinearity are involved. Our main aim is to shed light on the following two effects; static effects on optimal behavior caused by introducing heterogeneity and dynamical effects on a time-path caused by introducing nonlinearity. Three different kind of heterogeneity is taken into account; heterogeneity (i.e., difference) in production costs and heterogeneity in strategy (i.e., quantity or price) that each firm can choose, and heterogeneity in competition by which we mean that different firms are to take different strategies. For analytical convenience, demand is assumed to be isoelastic when nonlinearity is required.

In the classical duopoly theory, research interest was concentrated on competitions between firms which were homogeneous in the sense that they produce homogeneous goods, taking the same strategy (i.e., quantity or price). Among considerable efforts devoted to pull the classical model closer to reality, there are two important approaches; one is to introduce the notion of product differentiation, and the other is to consider a mixture of different strategies in a market. Following both research agendas, it has been demonstrated that a dominant strategy of each firm is to set quantity (price) in a differentiated duopoly if goods are substitutes (complements) [e.g., see Singh/Vives (1984), Okuguchi (1984) and Cheng (1985).] Although the conclusion is clear, it is derived under circumstances in which production costs are the same or homogeneous, and demand is linear. So far, not much yet has been revealed with respect to heterogeneity in production cost. The first result of this study adds a possibility that the price strategy is dominant if firms are heterogeneous in cost and goods are substitutes.

Turning our attentions to dynamical analysis, vast amounts of researches have been devoted to complex dynamics which nonlinear models intrinsically exhibit. For an overview of recent developments of complex dynamics in economics, see, e.g., three volumes edited by Rosser (2004) that contain widely selected topics on complexity in economics. Regarding nonlinear oligopoly dynamics, see Puu/Sushko (2002) and Chapter 7 of Puu (2003), in which mainly nonlinear Cournot dynamics is discussed from various points of view. On the contrary, only a few effort has been devoted to shed light on differentiated duopoly in heterogeneous competition: see Yousefi/Szidarovszky (2005) for a simulation study with a general isoelastic demand and Matsumoto/Onozaki (2005) for a theoretical study with a specific linear demand. The second result of this study is to reveal the circumstances under which the heterogeneous nonlinear duopoly model exhibits complex dynamics, relying on a simple iterative model.

The paper is organized as follows: Section 2 overviews heterogenous duopoly under a general form of demand function. Section 3 explains the static results obtained in a differentiated duopoly model with linear demand. Section 4 replaces linear demand with nonlinear demand and construct our model. Section 5 discusses dynamical implications. Section 6 makes concluding remarks.

2 Overview of Duopoly

Two distinct firms exist in a market: one is firm X that produces good x with a unit production cost a and sells goods at price p_x, and the other is firm Y that produces good y with a unit production cost b and sells goods at price p_y. Each firm chooses one of two strategies, quantity or price, and maximizes its profits, taking the competitor's strategic variable as given. With two firms and two strategies, there are four possible types of duopoly competition according as which firm takes which strategy. In *Cournot-Cournot* (henceforth CC) competition, each firm sets quantity, taking its competitor's output as given, and in *Bertrand-Bertrand* (BB) competition, each firms sets price, taking its competitor's price as given. In *Cournot-Bertrand* (CB) competition, firm X sets quantity, taking p_y as given while firm Y setz price, taking x as given. In *Bertrand-Cournot* (BC) competiton in which strategies are interchanged, firm X sets price, taking y as given while firm Y sets quantity, taking p_x as given. We call the former two types of competitions as *homogeneous* because both firms follow the same strategy. On the other hand, we call the latter two types as *heterogeneous* because the firms take different strategies.[1] In what follows, we present a somewhat general description of the above heterogeneous competition of duopoly.

Inverse demand functions of firm X and Y are given by

$$\begin{aligned} p_x &= f(x, y), \\ p_y &= g(x, y), \end{aligned} \tag{1}$$

which are assumed to be monotonically decreasing in x and y. Solving (1) simultaneously with respect to x and y yields direct demands

$$\begin{aligned} x &= u(p_x, p_y), \\ y &= v(p_x, p_y). \end{aligned} \tag{2}$$

[1] To the best of our knowledge, Bylka/Komar (1976) is the first to consider heterogeneous duopoly, what they call, "different mixed oligopolies."

For the sake of convenience, let us solve each equation of (1) with respect to x and y respectively and denote the solutions as

$$\begin{aligned} x &= f^{-1}(p_x, y), \\ y &= g^{-1}(x, p_y). \end{aligned} \tag{3}$$

Profit functions are

$$\begin{aligned} \pi_x &= (p_x - a)x, \\ \pi_y &= (p_y - b)y. \end{aligned} \tag{4}$$

In CB competition a strategic variable for firm X is quantity and that for firm Y is price, so that profit functions (4) should be rewrite, with the aid of (1) and (3), as

$$\begin{aligned} \pi_x = (p_x - a)x = (f(x, y) - a)\, x = \left(f\left(x, g^{-1}(x, p_y)\right) - a\right) x \\ = \Pi_x(x, p_y), \end{aligned}$$

$$\begin{aligned} \pi_y = (p_y - b)y = (g(x, y) - b)\, y = \left(g\left(x, g^{-1}(x, p_y)\right) - b\right) g^{-1}(x, p_y) \\ = \Pi_y(x, p_y). \end{aligned}$$

Reaction functions are derived by solving profit-maximizing conditions,

$$\frac{\partial \Pi_x(x, p_y)}{\partial x} = 0,$$

$$\frac{\partial \Pi_y(x, p_y)}{\partial p_y} = 0,$$

and a CB equilibrium is defined as[2]

$$x^{CB} = \underset{x}{\operatorname{argmax}}\ \Pi_x\left(x, p_y^{CB}\right),$$

$$p_y^{CB} = \underset{p_y}{\operatorname{argmax}}\ \Pi_y\left(x^{CB}, p_y\right).$$

In BC competition the strategic situation is reversed, so that profit func-

[2]Henceforth, superscript CB or BC is attached to a variable to denote that it is an equilibrium value of the corresponding type of duopoly.

tions are

$$\pi_x = (p_x - a)x = \left(f(x, y) - a\right)x = \Big(f\left(f^{-1}(p_x, y), y\right) - b\Big) f^{-1}(p_x, y)$$
$$= \Pi_x(p_x, y),$$

$$\pi_y = (p_y - b)y = \left(g(x, y) - b\right)y = \Big(g\left(f^{-1}(p_x, y), y\right) - b\Big) y$$
$$= \Pi_y(p_x, y).$$

Reaction functions are also obtained by solving profit-maximizing conditions,

$$\frac{\partial \Pi_x(p_x, y)}{\partial p_x} = 0,$$

$$\frac{\partial \Pi_y(p_x, y)}{\partial y} = 0,$$

and a BC equilibrium is defined as

$$p_x^{BC} = \underset{p_x}{\text{argmax}}\ \Pi_x\left(p_x, y^{BC}\right),$$

$$y^{BC} = \underset{y}{\text{argmax}}\ \Pi_y\left(p_x^{BC}, y\right).$$

3 Heterogeneous Strategy with Linear Demand

We recapitulate the essence of the differentiated duopoly model with linear demand proposed by Singh/Vives (1985) and summarize their main results in this section. Linear demand is replaced with nonlinear demand in the next section.

