

# Learning in Dynamic Oligopolies with Time Delay

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## Abstract

An  $N$ -firm oligopoly is considered where the firms are familiar with the technology of all competitors, so the cost functions are public information. However they do not have full knowledge of the linear price function, which is determined by two parameters; the maximum price and the market saturation point. It is assumed that they have full information on only one of these parameters and try to estimate and learn the value of the other. At each time period based on its current estimate of the price function each firm is able to compute its believed

equilibrium and the corresponding believed equilibrium price. Each firm then sends its own believed equilibrium output amount to the market, which is then produces an actual market price. The comparison of the actual market price and the believed equilibrium price is the basis of an adaptive adjustment process of the uncertain parameter. Two alternative learning schemes are introduced which are asymptotically stable if instantaneous market price information is available. That is, as time goes to infinity, the estimates of the price functions converge to the true function. The effect of information delay in the observed market prices is examined in both learning schemes. It is shown that asymptotical stability is preserved if the delay is below a certain threshold, and unstable otherwise. The stability region shrinks if either the number of firms increases or the firms select larger speeds of adjustments. It is also shown that at the stability switch Hopf bifurcation occurs giving the possibility of the birth of limit cycles.

## 1 Introduction

Dynamic systems theory is one of the most important research fields in quantitative sciences including engineering and mathematical economics. There is a large number of issues being examined in the literature. The asymptotic behavior of the state trajectories has been discussed in many textbooks and monographs. Different stability conditions have been derived and applied in particular models. For example, Szidarovszky and Bahill (1998) presents several applications in engineering and also in the social sciences.

In the literature of mathematical economics the different variants of oligopoly models play a significant role. This research area is based on the pioneering work of Cournot (1838). The classical Cournot model considers an industry where several firms produce the same item or offer the same service to a homogeneous market. This economic situation can be modeled as a non-cooperative game in which the players are the firms and the payoffs are the profits. A comprehensive summary of the earlier results is given in Okuguchi (1976) and their multiproduct extensions with several model variants and applications are presented in Okuguchi and Szidarovszky (1999). More recently an increasing attention has been given to nonlinear models and global dynamics. Recent developments in nonlinear oligopolies are reported for example in Bischi et al. (2010). In most dynamic economic models including oligopolies complete and instantaneous information is assumed when the

participants make their decisions. In real economies however there is always uncertainty in economic quantities and functional relations. For example, it is not realistic that in dynamic oligopolies the firms know the market demand and price functions. The effect of misspecified price function on the stability of the equilibrium is examined in Leonard and Nishimura (1999), Bischi et al. (2004) and Chiarella and Szidarovszky (2001) among others. However by repeated price informations the firms are able to learn and continuously update their beliefs of the price function. Fudenberg and Levine (1998) offer a general theory of learning in games, and special learning processes in dynamic oligopolies are discussed in Szidarovszky and Krawczyk (2004). These models are further extended in Bischi et al. (2010). The assumption of the availability of instantaneous information on prices and outputs of the competitors is unrealistic in real economies. The introduction of delayed information in dynamic oligopolies and in learning processes makes the dynamic properties of the models much more complicated. Information lags can be examined either as fixed or continuously distributed delays. In the case of fixed delays the dynamics is described by delay differential equations, the characteristic equations of which are mixed polynomial - exponential equations with infinite spectra. For example, Burger (1956), Cooke and Grossman (1982) and Bellman and Cooke (1956) offer complete stability analysis in many important cases with single delays. If continuously distributed delays are assumed, then Volterra-type integro-differential equations describe the dynamics. Cushing (1977) offers the theory of such models with applications to population dynamics.

In this paper the learning process discussed in Bischi et al. (2010) will be revisited with the additional assumption that only delayed price information is available to the firms. Fixed delays are assumed, the continuously distributed counterparts can be investigated in the same way as it is shown in Chiarella and Szidarovszky (2004).

This paper develops as follows. After the mathematical models and the learning process are revisited, fixed information delays are introduced into the price observations of the firms. Then stability conditions will be derived, and the occurrence of Hopf bifurcation is proved. The last section offers conclusion and future research directions.

