Cyclical Fluctuations in Continuous Time Dynamic Optimization Models : Survey of General Theory and an Application to Dynamic Limit Pricing

Toichiro Asada
Faculty of Economics, Chuo University
742-1 Higashinakano, Hachioji, Tokyo 102-0393, Japan
E-mail : asada@tamacc.chuo-u.ac.jp

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Abstract

In this paper, we reconsider the analytical results on the existence of cyclical fluctuations in continuous time dynamic optimization models with two state variables and their applications to dynamic economic theory. In the first part of the paper, we survey the useful analytical results which were obtained by Dockner and Feichtinger(1991), Liu(1994) and Asada and Yoshida(2003) on the general theory of cyclical fluctuations in continuous time dynamic optimizing and non-optimizing models. In the second part of the paper, we provide an application of these analytical results to a particular continuous time dynamic optimizing economic model, that is, a model of dynamic limit pricing with two state variables, which is an extension of Gaskins' (1971) prototype model.

Key words : Cyclical fluctuations, Continuous time dynamic optimization models, Hopf Bifurcation, Dynamic limit pricing.

1. Introduction

It is well known that the typical continuous time dynamic optimization model with only one state variable, which is very popular in economics, does not produce the cyclical fluctuations but it produces the monotonic convergence to the equilibrium point. On the other hand, some economic theorists provided various types of continuous time dynamic optimization models with two state variables which entails cyclical fluctuations. Some examples of such works are Benhabib and Nishimura(1979), Benhabib and Rustichini(1990), Asada and Semmler(1995), and Asada and Semmler(2004). The above-mentioned works showed the existence of closed orbits as the optimal trajectories analytically as well as numerically by applying the Hopf Bifurcation theorem.[1]

All of the above-mentioned works are the studies of particular economic models rather than the systematic investigations of the general continuous time dynamic optimization models with two state variables. On the other hand, Dockner and Feichtinger(1991) provided an exhaustive classification of the nature of the solution of such a general model including the conditions for the occurrence of the Hopf Bifurcation. Feichtinger, Novak and Wirl(1994) and Faria and Andrade(1998) are examples of the applications of Dockner and Feichtinger's (1991) theorem to the economic models. Asada and Yoshida(2003) discussed on the analytical results of Dockner and Fichtinger(1991) from a particular point of view.

In this paper, we reconsider the analytical results on the existence of cyclical fluctuations in continuous time dynamic optimization models with two state variables and their applications to dynamic economic theory. Our strategy is to take up a particular economic model from the viewpoint of an application of the general theory of dynamic optimization. In section 2, we survey the useful analytical results which were obtained by Dockner and Feichtinger(1991), Liu(1994) and Asada and Yoshida(2003) on the general theory of cyclical fluctuations in continuous time dynamic optimizing and non-optimizing models. In section 3, we provide an application of these analytical results to a particular continuous time dynamic optimization model, that is a model of dynamic limit pricing with two state variables, which is an extension of Gaskin's (1971) prototype model. Section 4 is devoted to an interpretation of the analytical results obtained in section 3.

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[1] This does not necessarily mean that every continuous time dynamic optimization model with two state variables produces cyclical fluctuations. For example, Asada, Semmler and Novak(1998) proved analytically that Romer's (1990) continuous time dynamic optimization model of endogenous growth with two state variables entails only the monotonic convergence to the equilibrium point.

2. Survey of general theory

In this section, we survey some useful analytical results on the existence of cyclical fluctuations in continuous time dynamic optimization and non-optimization models. First, let us quote the following 'Hopf Bifurcation theorem' which describes a set of sufficient conditions for the existence of the closed orbits in a general n-dimensional system of nonlinear differential equations (cf. Gandolfo 1996 Chap. 25 and Asada, Chiarella, Flaschel and Franke 2003 Mathematical Appendix).

Theorem 1. (Hopf Bifurcation theorem)

Let $\dot{x} = f(x;\varepsilon)$, $x \in R^n$, $\varepsilon \in R$ be an n-dimensional system of differential equations depending upon a parameter ε. Suppose that the following conditions (H1) – (H3) are satisfied.

(H1) The system has a smooth curve of equilibria given by $f(x^*(\varepsilon);\varepsilon) = 0$,

(H2) The characteristic equation $|\lambda I - Df(x^*(\varepsilon_0);\varepsilon_0)| = 0$ has a pair of pure imaginary roots $\lambda(\varepsilon_0)$, $\bar{\lambda}(\varepsilon_0)$ and no other roots with zero real parts, where $Df(x^*(\varepsilon_0);\varepsilon_0)$ is the Jacobian matrix of the above system at $(x^*(\varepsilon_0),\varepsilon_0)$ with the parameter value ε_0,

(H3) $\left.\dfrac{d\{\mathrm{Re}\,\lambda(\varepsilon)\}}{d\varepsilon}\right|_{\varepsilon=\varepsilon_0} \neq 0$, where $\mathrm{Re}\,\lambda(\varepsilon)$ is the real part of $\lambda(\varepsilon)$.