Inverse demand is assumed to be linear and given by

$$p_x = p_M - x - \theta y,$$
$$p_y = p_M - \theta x - y, \tag{5}$$

in which θ denotes the degree of product differentiation and p_M is a positive constant indicating the maximum price to be attained. If $\theta = 1$, (5) are reduced to demand functions for homogenous goods (i.e., perfect substitutes). If $\theta = 0$, it means the case of independent products in which the duopoly market turns to be two monopoly markets. A positive (negative) θ implies that goods are substitutes (complements). Since we get, due to the symmetry, complementarity from substitutability only by changing the sign of θ, we confine our analysis to the case in which the goods are substitutes in this study. Hence we assume

Assumption 1 $0 < \theta < 1$.

Solving (5) simultaneously with respect to x and y yields direct demands,

$$x = \frac{1}{1-\theta^2}\left\{(1-\theta)p_M - p_x + \theta p_y\right\},$$

$$y = \frac{1}{1-\theta^2}\left\{(1-\theta)p_M + \theta p_x - p_y\right\}.$$

In this section, marginal costs are zero, which is the usual assumption in the context of linear model.[3] In CB competition, firm X is a quantity-setter and firm Y is a price-setter. With the aid of the manner described in the previous section, we get the reaction functions,

$$2(1-\theta^2)x - \theta p_y = (1-\theta)p_M \qquad \text{for firm } X,$$

$$\theta x + 2p_y = p_M \qquad \text{for firm } Y.$$

Although given in implicit forms, the reaction function of firm X can be seen to be upward sloping and that of firm Y downward sloping. These reaction functions intersect only once to yield a CB equilibrium in which outputs and prices are

$$\left(x^{CB}, y^{CB}\right) = \left(\frac{2-\theta}{4-3\theta^2}p_M, \frac{(2+\theta)(1-\theta)}{4-3\theta^2}p_M\right),$$

$$\left(p_x^{CB}, p_y^{CB}\right) = \left(\frac{(2-\theta)(1-\theta^2)}{4-3\theta^2}p_M, \frac{(2+\theta)(1-\theta)}{4-3\theta^2}p_M\right).$$

Substituting into the profit functions yields CB profits,

$$\left(\pi_x^{CB}, \pi_y^{CB}\right) = \left(\frac{(2-\theta)^2(1-\theta^2)}{(4-3\theta^2)^2}{p_M}^2, \frac{(2+\theta)^2(1-\theta)^2}{(4-3\theta^2)^2}{p_M}^2\right)$$

[3]The results to be obtained in this section hold if cost are positive and identical.

The ratios of each variables are

$$\frac{x^{CB}}{y^{CB}} = \frac{2-\theta}{2-\theta-\theta^2} > 1,$$

$$\frac{p_x^{CB}}{p_y^{CB}} = \frac{2+\theta-\theta^2}{2+\theta} < 1, \tag{6}$$

$$\frac{\pi_x^{CB}}{\pi_y^{CB}} = \frac{4-3\theta^2+\theta^3}{4-3\theta^2-\theta^3} > 1.$$

Inequalities imply that firm X produces more output, faces a lower price and makes more profits.

In BC competition in which firm X is a price-setter and firm Y is a quantity-setter, firm X chooses p_x, taking y as given while firm Y chooses y, taking p_x as given. Symmetrically, we get the following ratios of BC variables,

$$\frac{x^{BC}}{y^{BC}} = \frac{2-\theta-\theta^2}{2-\theta} < 1,$$

$$\frac{p_x^{BC}}{p_y^{BC}} = \frac{2+\theta}{2+\theta-\theta^2} > 1, \tag{7}$$

$$\frac{\pi_x^{BC}}{\pi_y^{BC}} = \frac{4-3\theta^2-\theta^3}{4-3\theta^2+\theta^3} < 1.$$

It is noted that all inequalities are reversed, which implies that firm Y produces more output, faces a lower price and makes larger profits in BC competition. From (6) and (7), we conclude that if goods are substitutes, a quantity strategy is preferable in heterogeneous competition, which coincides with what Singh/Vives (1984) show. The results are then summarized as

Proposition 1 *Given Assumption 1, quantity-setter produces more output, faces a lower prices and makes larger profits than price-setter in CB competition as well as in BC competition.*

4 Heterogeneous Competition with Nonlinear Demand

In this section we replace the linear demand with the nonlinear demand and find out effects on optimal behavior of firms caused by such a replacement.

The inverse demand functions of firm X and Y are assumed to be isoelastic,

$$
\begin{aligned}
p_x &= \frac{1}{x+\theta y}, \\
p_y &= \frac{1}{\theta x + y}.
\end{aligned}
\tag{8}
$$

where θ, again, denotes the degree of product differentiation which fulfills Assumption 1.[4] Solving (8) with respect to x and y yields direct demands,

$$
\begin{aligned}
x &= \frac{1}{1-\theta^2}\left(\frac{1}{p_x} - \frac{\theta}{p_y}\right), \\
y &= \frac{1}{1-\theta^2}\left(\frac{\theta}{p_x} - \frac{1}{p_y}\right).
\end{aligned}
$$

Assumption 1 assures the normal situation that quantity demanded is negatively related to its own price and does not go to infinity. It also implies the substitutability between goods because the cross derivatives are positive.