## 2 The Mathematical Models of Learning

An  $N$ -firm oligopoly is considered in which the firms offer the same product to a homogeneous market. Let  $p(s) = B - As$  be the unit price, where  $B$  is the maximum price and  $A$  the marginal price,  $s$  is the total output of the industry. If  $x_k$  is the output of firm  $k$ , then  $s = \sum_{k=1}^N x_k$  is the total output of the industry. Let  $C_k(x_k) = c_k x_k + d_k$  be the cost function of firm  $k$ , where  $d_k$  is the fixed cost and  $c_k$  is the marginal cost, then its profit is given as

$$\varphi_k = x_k(B - As) - (c_k x_k + d_k). \quad (1)$$

The uncertainty in the price function can be modeled in several ways. In this paper we consider the following two cases:

1. Each firm knows the marginal price  $A$  but has its own estimate of the maximum price, which is denoted by  $B_k(t)$  at time period  $t$ ;
2. Each firm  $k$  knows the market saturation point  $B/A$  but has its own estimate of the slope,  $A_k(t)$ .

We will now revisit the corresponding learning processes and introduce fixed delays into the price observations.

1. At each time period  $t$  each firm  $k$  believes that the profit of any firm  $l$  (including itself) is

$$\bar{\varphi}_l = x_l(B_k(t) - As) - (c_l x_l + d_l), \quad (2)$$

so firm  $k$  believes that the best response of firm  $l$  is

$$x_l = \frac{B_k(t) - As - c_l}{A}, \quad (3)$$

and believes that the equilibrium is obtained by adding these equations for all firms:

$$s = \frac{1}{A}(NB_k(t) - NAs - \sum_{i=1}^N c_i).$$

Therefore firm  $k$  believes that at the equilibrium

$$\bar{s} = \frac{NB_k(t) - \sum_{i=1}^N c_i}{A(N+1)} \quad (4)$$

and the equilibrium price is

$$\bar{p}_k = B_k(t) - As = \frac{B_k(t) + \sum_i c_i}{N+1}. \quad (5)$$

The corresponding equilibrium output of firm  $k$  is believed as

$$\bar{x}_k = \frac{B_k(t) - c_k - A\bar{s}}{A} = \frac{B_k(t) + \sum_{i=1}^N c_i - (N+1)c_k}{A(N+1)}. \quad (6)$$

Each firm thinks in the same way as shown above, so the total output of the industry becomes

$$s = \sum_{k=1}^N \bar{x}_k = \frac{1}{A(N+1)} \left( \sum_{k=1}^N B_k(t) - \sum_{k=1}^N c_k \right) \quad (7)$$

with actual market price

$$p = B - As = B - \frac{1}{N+1} \left( \sum_{k=1}^N B_k(t) - \sum_{k=1}^N c_k \right).$$

If the price information is delayed, then the firms believe that the actual price is

$$p^* = B - \frac{1}{N+1} \left( \sum_{k=1}^N B_k(t - \tau) - \sum_{k=1}^N c_k \right), \quad (8)$$

where  $\tau$  denotes the fixed delay.

Comparing the believed price  $\bar{p}_k$  and the actual price  $p^*$  each firm adjusts its belief on the maximum price as

$$\dot{B}_k(t) = K_k (p^* - \bar{p}_k), \quad (9)$$

where  $K_k$  is the speed of adjustments of the firm. This process can be explained as follows. If the actual price is higher than the believed price, then the firm wants to increase its estimate of the price function by increasing the value of  $B_k(t)$ . If the actual price is smaller, then the firm wants to decrease its estimate  $B_k(t)$ , and if the two prices are equal, then the firm wants to keep its estimate of the maximum price. By introducing the notations  $\Delta_k(t) = B_k(t) - B$  and  $\alpha_k = \frac{K_k}{N+1}$  equation (9) can be rewritten as

$$\dot{\Delta}_k(t) + \alpha_k \Delta_k(t) + \alpha_k \sum_{i=1}^N \Delta_i(t - \tau) = 0 \quad (1 \leq k \leq N), \quad (10)$$

which is a system of delayed ordinary differential equations. The only steady state of this system is  $\bar{\Delta}_k = 0$  for all  $k$ , that is,  $\bar{B}_k = B$ , the full knowledge of the price function.