4.1 Cournot-Bertrand Competition

In this subsection, we consider CB competition in which firm X is a quantity-setter and firm Y is a price-setter. Let us remind that a and b denote constant marginal costs. Following the procedure described in Section 2, we get the CB reaction functions,

$$
\begin{aligned}
\theta p_y &= a\left((1-\theta^2)p_y x + \theta\right)^2 && \text{for firm } X, \\
\theta x p_y^2 &= b && \text{for firm } Y,
\end{aligned}
\tag{9}
$$

both of which are defined in (x, p_y)-space and presented as implicit forms only for the sake of later analysis. To characterize CB equilibrium in the quantity space, (x, y), it is convenient to transform (9) using the inverse demand for y as

$$
\begin{aligned}
\theta(\theta x + y) &= a(x+\theta y)^2 && \text{for firm } X, \\
\theta x &= b(\theta x + y)^2 && \text{for firm } Y.
\end{aligned}
\tag{10}
$$

[4] Puu (2003) introduces the same type of isoelastic demand function into a traditional CC duopoly model and extensively studies its dynamics when goods are homogenous.

Dividing the first equation of (10) by the second gives, after arrangements,

$$c(\theta + z) = \left(\frac{1+\theta z}{\theta + z}\right)^2. \tag{11}$$

where $z = \frac{y}{x}$ and $c = \frac{b}{a}$.

Let us denote the right hand side of (11) by $h(z)$ and the left hand side by $w_{CB}(z)$. An intersection of these two functions, if any, which is a solution of (11), determines a CB equilibrium. The function $h(z)$ is positive, declines monotonically (i.e., $h'(z) < 0$), and is bounded from above (i.e., $h(0) = \frac{1}{\theta^2} > 1$) as well as below ($\lim_{z\to\infty} h(z) = \theta^2 < 1$) under Assumption 1. The function $w_{CB}(z)$ has a positive intercept ($w_{CB}(0) = c\theta > 0$) and increases monotonically. Thus, to ensure the existence of intersection (and, at the same time, its uniqueness) in the nonnegative quadrant, we need to impose $w_{CB}(0) < h(0)$ or equivalently

Assumption 2 $c < \dfrac{1}{\theta^3}$.

Let us denote the solution of (11) by γ that is the ratio of CB outputs,

$$\gamma = \gamma(\theta, c) \tag{12}$$

where $c(\theta+\gamma) = \left(\frac{1+\theta\gamma}{\theta+\gamma}\right)^2$ and $\gamma = \frac{y^{CB}}{x^{CB}}$. Since the explicit form of $\gamma(\theta, c)$, which can be constructed, seems to be unmanageable, we do not exhibit it.

Substituting x^{CB} and $y^{CB} = \gamma x^{CB}$ into (10) and arranging terms give explicit forms of CB outputs in terms of parameters,

$$\begin{aligned} x^{CB} &= \frac{\theta(\theta+\gamma)}{a(1+\theta\gamma)^2} = \frac{\theta}{b(\theta+\gamma)^2} > 0, \\ y^{CB} &= \frac{\gamma\theta(\theta+\gamma)}{a(1+\theta\gamma)^2} = \frac{\gamma\theta}{b(\theta+\gamma)^2} > 0. \end{aligned} \tag{13}$$

Substituting (13) into (8) yields CB prices and then their ratio as

$$\begin{aligned} p_x^{CB} &= \frac{1}{(1+\theta\gamma)x^{CB}}, \\ p_y^{CB} &= \frac{1}{(\theta+\gamma)x^{CB}}, \\ \frac{p_y^{CB}}{p_x^{CB}} &= \frac{1+\theta\gamma}{\theta+\gamma}. \end{aligned} \tag{14}$$

Finally, substituing (13) and (14) into profit functions yields CB profits and then their ratio as

$$\pi_x^{CB} = \frac{1-\theta^2}{(1+\theta\gamma)^2},$$

$$\pi_y^{CB} = \frac{\gamma^2 c(\theta+\gamma)}{(1+\theta\gamma)^2}, \tag{15}$$

$$\frac{\pi_y^{CB}}{\pi_x^{CB}} = \frac{\gamma^2}{1-\theta^2}\left(\frac{1+\theta\gamma}{\theta+\gamma}\right)^2.$$

In what follows we concentrate on determining relative magnitude of CB variables, which depends upon the configuration of parameters (θ, c). From (12) $w_{CB}(\gamma) = h(\gamma)$. Since $h(1) = 1$, we get

$$\gamma \lesseqqgtr 1 \quad \text{according as} \quad w_{CB}(1) \gtreqqless 1.$$

It is found that from (14)

$$\frac{1+\theta\gamma}{\theta+\gamma} \gtreqqless 1 \quad \text{according as} \quad \gamma \lesseqqgtr 1.$$

Solving $w_{CB}(1) = 1$ with respect to c yields a critical value of production cost ratio denoted by $\bar{c}$, and under Assumption 1 we obtain

$$\bar{c} = \frac{1}{1+\theta} < 1.$$

Now it is easy to see the relations with respect to the output ratio and the price ratio,

$$\frac{y^{CB}}{x^{CB}} = \gamma \lesseqqgtr 1 \quad \text{and} \quad \frac{p_y^{CB}}{p_x^{CB}} = \frac{1+\theta\gamma}{\theta+\gamma} \gtreqqless 1 \quad \text{according as} \quad c \lesseqqgtr \bar{c}, \tag{16}$$

which implies that the equi-output curve is identical with the equi-price curve in (θ, c)-space. The $c = \bar{c}$ curve is downward-sloping and divides the parameter region to be considered $\{(\theta, c) \mid 0 < \theta < 1 \text{ and } 0 < c\}$ into two subregions. It can be verified that $x^{CB} > y^{CB}$ and $p_x^{CB} < p_y^{CB}$ for (θ, c) in the region above the curve and the inequality is reversed in the region below. This fact means that a firm which produces more output selles it at lower prices.

Now let us turn to profit comparison. It is difficult to derive the explicit form of the equi-profit curve, $\pi_y^{CB} = \pi_x^{CB}$, due to the complicated expression of γ. In spite of doing so, we depict the equi-profit curve with the aid of numerical calculations as a bold line in Figure 1(A). It exhibits a U-shaped profile and divides the parameter region into two. Profit contours with various ratio values of ρ are also illustrated. It is easy to see that $\pi_y^{CB} < \pi_x^{CB}$ in the upper subregion with $\rho < 1$ and $\pi_y^{CB} > \pi_x^{CB}$ in the lower subregion with $\rho > 1$. Production of firm Y, y^{CB}, is negative in the shaded region at the upper-right corner of both Figure 1(A) and (B). That region will be neglected in the further considerations.

To sum up, since the equi-output curve and the equi-profit curve are different and have no intersection within the parameter region to be considered, they divide the region into three subregions labelled as CB_1, CB_2 and CB_3 in Figure 1(B). The order of magnitude of each firm's CB variables in each region is as follows:

$$CB_1 = \{(\theta, c) \mid x^{CB} > y^{CB},\ p_x^{CB} < p_y^{CB},\ \pi_x^{CB} > \pi_y^{CB}\},$$

$$CB_2 = \{(\theta, c) \mid x^{CB} > y^{CB},\ p_x^{CB} < p_y^{CB},\ \pi_x^{CB} < \pi_y^{CB}\},$$

$$CB_3 = \{(\theta, c) \mid x^{CB} < y^{CB},\ p_x^{CB} > p_y^{CB},\ \pi_x^{CB} < \pi_y^{CB}\}.$$

Let us say that a firm is efficient (inefficient) if its marginal cost is lower (higher). In the region CB_1 with $c > 1$, firm X is efficient, produces more output, faces lower prices and earns more profits than firm Y. In the region CB_1 with $c < 1$, it is inefficient, still produces more output and earns more profits. In the region CB_2 in which $c < 1$, it is inefficient, still produces more output but earns less profits. Finally, in the region CB_3, firm X is inefficinet, produces lower output, faces a higher price and makes less profits than firm Y. In a duopoly situation, the results display the symmetry. Thus from the point of view of firm Y, the reversed results applies to firm Y. In the region CB_3, for example, firm Y is efficient, produces more output, sets lower prices and makes more profits than firm X. The same is said of regions CB_2 and CB_1. We summarize the results of CB competition as follows:

Proposition 2 *Firm X (firm Y) produces more output and makes more profits in CB_1 (CB_3) whereas firm X produces more output but is inefficient and less profitable while firm Y produces less output but is efficient and more profitable in CB_2.*

***** Figure 1 *****

4.2 Bertrand-Cournot Competiton

In this subsection, we consider BC competition in which firm X is a price-setter and firm Y is a quantitiy-setter. Due to the symmetry between CB and BC competitions, by replacing x with y, p_x with p_y, and a with b in (10), reaction functions in the BC competition are

$$\begin{aligned} \theta y p_x^2 &= a && \text{for firm } X \\ \theta p_x &= b((1-\theta^2)p_x y + \theta)^2 && \text{for firm } Y \end{aligned} \tag{17}$$

which are defined in (p_x, y)-space. As in CB competition, we convert BC equilibrium into the quantity space, so that (17) is rewritten, by substituting the inverse demand function of firm X, as

$$\begin{aligned} \theta y &= a(x+\theta y)^2 && \text{for firm } X, \\ \theta(x+\theta y) &= b(\theta x + y)^2 && \text{for firm } Y, \end{aligned} \tag{18}$$

which is defined in (x, y)-space. Dividing the first equation of (18) by the second and applying again $z = \frac{x}{y}$ and $c = \frac{b}{a}$ yields

$$\frac{cz}{(1+\theta z)} = \left(\frac{1+\theta z}{\theta + z}\right)^2. \tag{19}$$

The right hand side of (19) is the same as that in (11) and denoted again by $h(z)$ while the left hand side by $w_{BC}(z)$. An intersection of these two functions, if any, which is a solution of (19), determines a BC equilibrium. Since $w_{BC}(z)$ is monotonically increasing and bounded from above (i.e., $\lim_{z\to\infty} w_{BC}(z) = \frac{c}{\theta}$), we need to impose $\lim_{z\to\infty} w_{BC}(z) > \lim\limits_{z\to\infty} h(z)$, or equivalently $c > \theta^3$, to ensure a positive solution.[5] Thus we require

Assumption 3 $c > \theta^3$.

Let us denote a solution of (19) by δ that is the ratio of BC outputs,

$$\delta = \delta(\theta, c)$$

where $\frac{c\delta}{(1+\theta\delta)} = \left(\frac{1+\theta\delta}{\theta+\delta}\right)^2$ and $\delta = \frac{y^{BC}}{x^{BC}}$. Substituting x^{BC} and $y^{BC} = \delta x^{BC}$ into (18) and arranging terms give explicit forms of BC outputs in terms of

[5]It should be reminded that $h(z)$ declines monotonically and is bounded from below, i.e., $\lim\limits_{z\to\infty} h(z) = \theta^2 < 1$.

parameters,

$$
\begin{aligned}
x^{BC} &= \frac{\theta\delta}{a(1+\theta\delta)^2} = \frac{\theta(1+\theta\delta)}{b(\theta+\delta)^2} > 0, \\
y^{BC} &= \frac{\theta\delta^2}{a(1+\theta\delta)^2} = \frac{\delta\theta(1+\theta\delta)}{b(\theta+\delta)^2} > 0.
\end{aligned}
\tag{20}
$$

Substituting (20) into (8) yields BC prices and then their ratio as

$$
\begin{aligned}
p_x^{BC} &= \frac{1}{(1+\theta\delta)x^{BC}}, \\
p_y^{BC} &= \frac{1}{(\theta+\delta)x^{BC}}, \\
\frac{p_y^{BC}}{p_x^{BC}} &= \frac{1+\theta\delta}{\theta+\delta}.
\end{aligned}
\tag{21}
$$

By the same reason as in the analysis of CB competition, the $w_{BC}(1) = 1$ locus is equivalent to the equi-output curve as well as the equi-price curve. Solving it with respect to c yields a critical value of the production cost ratio denoted by $\tilde{c}$, and we obtain $\tilde{c} = 1 + \theta$. The following relations are also derived:

$$
\delta \gtreqless 1 \quad \text{and} \quad \frac{1+\theta\delta}{\theta+\delta} \lesseqgtr 1 \qquad \text{according as} \quad c \lesseqgtr \tilde{c},
$$

which implies that firm X produces more output and sets a lower price in the region above the critical line $c = 1 + \theta$ while firm Y produces more output and faces a lower price in the region below.