2. In this case firm  $k$  believes that each firm  $l$  has the profit function

$$\bar{\varphi}_l = x_l A_k(t) \left( \frac{B}{A} - s \right) - (c_l x_l + d_l) \quad (11)$$

where only the ratio  $\frac{B}{A}$  is known without knowing the individual values of  $A$  and  $B$ . The believed best response of firm  $l$  is

$$x_l = \frac{B}{A} - s - \frac{c_l}{A_k(t)}, \quad (12)$$

so the believed industry output is the solution of equation

$$s = \frac{NB}{A} - Ns - \frac{1}{A_k(t)} \sum_{l=1}^N c_l$$

which is

$$\bar{s} = \frac{1}{N+1} \left( \frac{NB}{A} - \frac{1}{A_k(t)} \sum_{l=1}^N c_l \right). \quad (13)$$

So the believed equilibrium price is

$$\bar{p}_k = A_k(t) \left( \frac{B}{A} - \bar{s} \right) = \frac{1}{N+1} \left( \frac{B}{A} A_k(t) + \sum_{l=1}^N c_l \right) \quad (14)$$

and firm  $k$  produces the amount

$$\bar{x}_k = \frac{B}{A} - \bar{s} - \frac{c_k}{A_k(t)} = \frac{B}{A(N+1)} - \frac{c_k}{A_k(t)} + \frac{1}{A_k(t)(N+1)} \sum_{l=1}^N c_l. \quad (15)$$

Then the actual industry output becomes

$$s = \sum_{k=1}^N \bar{x}_k = \frac{NB}{A(N+1)} - \sum_{k=1}^N \frac{c_k}{A_k(t)} + \frac{\sum_{l=1}^N c_l}{N+1} \sum_{k=1}^N \frac{1}{A_k(t)} \quad (16)$$

with the corresponding market price

$$p = B - As = \frac{B}{N+1} + A \sum_{k=1}^N \frac{c_k}{A_k(t)} - \frac{A}{N+1} \left( \sum_{l=1}^N c_l \right) \left( \sum_{k=1}^N \frac{1}{A_k(t)} \right).$$

If the price has a delay  $\tau > 0$ , then the firms actually observe the price

$$p^* = \frac{B}{N+1} + A \sum_{k=1}^N \frac{c_k}{A_k(t-\tau)} - \frac{A}{N+1} \left( \sum_{l=1}^N c_l \right) \left( \sum_{k=1}^N \frac{1}{A_k(t-\tau)} \right). \quad (17)$$

The comparison of the believed price  $\bar{p}_k$  and actually observed price  $p^*$  leads to the adjustment process

$$\dot{A}_k(t) = K_k(p^* - \bar{p}_k) \quad (18)$$

where  $K_k$  is the speed of adjustments of firm  $k$ . This is a system of nonlinear delayed differential equations

$$\begin{aligned} \dot{A}_k(t) = K_k \left( \frac{B}{N+1} + A \sum_{l=1}^N \frac{c_l}{A_l(t-\tau)} - \frac{A}{N+1} \left( \sum_{l=1}^N c_l \right) \left( \sum_{l=1}^N \frac{1}{A_l(t-\tau)} \right) \right. \\ \left. - \frac{1}{N+1} \left( \frac{B}{A} A_k(t) + \sum_{l=1}^N c_l \right) \right). \quad (19) \end{aligned}$$

Clearly at the steady state the  $A_k$  values are equal, and the common value,  $\bar{A}$ , satisfies equation

$$\begin{aligned} 0 &= \frac{B}{N+1} + \frac{A}{\bar{A}} \sum_{l=1}^N c_l - \frac{A}{N+1} \left( \sum_{l=1}^N c_l \right) \frac{N}{\bar{A}} - \frac{1}{N+1} \left( \frac{B\bar{A}}{A} + \sum_{l=1}^N c_l \right) \\ &= \frac{B}{N+1} \left( 1 - \frac{\bar{A}}{A} \right) + \left( \sum_{l=1}^N c_l \right) \frac{1}{N+1} \left( -1 + \frac{A}{\bar{A}} \right) \end{aligned}$$