Substituing (20) and (21) into the profit functions yields BC profits and then their ratio as

$$
\begin{aligned}
\pi_x^{BC} &= \frac{1}{(1+\theta\delta)^2}, \\
\pi_y^{BC} &= \frac{\delta^2(1-\theta^2)}{(\theta+\delta)^2}, \\
\frac{\pi_y^{BC}}{\pi_x^{BC}} &= \delta^2(1-\theta^2)\left(\frac{1+\theta\delta}{\theta+\delta}\right)^2.
\end{aligned}
\tag{22}
$$

Although it is, again, difficult to derive the explicit form of the equi-profit curve, $\pi_y^{BC} = \pi_x^{BC}$, due to the complicated expression of δ, it is possible, with the aid of numerical calculations, to depict the curve that exhibits an inverted U-shaped profile and divides the parametric region into two subregions, as seen in Figure 2(A). There, σ denotes the profit ratio and thus $\sigma = 1$ means the equi-profit curve. It is easy to see that $\pi_y^{BC} < \pi_x^{BC}$ in the upper subregion with $\sigma < 1$, and $\pi_y^{BC} > \pi_x^{BC}$ in the lower subregion with $\sigma > 1$.

To sum up, since the equi-output curve is different from the equi-profit curves, the whole parameter region is finally divided into three subregions labeled as BC_1, BC_2 and BC_3 in Figure 2(B). Production of firm X, x^{BC}, is negative in the shaded region at the lower-right corner in each of Figure 2(A) and (B). That region will be neglected in the further considerations. In each subregion, the orderings between BC variables are as follows,

$$
\begin{aligned}
BC_1 &= \{(\theta, c) \mid x^{BC} < y^{BC},\ p_x^{BC} > p_y^{BC},\ \pi_x^{BC} < \pi_y^{BC}\}, \\
BC_2 &= \{(\theta, c) \mid x^{BC} < y^{BC},\ p_x^{BC} > p_y^{BC},\ \pi_x^{BC} > \pi_y^{BC}\}, \\
BC_3 &= \{(\theta, c) \mid x^{BC} > y^{BC},\ p_x^{BC} < p_y^{BC},\ \pi_x^{BC} > \pi_y^{BC}\}.
\end{aligned}
$$

The same observation as in CB competition applies to the orderings in BC competition. Thus we summarize the results of BC competition as follows:

Proposition 3 *Firm Y (firm X) produces more output and makes more profits in BC_1 (BC_3) whereas firm Y produces more output but is inefficient and less profitable while firm X produces less output but is efficient and more profitable in BC_2.*

***** Figure 2 *****

According to Proposition 1, a quantity-setter makes larger profit than a price-setter in heterogeneous competitions with the same (i.e., homogeneous) production costs. In this section, we consider the optimal behavior of duopolists in heterogeneous competitions but with the different (i.e., heterogeneous) production costs. Propositions 2 and 3 indicates that our results are richer and summarised as follows:

Summary 1 *(i) A quantity-setter produces more output and makes more profits if it is efficient. (ii) A quantity-setter may produce more output and makes more profits even if it is inefficient. (iii) A price-setter may be more profitable only if it is efficient.*

Two remarks are given concerning Summary 1. (i) For consumers of good y, the second result is the worst because they buy smaller amounts of good y at a higher price and contribute to higher profits of firm Y. (ii) The qualitatively same results as the summary are shown by Matsumoto/Onozaki (2005) in a linear duopoly model with heterogeneous costs. Although the heterogenenous costs and nonlinear demand are simultaneously introduced in this section, the results obtained here are due to the heterogeneity in costs.

4.3 Profit Comparison in CB and BC Competitions

In this subsection, our main concerns are on the effect on profits caused by a choice of particular strategy, namely, quantity or price. Thus the following discussion is focused only on profit. In doing so, we compare profits of each firm in CB competition with those in BC competition. Such comparison will allow us to see the impact of strategic behavior on profits in a differentiated market.

From (15) and (22), we obtain the profit ratios of each firm across different competitions as

$$\begin{aligned} \frac{\pi_x^{CB}}{\pi_x^{BC}} &= (1-\theta^2)\left(\frac{1+\theta\delta}{1+\theta\gamma}\right)^2, \\ \frac{\pi_y^{CB}}{\pi_y^{BC}} &= \frac{1}{1-\theta^2}\left(\frac{\gamma}{\delta}\frac{\theta+\delta}{\theta+\gamma}\right)^2. \end{aligned} \tag{23}$$

Since $w_{CB}(z) > w_{BC}(z)$ for all z, and $h(z)$ is decreasing in z, the intersection of $w_{CB}(z)$ with $h(z)$ is always located at left side of the intersection of $w_{BC}(z)$ with $h(z)$, which imply that $\gamma < \delta$ for $z > 0$. As seen in (23), this result alone, however, is not enough to determine profitability of competition. Thus, we use graphical representation to make comparisons in Figure 3, which is drawn again with the aid of numerical calculations. The shaded region located at either the upper-right corner or the lower-right indicates the set of parameters that violates either Assumption 2 or 3. Thus, these two regions are neglected in further considerations.

***** Figure 3 *****

The $\pi_x^{CB} = \pi_x^{BC}$ curve is upward-sloping and the $\pi_y^{CB} = \pi_y^{BC}$ curve is slightly downward-sloping. It can be confirmed that $\pi_x^{CB} < \pi_x^{BC}$ above the former equi-profit curve and $\pi_y^{CB} < \pi_y^{BC}$ above the latter equi-profit curve, and the inequalities are reversed below. The $\pi_x^{CB} = \pi_y^{CB}$ curve and the

$\pi_x^{BC} = \pi_y^{BC}$ curve are also drawn in Figure 3. By these four equi-profit curves, the parameter region is divided into five subregions, which are labelled as I, II, III, IV, and V. Since the vertical axis represents the production cost ratio, the difference in production cost gets larger as the ratio is larger or smaller than unity. For convenience, we say that cost difference is *major* in the regions I and II, *moderate* in the regions III and IV, and *minor* in the region V.