which can occur only if  $\bar{A} = A$ , which corresponds to the full knowledge of the price function. We can examine the asymptotical behavior of system (19) by linearizing it around the steady state. Simple differentiation shows that

$$\frac{\partial}{\partial A_k(t)} (p^* - \bar{p}_k) = -\frac{B}{A(N+1)}, \quad \frac{\partial}{\partial A_l(t)} (p^* - \bar{p}_k) = 0 \quad (l \neq k)$$

$$\begin{aligned} \frac{\partial}{\partial A_k(t-\tau)} (p^* - p_k) &= \frac{A}{A_k^2(t-\tau)} \left( -c_k + \frac{1}{N+1} \sum_{l=1}^N c_l \right) \\ &= \frac{1}{A} \left( -c_k + \frac{1}{N+1} \sum_{l=1}^N c_l \right) \end{aligned}$$

and similarly

$$\frac{\partial}{\partial A_l(t - \tau)}(p^* - p_k) = \frac{1}{A} \left( -c_l + \frac{1}{N + 1} \sum_{i=1}^N c_i \right)$$

at the steady state. Therefore the linearized equation (19) can be written as

$$\dot{\Delta}_k(t) = K_k \left( -\frac{B}{A(N + 1)} \Delta_k(t) - \sum_{l=1}^N \frac{\gamma_l}{A} \Delta_l(t - \tau) \right) \quad (20)$$

with

$$\gamma_l = c_l - \frac{1}{N + 1} \sum_{i=1}^N c_i, \quad \Delta_l(t) = A_l(t) - A$$

for all  $l$ . By letting

$$\alpha_k = \frac{K_k B}{A(N + 1)} \quad \beta_{kl} = \frac{K_k \gamma_l}{A}$$

system (20) can be written as

$$\dot{\Delta}_k(t) + \alpha_k \Delta_k(t) + \sum_{l=1}^N \beta_{kl} \Delta_l(t - \tau) = 0. \quad (21)$$

### 3 Stability Analysis

For the sake of simplicity the symmetric case is considered, when  $c_k \equiv c$ ,  $K_k \equiv K$  for all  $k$ . In this special case equation (10) is simplified as

$$\dot{\Delta}_k(t) + \alpha \Delta_k(t) + \alpha \sum_{i=1}^N \Delta_i(t - \tau) = 0$$

with  $\alpha = K/(N + 1)$ . Therefore with identical initial values  $\Delta_i(-t)$  ( $0 \leq t \leq \tau$ ) the trajectories are also identical and equal to the solution of the one-dimensional equation

$$\dot{\Delta}(t) + \alpha \Delta(t) + \alpha N \Delta(t - \tau) = 0. \quad (22)$$



In the symmetric case equation (21) also specializes to the following:

$$\dot{\Delta}(t) + \alpha\Delta(t) + \beta N\Delta(t - \tau) = 0 \quad (23)$$

with  $\alpha = KB/(A(N + 1))$  and  $\beta = Kc/(A(N + 1))$ . Since (22) is a special case of (23) with  $\beta = \alpha$ , we will first concentrate on equation (23).

In order to guarantee the non-negativity of the equilibrium outputs at the steady we have to assume that  $B \geq c$ , that is,  $\alpha \geq \beta$ . Let's look for the solution in the usual exponential form  $\Delta(t) = e^{\lambda t}u$ , and by substituting it into (23) the characteristic equation of the system is obtained:

$$\lambda + \alpha + \beta N e^{-\lambda\tau} = 0. \quad (24)$$

By introducing the new variables  $\Lambda = \lambda\tau$ ,  $C = \alpha\tau$  and  $D = \beta\tau$ , we have

$$\Lambda + C + D N e^{-\Lambda} = 0 \quad (25)$$

where  $C \geq D$ . Notice first that without delay  $\tau = 0$  and equation (24) becomes

$$\lambda + \alpha + \beta N = 0,$$

so the only eigenvalue is negative. Assume that  $\tau > 0$  and let  $\Lambda = q + ir$  be a complex solution. We can assume that  $r > 0$ , since if  $\Lambda$  is a solution, then its conjugate is also a solution. Then