In regions I and II in which the cost difference is major, the order of magnitude of profits are as follows:

$$\begin{array}{ccc} \pi_x^{CB} & < & \pi_x^{BC} \\ \vee & & \vee \\ \pi_y^{CB} & < & \pi_y^{BC} \end{array} \qquad\qquad \begin{array}{ccc} \pi_x^{CB} & > & \pi_x^{BC} \\ \wedge & & \wedge \\ \pi_y^{CB} & > & \pi_y^{BC} \end{array}$$

region I region II

In region I, both firms prefers BC competition in which they can make more profits than in CB competition. Within BC competition, firm X is efficient and more profitable. In region II, we have the reverse results; both firms prefers CB competition in which they can make more profits than in BC competition. Regarding the selection of competition what one firm intends is the same as what the other firm intends. In this sense, both firm can be cooperative. Within CB competition, firm X is efficient and more profitable. Firm X takes a price strategy in BC competition and so does firm Y in CB competition. We summarize these results as

Proposition 4 *If the difference in production cost is major, an efficient firm prefers taking a price strategy and can make more profits in a heterogeneous competition while an inefficient firm prefers taking a quantity strategy and its profit is less than that of the competitor but more than its own profit obtained in the other competition.*

In regions III and IV in which the cost difference is moderate, the ordering of magnitude of profits are as follows:

$$\begin{array}{ccc} \pi_x^{CB} & > & \pi_x^{BC} \\ \vee & & \vee \\ \pi_y^{CB} & < & \pi_y^{BC} \end{array} \qquad\qquad \begin{array}{ccc} \pi_x^{CB} & > & \pi_x^{BC} \\ \wedge & & \wedge \\ \pi_y^{CB} & < & \pi_y^{BC} \end{array}$$

region III region IV

In region III, firm X prefers CB competition in which it makes more profits than in BC competion. On the other hand, firm Y prefers BC competition in which it makes more profits than in CB competition. Both firms prefer being a quantity-setter, which is not allowed in heterogeneous competition.[6] However, no matter which competition is selected, efficient firm X is always more profitable. Thus it can be said that efficiency implies profitability. It is noted that the results are reversed in the region IV. We summarize the results in

Proposition 5 *If the difference in production cost is moderate, an efficient firm prefers taking a quanity strategy and can be profitable in a heterogeneous competition while an inefficient firm also prefers taking a quantity strategy but its profit gets smaller if it plays a price-setter.*

In region V where the cost difference is minor, the order of magnitude of profits are as follows:

$$\begin{array}{ccc} \pi_x^{CB} & > & \pi_x^{BC} \\ \vee & & \wedge \\ \pi_y^{CB} & < & \pi_y^{BC} \end{array}$$

region V

Similarly as in the case with moderate costs, preference on competition of one firm differs from that of the other firm. Indeed, on one hand, firm X prefers CB competition because it can make more profits than in BC competition. On the other hand, firm Y prefers BC competition because it can make more profits than in CB competition. Differently from this, however, an inefficient firm can be profitable in region V. In this sense, efficiency does not necessarily imply profitability. We summarize as follows:

Proposition 6 *If the cost difference is minor, profitability of firm depends on which competition is selected.*

From Propositions 4, 5, and 6, profit comparison between heterogeneous competitions are summarizes as follows:

Summary 2 *Provided the goods are substitutes and production costs are heterogeneous, a price strategy dominates a quantity strategy if cost difference is major, and the strategy dominance is reversed if minor or moderate.*

[6] Here is room for discussing homogeneous competition, but we limit our analysis of this study only to heterogeneous competition.

5 Dynamical Analysis of Heterogeneous Competition

In this section, we carry out an analysis of dynamics resulting from heterogeneous competition, based upon a simple iterative model. In Section 5.1, we reveal the circumstances under which heterogeneous competition exhibits complex nonlinear dynamics, and further, we consider economic implications of nonlinear dynamics from a long-run point of view in Section 5.2.

5.1 Complex Dynamics of CB Competition

Solving the first-order conditions (9) with respect to the decision variables, each firm obtains an explicit form of its reaction function in CB Competition,

$$\begin{aligned} x &= \frac{1}{1-\theta^2}\left(\sqrt{\frac{\theta}{ap_y}} - \frac{\theta}{p_y}\right), \\ p_y &= \sqrt{\frac{b}{\theta x}}. \end{aligned} \tag{24}$$

Although there are some ways to derive dynamical models from the static reaction functions (24), we concentrate on a simple iterative model by taking lag of the variables of the right hand sides,[7]

$$\begin{aligned} x(t+1) &= \frac{1}{1-\theta^2}\left(\sqrt{\frac{\theta}{ap_y(t)}} - \frac{\theta}{p_y(t)}\right), \\ p_y(t+1) &= \sqrt{\frac{b}{\theta x(t)}}. \end{aligned} \tag{25}$$

[7] We are now preparing another paper to study dynamical properties of an adaptive model such that

$$\begin{aligned} x(t+1) &= (1-k_x)x(t) + \frac{k_x}{1-\theta^2}\left(\sqrt{\frac{\theta}{ap_y(t)}} - \frac{\theta}{p_y(t)}\right), \\ p_y(t+1) &= k_y\sqrt{\frac{b}{\theta x(t)}} + (1-k_y)p_y(t), \end{aligned}$$

where k_x and k_y are adjustment coefficients and $0 < (k_x, k_y) \leq 1$. The model exhibits Neimark-Sacker bifurcations when $(k_x, k_y) < 1$. The present model is a special case of the above where $k_x = k_y = 1$.

Let us linearize (25) in the neighborhood of CB equilibrium to study its local stability. The Jacobian matrix evaulated at CB equilibrium is

$$J^{CB} = \begin{pmatrix} 0 & \dfrac{1}{2(1-\theta^2)}\left(\dfrac{\theta}{a}\right)^2 \dfrac{2\theta^2+\theta\gamma-1}{c(1+\theta\gamma)^2} \\ -\dfrac{1}{2}\left(\dfrac{a}{\theta}\right)^2 c(1+\theta\gamma)^2 & 0 \end{pmatrix}. \quad (26)$$

The eigenvalues of (26) are the solutions of the characteristic equation

$$\lambda^2 - \mathrm{tr} J^{CB}\lambda + \det J^{CB} = 0$$

where

$$\begin{aligned} \mathrm{tr} J^{CB} &= 0, \\ \det J^{CB} &= \frac{2\theta^2+\theta\gamma-1}{4(1-\theta^2)}. \end{aligned} \quad (27)$$

It is well-known that the equilibrium point is asymptotically stable if and only if both eigenvalues have modulus smaller than unity. It is also well-known that the characteristic equation has eigenvalues less than unity in absolute value if and only if

$$\pm\mathrm{tr} J^{CB} + \det J^{CB} + 1 > 0 \quad \text{and} \quad 1 - \det J^{CB} > 0. \quad (28)$$