$$q + ir + C + D N e^{-q}(\cos r - i \sin r) = 0.$$

Separating the real and imaginary parts we have

$$q + C + D N e^{-q} \cos r = 0 \quad (26)$$

and

$$r - D N e^{-q} \sin r = 0. \quad (27)$$

If  $\sin r = 0$ , then  $r = 0$ , so from (26)

$$q = -C - D N e^{-q}$$

showing that  $q < 0$ . If  $\sin r \neq 0$ , then

$$e^{-q} = \frac{r}{D N \sin r}$$

and substituting it into (26) gives relation

$$q + C + r \cot r = 0$$

so

$$q = -C - r \cot r. \quad (28)$$

If we substitute this equation into (27), a single variable equation is obtained for  $r$ :

$$r - DNe^{C+r \cot r} \sin r = 0$$

or

$$\frac{r}{DNe^C} = e^{r \cot r} \sin r. \quad (29)$$

The steady state is asymptotically stable if  $q < 0$ , which occurs if

$$r \cot r > -C. \quad (30)$$

Let  $f(r)$  denote the right hand side of (29) and let  $g(r) = r \cot r$ . Simple differentiation shows that

$$g'(r) = \left( \frac{r \cos r}{\sin r} \right)' = \frac{\sin 2r - 2r}{2 \sin^2 r} < 0$$

for all  $r > 0$  implying that  $g(r)$  strictly decreases in  $r$ . Furthermore

$$\lim_{r \rightarrow 0} \frac{r \cos r}{\sin r} = 1,$$

$$\lim_{r \rightarrow \pi/2 + k\pi} \frac{r \cos r}{\sin r} = 0$$

and

$$\lim_{r \rightarrow k\pi} \frac{r \cos r}{\sin r} = \begin{cases} -\infty & \text{from the left hand side} \\ +\infty & \text{from the right hand side.} \end{cases}$$

The graph of  $g(r)$  is shown in Figure 1.

Notice that in each subinterval  $(\frac{\pi}{2} + k\pi, (k+1)\pi)$  there is a unique intercept  $r_k$  of the graph of  $g(r)$  and the horizontal line of  $-C$ , and (30) implies that the real part of the solution is negative if  $k\pi < r < r_k$ ,  $k = 0, 1, 2, \dots$ . It is also easy to see that

$$\lim_{r \rightarrow 0} f(r) = 0$$

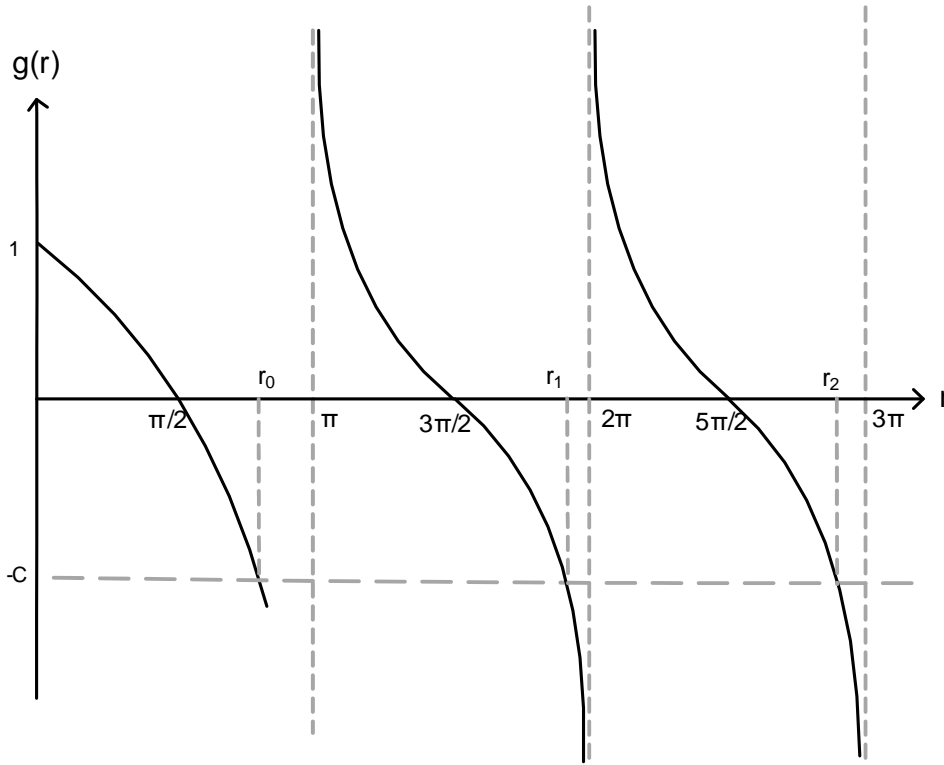


Figure 1: Graph of  $g(r)$

and for all  $k \geq 1$ ,

$$\lim_{r \rightarrow k\pi - 0} f(r) = 0,$$

since  $\cot r \rightarrow -\infty$  as  $r \rightarrow k\pi - 0$ .

Furthermore

$$\lim_{r \rightarrow k\pi + 0} f(r) = \begin{cases} \infty & \text{if } k \text{ is even} \\ -\infty & \text{if } k \text{ is odd} \end{cases}$$

and

$$f\left(\frac{\pi}{2} + k\pi\right) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd.} \end{cases}$$

By differentiation

$$f'(r) = \frac{1}{\sin r} e^{r \cot r} (\sin 2r - r). \quad (31)$$

There is a unique  $r^* \in (0, \frac{\pi}{2})$  such that  $\sin 2r^* = r^*$ , and for  $r < r^*$  we have  $\sin 2r > r$ , and for  $r > r^*$ , clearly  $\sin 2r < r$ . Therefore  $f'(r) > 0$  if and only if  $r \in (0, r^*)$  or  $r \in ((2k-1)\pi, 2k\pi)$  for  $k \geq 1$ . Similarly  $f'(r) < 0$  if and only if  $r \in (r^*, \pi)$  or  $r \in (2k\pi, (2k+1)\pi)$  for  $k \geq 1$ . The graph of  $f(r)$  is shown in Figure 2.

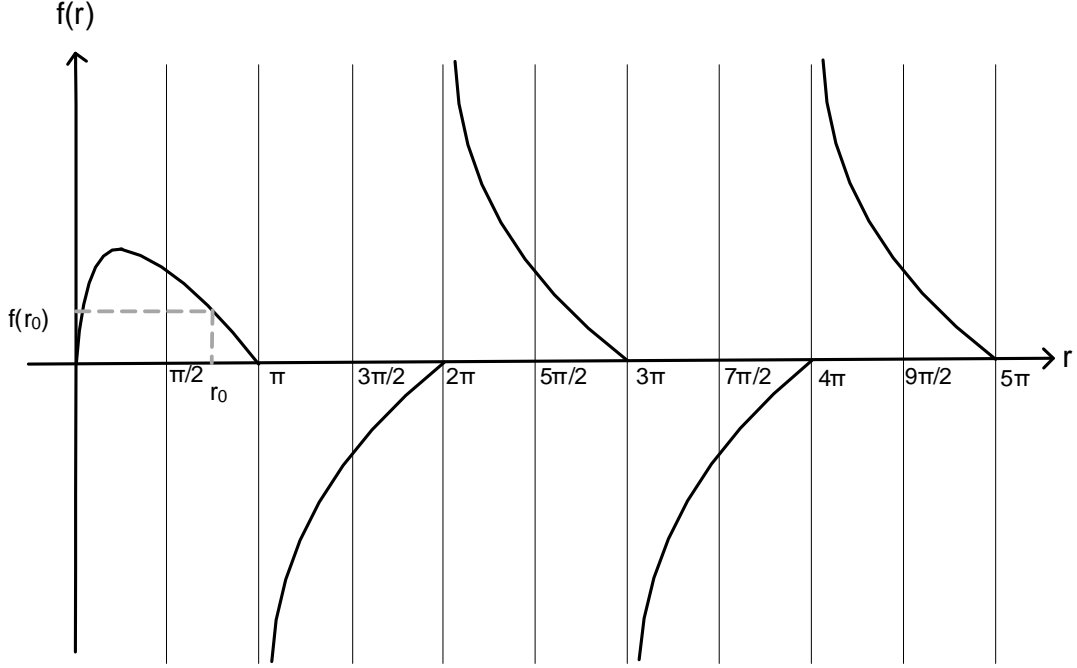


Figure 2: Graph of  $f(r)$

The solution of equation (29) is the intercept of this graph with the linear function  $r/(DNe^C)$ . In order to satisfy relation (30) the solution has to be in the union of intervals  $[0, r_0)$ ,  $[2\pi, r_2]$ ,  $[4\pi, r_4], \dots$ , which is the case when