Thus these three conditions define the parameter domain of asymptotic stability of CB equilibrium. To examine the stability, we substitute (27) into the left-hand sides of the first two inequalities in (28) to have

$$1 + \frac{2\theta^2+\theta\gamma-1}{4(1-\theta^2)} > 0,$$

which always holds for positive a, b, and θ. Regarding the last inequiality, the $\det J^{CB} = 1$ boundary is given by

$$6\theta^2 + \theta\gamma - 5 = 0.$$

On the boundary, the eigenvalues are pure imaginary and unity in absolute value, which means $\lambda_{1,2} = e^{\pm i2\pi\varepsilon}$, $0 \le \varepsilon \le 1$. Therefore,

$$tr J^{CB} = \lambda_1 + \lambda_2 = 2\cos 2\pi\varepsilon.$$

On the other hand, $\mathrm{tr} J^{CB} = 0$. Thus $2\cos 2\pi\varepsilon = 0$ from which we derive $\varepsilon = 0.25$. This implies the emergence of a four-period cycle on this bounday, as shown by Sonis (2000). As a set of parameters crosses the boundary, the fixed point is replaced by a four-period cycle, and then proceeds to chaos through a period-doubling way.

We need to prevent a solution generated by the model from being negative. Let us denote the reaction functions of firm X and Y (i.e., the first and the second equation of (24)) by $r_x(p_y)$ and $r_y(x)$, respectively, and let us also denote a maximizer and maximum of $r_x(p_y)$ by p_y^m and x^m. Then we obtain

$$p_y^m = 4a\theta \quad \text{and} \quad x^m = \frac{1}{4a(1-\theta^2)}.$$

If $r_x(0) \geq r_y(x^m)$ holds, then a solution of (25) is non-negative for any $t \geq 1$. Solving the last inequality condition gives the following non-negative condition for a time-path,

$$c \geq \frac{\theta^3}{4(1-\theta^2)}.$$

Since we assume Assumption 2, the feasible domain of *instability* is given by the following inequalities:

$$6\theta^2 + \theta\gamma(\theta, c) - 5 > 0, \quad c > \frac{\theta^3}{4(1-\theta^2)}, \quad \text{and} \quad c < \frac{1}{\theta^3},$$

which is a strip-wise region denoted by [U] in Figure 4(A). It can be seen that crossing the boundary into this region allows for a period doubling cascade to chaos. Each color in Figure 4 corresponds to period's number of cycles as displayed in the table below the figure; the red area labelled as [S] exhibits a set of (θ, c) for which trajectories converge to a unique stable fixed point. The yellow area corresponds to a period-4 cycle, the light blue area to a period-8 cycle, the light red area to a cycle of period-16 and so forth. The black area corresponds to a cycle of period-17 up to -64 and the white area corresponds to a cycle of period-over-64 or an aperiodic (including chaotic) orbit.[8] CB output is negative in the light-gray triangle-wise region at the upper-left, and a dynamical solution becomes negative in the dark-gray distorted rectangle region. An enlargement of the unstable region is shown in Figure 4(B), which facilitate taking a view of the aperiodic region. It should be noted that, due to the symmetry, similar figures are obtained concerning dynamics in BC competition.

***** Figure 4 *****

[8]For practical purposes, it is enough to check up to period-64 cycle in order to detect aperiodic area.

5.2 Long-Run Aspects of Complex Dynamics

It is now well-known that chaotic dynamics has the sensitive dependence on initial conditions. A consequence of the dependence is that even a slightly different choice of initial conditions can drastically alter the whole future evolution of the trajectories. Therefore it is meaningless to investigate individual chaotic trajectories. One way to characterize such chaotic dynamics is to turn our attention to statistical or long-run behavior of trajectories. Under rather weak mathematical conditions, frequencies of a chaotic trajectory may converge to a stable density function. Once the explicit form of the density function is constructed, it can be shown that for any continuous function of a variable, its time average calculated along a chaotic trajectory is equal to its space average.[9] This implies that it is possible to analytically calculate the long-run average behavior that is independent from a choice of initial conditions. However, it is, in general, difficult to construct an explicit form of the density function, even if its existence is mathematically confirmed. We thus numerically calculate the long-run average behavior over sufficiently a long period of time as a proxy for the analytical value of the long-run behavior.

Given the simple iterative dynamical sytem, the average profit of a time path with T-period is defined by

$$\bar{\pi}_i = \frac{1}{T}\sum_{t=0}^{T-1} \pi(i(0), p_j(0)) \qquad i, j = x, y, \quad i \neq j.$$

We perform numerical simulations of CB competition and then BC competition. Figure 5 illustrates the results of the numerical simulations in which the degree of product differentiation is measured on the horizontal axis and the long-run average profits of firm X and Y are on the horizontal axis. For comparison, graphs of the CB equilibrium profits of both firms are also depicted as slightly downward sloping dashed curves. In each simulation, the average is calculated from the last 1000 out of 3000 iterations, and θ is increased in steps of 0.0025 (= 1/400) from $\theta_s - k$ to θ_e. Here k is an arbitrarily

[9]Let $x_{t+1} = \phi(x_t)$ be a dynamical process where x_t is a variable at time t. Suppose the frequencies of the trajectory $\{x_t\}_{t=0}^{\infty}$ converge to a density function, φ. Then for any continuous function f and for any initial point x_0,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} f(\phi^t(x_0)) = \int f(x)\varphi(x)ds$$

where $\phi^t = \phi^{t-1} \cdot \phi$ and $\phi^0 = 1$. See, for example, Chapter 8 of Day (1994) for more details.

small number ($k = 0.005$ in the simulations), θ_s is the value for which loss of stability occurs, and θ_e is the value for which a trajectory becomes zero.[10]

It can be observed in Figure 5 that the average profit of each firm is identical to its CB profit when θ is less than θ_s, that is, when the dynamical system is stable. In Figure 5(A), the production cost ratio is chosen to be unity for which firm X is more profitable than firm Y. It can be seen that along a unstable time-path, the average profit of firm X is larger than its CB profit while the average profit of firm Y is not only less than its CB profit but negative for a little larger value than θ_s. In Figure 5(B), the production cost ratio is chosen to be 0.5, which means that firm X is inefficient and firm Y is more efficient. In consequence, CB profit of firm$Y$ is larger than that of firm X. However, as seen in the figure, the average profit of firm Y gets decreasing very rapidly and sooner or later becomes negative as θ increases from θ_s. On the other hand, the average profit of firm X increases in θ and becomes larger than CB profit of firm X. Due to the symmetry between CB competition and BC competition, the numerical results obtained here are reversed in BC competition. Although we do not represent simulation results in BC competition, we have exactly the same illustrations of long-run average profits as shown in Figure 5(A) and (B) if we put $\tilde{c} = 1$ and $\tilde{c} = 2$ and replace x and y by y and x, and CB by BC. These numerical simulations indicate the following result:

Proposition 7 *When an equilibrium is unstable in heterogenous competitions, a quantity-setting firm is more profitable while the long-run average profit of a price-setting firm is negative soon after unstability occurs.*

***** Figure 5 *****

It can be explained why the long-run average profit of firm Y becomes negative. By definition, the profit of firm Y taken along a time path is positive if

$$p_y(t) > b \quad \text{and} \quad y(t) > 0.$$

[10]Given $c = \bar{c}$, θ_s and θ_e are defined by

$$6\theta_s + \theta_s\gamma(\theta_s, \bar{c}) - 5 = 0 \quad \text{and} \quad \bar{c} = \frac{\theta_e^3}{4(1-\theta_e^2)}.$$

Solving the inverse demand of good y for y and substituting the first equation of the dynamical system yields, after arrangements,

$$y(t) = \frac{\theta}{1-\theta^2}\left(\frac{1}{\theta p_y(t)} - \sqrt{\frac{\theta}{ap_y(t)}}\right),$$

which implies the following condition for a positive $y(t)$:

$$p_y(t) < \frac{a}{\theta^3}.$$

Thus as far as the time path of p_y stays inside of the interval $(b, \frac{a}{\theta^3})$, the profit of firm Y is positive. However, as can be seen in Figure 6 in which a part of the return map is illustrated, a chaotic path of p_y is either larger than $\frac{a}{\theta^3}$ or less than b.[11] In consequence, the profit taken along the chaotic path is negative except in a neighborhood of the starting point. On the other hand, since it can be verified that $\pi_x(t) = (p_x(t) - a)x(t) > 0$ if $p_y(0) > a\theta$ and $x(0) > 0$, the long-run average profit of firm X is positive along a chaotic path.

***** Figure 6 *****

A natural question that arises is what firm Y should do to cope with this unfavorable situation. There are several possible answers: the first is that a price-setting firm exits the unstable market because it makes losses. Since its long-run average profit is much larger than an equilibrium profit, a quantity-setting firm is in favor of the unstable market and has a strong reason to remain in the market. In this case, a duopoly market turns out to be a monopoly market as a result of unstable heterogeneous competition.[12] The second is to get rid of the assumption of full information and to reconsider dynamics under boundedly rational circumstances in which the firms

[11] In this simulation, we set $a = b = 2$, $\theta = 0.9027$, $x(0) = 0.35$ and $p_y(0) = 2.52$. 750 iterations are performed and the trajectory obtained after 500 iterations is depicted. The upper side of the return map is eliminated only for convenience.

[12] It is well-known that the isoelastic demand is problematic in a monopoly case. To remedy this problem, we can consider the following inverse demand function after firm Y leaves the market,

$$p = \frac{1}{x+\varepsilon}$$

where ε is a positive autonomous demand. When the monopoly firm is required to produce more than the autonomous demands, its profit is $\pi_x = (1-\sqrt{a\varepsilon})^2 > 0$.

have only misspecified knowledge of the demand function. In fact, this line of research has already launched. One interesting approach is proposed by Bischi et al. (2004) in which a duopoly game with production differentiation is considered in CB competition. Under the assumption of limited information such that the firms do not know the demand function of the market, a profit maximization problem is solved with the aid of, what they call a *local and monopolistic approximation* of the demand function. The result is that dynamics based on this approach always converges to an equilibrium. The third is to stabilize chaotic fluctuations through public effort such as government intervention to the market or individual effort such as changing the formation of expectations. See Matsumoto (2006) for controlling chaos in a nonlinear duopoly model. The last is to change a strategy from price to quantity, which implies to change a type of competition from heterogeneity to homegeneity.

6 Concluding Remarks

In this study, we have two main aims, the first of which is to shed light on the roles of production cost difference on the optimal behavior in a differentiated duopoly model, and the second of which is to shed light on the roles of nonlinearity on unstable dynamics possibly generated in the model.

Within a heterogeneous competition, heterogeneities in production cost do matter in the sense that a price strategy can dominate a quantity strategy (roughly speaking) if a price setter is efficient with regard to production cost. This is not observed in the traditional differentiated duopoly model in which homogeneous cost assumption is adapted. Second, it is shown that a heterogeneous competition is a natural consequence of the optimal behavior of each firm if the cost difference is large. This result is new and is obtainable only when heterogeneous costs are taken into account.

It have been demonstrated that a homogeneous competition can generate complex dynamic involving chaos in duopolistic as well as oligopolistic circumstance in which demand is nonlinear. Turning our attentions to dynamics of a nonlinear duopoly with product differentiation in a heterogeneous competition, we have shown that heterogeneous competition may follow complex nonlinear dynamics. In consideration of the result, we have constructed a feasible domain of unstable equilibrium, which provides parametric configurations to give rise to chaotic dynamics. We have also demonstrated that even if a price-setter can make more profits at equilibrium than a quantity setter, its long-run average profit taken along the unstable but bounded time path becomes not only less than the equilibrium profit but also negative.

We have focused on the heterogeneous case in which firms are not allowed to take the same strategy. It is possible to extend our results in the general case that involves homogeneous as well as heterogeneous competition.

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Captions of Figures

Figure 1 Regime classification of CB competitions

(A)
(B)

Figure 2 Regime classification of BC competitions

(A) Profit Contour Curves
(B) Division of Parameter Region

Figure 3 Profitability classification in parameter region

Figure 4 Division of the parameter region in CB competition when $k_x = k_y = 1$

(A) Division of the parameter region
(B) Enlargement of (A)

Figure 5 Long-run average profits in CB competition

(A) $\bar{c} = 1$ and $\pi_x^{CB} > \pi_y^{CB}$
(B) $\bar{c} = 0.5$ and $\pi_x^{CB} < \pi_y^{CB}$

Figure 6 Return map of the simple iterative dynamical model

Parameters and initial condition are as follows: $a = b = 2$, $\theta = 0.9027$, $x(0) = 0.35$, $p_y(0) = 2.52$. The upper side of the return map is eliminated.