$$\frac{1}{DNe^C} > \frac{f(r_k)}{r_k} \quad (k = 0, 2, 4, \dots).$$

This can be rewritten as

$$\begin{aligned} 1 &> \frac{e^{r_k \cot r_k} \sin r_k DNe^C}{r_k} = \frac{e^{-C} e^C DN \cos r_k}{r_k \cot r_k} \\ &= -\frac{ND}{C} \cos r_k. \end{aligned} \quad (32)$$

Notice that  $r_0 < r_2 < r_4 < \dots$  and  $r_k \cot r_k = -C$  implying that  $\cos r_0 < \cos r_2 < \cos r_4 < \dots < 0$ , so if (32) holds with  $k = 0$ , then it holds for all even values of  $k$ . We can therefore conclude that the steady state is asymptotically stable if

$$1 > -\frac{ND}{C} \cos r_0 \quad (33)$$

and unstable if

$$1 < -\frac{ND}{C} \cos r_0.$$

If  $C > ND$ , then (33) holds implying that the system is asymptotically stable. We will assume that  $C \leq ND$  in the following discussion.

Clearly  $r_0$  is a strictly increasing function of  $C$ ,  $r_0 = r_0(C)$ , so (33) holds if

$$r_0(C) < \cos^{-1}\left(-\frac{C}{ND}\right)$$

or

$$g(r_0(C)) > g\left(\cos^{-1}\left(-\frac{C}{ND}\right)\right)$$

which can be rewritten as

$$\begin{aligned} -C &> \cos^{-1}\left(-\frac{C}{ND}\right) \cot\left(\cos^{-1}\left(-\frac{C}{ND}\right)\right) \\ &= \cos^{-1}\left(-\frac{C}{ND}\right) \frac{-\frac{C}{ND}}{\sqrt{1 - \frac{C^2}{N^2D^2}}} \end{aligned}$$

or

$$C < \cos^{-1}\left(-\frac{C}{ND}\right) \frac{C}{\sqrt{D^2N^2 - C^2}}. \quad (34)$$

In the case of system (22),  $\alpha = \beta = K/(N+1)$ ,  $C = D = K\tau/(N+1)$ , so this stability condition can be written as

$$\frac{K\tau}{N+1} < \cos^{-1}\left(-\frac{1}{N}\right) \frac{1}{\sqrt{N^2 - 1}}$$

or

$$\tau < \frac{\cos^{-1}\left(-\frac{1}{N}\right)(N+1)}{K\sqrt{N^2 - 1}} = \frac{\cos^{-1}\left(-\frac{1}{N}\right)\sqrt{\frac{N+1}{N-1}}}{K}. \quad (35)$$

Hence we have the following result.

**Theorem 1** *The steady state of system (22) is globally asymptotically stable if  $\tau < \tau^*$  and unstable if  $\tau > \tau^*$  where*

$$\tau^* = \frac{\cos^{-1}\left(-\frac{1}{N}\right)\sqrt{\frac{N+1}{N-1}}}{K}.$$

Notice that  $\tau^*$  is decreasing in both  $N$  and  $K$ , and converges to zero as  $K \rightarrow \infty$ , furthermore it converges to  $\pi/(2K)$  as  $N \rightarrow \infty$ .

Consider next system (23), where  $\alpha = KB/(A(N+1))$ ,  $\beta = Kc/(A(N+1))$  and so  $C = KB\tau/(A(N+1))$  and  $D = Kc\tau/(A(N+1))$ . Since

$$\frac{C}{ND} = \frac{B}{cN},$$

the stability condition (34) can be rewritten as

$$\frac{KB\tau}{A(N+1)} < \cos^{-1}\left(-\frac{B}{cN}\right) \frac{\frac{B}{cN}}{\sqrt{1 - \frac{B^2}{c^2N^2}}}.$$

That is,

$$\tau < \frac{A(N+1)}{K} \cos^{-1}\left(-\frac{B}{cN}\right) \frac{1}{\sqrt{c^2N^2 - B^2}}, \quad (36)$$

which proves the following result.

**Theorem 2** *The steady state of system (23) is locally asymptotically stable if  $\tau < \tau^{**}$  and unstable if  $\tau > \tau^{**}$ , where*

$$\tau^{**} = \frac{A(N+1)}{K} \cos^{-1}\left(-\frac{B}{cN}\right) \frac{1}{\sqrt{c^2N^2 - B^2}}.$$

Notice that  $\tau^{**}$  strictly decreases in both  $K$  and  $N$ , it converges to zero as  $K \rightarrow \infty$  and converges to  $A\pi/(2Kc)$  as  $N \rightarrow \infty$ .

## 4 Occurance of Hopf bifurcation

Stability switch may occur when  $\Lambda = ir$  ( $r > 0$ ) is an eigenvalue of equation (25). By substitution

$$ir + C + DN(\cos r - i \sin r) = 0.$$

Separating the real and imaginary parts gives two equations:

$$C + DN \cos r = 0 \quad (37)$$

$$r - DN \sin r = 0. \quad (38)$$

From (37),

$$\cos r = -\frac{C}{DN}$$

so

$$r = \cos^{-1}\left(-\frac{C}{DN}\right) + 2n\pi$$

since  $\sin r > 0$  by (38). Then (38) implies that

$$\cos^{-1}\left(-\frac{C}{DN}\right) + 2n\pi - DN\sqrt{1 - \frac{C^2}{D^2N^2}} = 0.$$

It is easy to see that with  $n = 0$  this equation is the same as the equality versions of (35) and (36). Consider  $\tau$  as the bifurcation parameter with all other constants being fixed. Then we can assume that  $\Lambda = \Lambda(\tau)$ . By implicitly differentiating equation (25) with respect to  $\tau$  we have

$$\Lambda' + \alpha + \beta N e^{-\Lambda} - \beta \tau N e^{-\Lambda} \Lambda' = 0$$

implying that

$$\Lambda' = \frac{\alpha + \beta N e^{-\Lambda}}{\beta \tau N e^{-\Lambda} - 1}.$$

Combining this equation with (25) gives

$$\Lambda' = \frac{\alpha + \beta N(-\Lambda - C)/DN}{\beta \tau N(-\Lambda - C)/DN - 1} = \frac{-\Lambda}{-\tau \Lambda - \tau^2 \alpha - \tau}$$

where we used the notations  $C = \alpha\tau$  and  $D = \beta\tau$ . If  $\Lambda = ir$ , then

$$\Lambda' = \frac{ir}{\tau^2 \alpha + \tau + i\tau r}$$

with real part

$$Re\Lambda' = \frac{r^2}{\tau(\tau\alpha + 1)^2 + \tau r^2} > 0$$

showing that at the stability switches Hopf bifurcation occurs giving the possibility of the birth of limit cycles.

## 5 Conclusions

In this paper two special learning schemes were examined under the additional assumption that only delayed information is available about the market price. Linear price and cost functions were assumed, and fixed time delay was incorporated into the dynamic models. The mathematical models were formulated in general, and the stability analysis was performed in the symmetric case, when the firms had identical marginal costs, speeds of adjustments and initial output quantities. If the delay is sufficiently small, less than  $\tau^*$  or  $\tau^{**}$ , then the steady state is asymptotically stable meaning that the estimates of the price function converge to the true function, which shows the possibility of learning. If the delay is larger than the given threshold, then the steady state is unstable. At the stability switches Hopf bifurcation occurs. Notice that model (10) is linear, where asymptotic stability is global, however system (19) is nonlinear, where only local asymptotic stability can be guaranteed under the derived conditions.

Nonlinear and nonsymmetric oligopolies will be the subjects of our future research when recent results of nonlinear dynamics can be applied.

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