Then, there exists a continuous function $\varepsilon(\gamma)$ with $\varepsilon(0) = \varepsilon_0$, and for all sufficiently small values of $\gamma \neq 0$ there exists a continuous family of non-constant periodic solution $x(t,\gamma)$ for the above dynamical system, which collapses to the equilibrium point $x^*(\varepsilon_0)$ as $\gamma \to 0$. The period of the cycle is close to $2\pi / \mathrm{Im}\,\lambda(\varepsilon_0)$, where $\mathrm{Im}\,\lambda(\varepsilon_0)$ is the imaginary part of $\lambda(\varepsilon_0)$.

The point $\varepsilon = \varepsilon_0$ that satisfies all of the above conditions (H1) – (H3) is called the 'Hopf Bifurcation point'. An important necessary condition for the occurrence of Hopf Bifurcation is that the characteristic equation of the above system has a pair of pure imaginary roots at $\varepsilon = \varepsilon_0$. It is well known that the typical continuous time dynamic optimization model with single state variable has two characteristic roots and at least one of which has positive real part, so that the Hopf Bifurcation cannot occur in such a model. But, Dockner and Feichtinger (1991) and Asada and Yoshida (2003) proved analytically that the existence of Hopf Bifurcation is at least potentially possible if we consider the continuous time dynamic optimization model with two state variables.

Following Dockner and Feichtinger(1991) and Asada and Yoshida(2003), let us consider the following typical continuous time dynamic optimization problem with two state variables.

$$\text{Maximize } \int_0^\infty F(k_1,k_2,u_1,u_2,\cdots,u_n)e^{-rt}dt \tag{1}$$

subject to

$$\dot{k}_1 = f(k_1,k_2,u_1,u_2,\cdots,u_n), \quad \dot{k}_2 = g(k_1,k_2,u_1,u_2,\cdots,u_n;\varepsilon), \tag{2}$$

$$k_1(0) = k_{10} = given, \quad k_2(0) = k_{20} = given, \tag{3}$$

where $k_i (i = 1,2)$ are two state variables, $u_j (j = 1,2,\cdots,n)$ are control variables, r is the rate of discount that is a positive parameter, and ε is another parameter.[2] We assume that the functions F, f, and g are at least twice continuously differentiable.

We can solve this problem by means of Pontryagin's maximum principle (cf. Chiang 1992 and Dockner, Jorgensen, Van Long and Sorger 2000). First, let us define the current value Hamiltonian as

$$H = F(k_1,k_2,u_1,u_2,\cdots u_n) + \mu_1 f(k_1,k_2,u_1,u_2,\cdots,u_n) + \mu_2 g(k_1,k_2,u_1,u_2,\cdots,u_n;\varepsilon), \tag{4}$$

where μ_1 and μ_2 are two costate variables which correspond to two state variables k_1 and k_2 respectively. Then, a set of necessary conditions of the optimality becomes as follows.

(i) $\dot{k}_i = \partial H / \partial \mu_i \quad (i = 1,2)$

(ii) $\dot{\mu}_i = r\mu_i - \partial H / \partial k_i \quad (i = 1,2)$

(iii) $\underset{(u_1,u_2,\cdots,u_n)}{Max} H$

(iv) $\lim_{t\to\infty} k_i \mu_i e^{-rt} = 0 \quad (i = 1,2)$ (5)

The conditions (5)(i) are equivalent to the dynamic constraints (2). The conditions (5)(ii) are a set of differential equations which describe the dynamics of the costate variables. We suppose that the condition (5)(iii) are equivalent to the following first

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2 We can introduce other parameters which affect functions F and f, but the formulation in the text is sufficient for our purpose.

order conditions.[3]

$$\partial H / \partial u_j = 0 \quad (j = 1,2,\cdots,n) \tag{6}$$

This is a set of simultaneous equations with respect to the control variables. We assume that its solution is uniquely determined and it can be expressed by the following continuously differentiable functions.

$$u_j = u_j(k_1, k_2, \mu_1, \mu_2; \varepsilon) \quad (j = 1,2,\cdots,n) \tag{7}$$

The conditions (5)(iv) are called the 'transversality conditions'.

Substituting the relationships (7) into (5)(i) and (5)(ii), we obtain the following four-dimensional system of linear or nonlinear differential equations.

(i) $\dot{k}_1 = G_1(k_1, k_2, \mu_1, \mu_2; \varepsilon)$

(ii) $\dot{k}_2 = G_2(k_1, k_2, \mu_1, \mu_2; \varepsilon)$

(iii) $\dot{\mu}_1 = G_3(k_1, k_2, \mu_1, \mu_2; r, \varepsilon)$

(iv) $\dot{\mu}_2 = G_4(k_1, k_2, \mu_1, \mu_2; r, \varepsilon)$ (8)

We shall consider the dynamics of this system around the equilibrium point by *assuming* that there exists a meaningful equilibrium solution $(k_1^*, k_2^*, \mu_1^*, \mu_2^*)$ of this system such that $\dot{k}_1 = \dot{k}_2 = \dot{\mu}_1 = \dot{\mu}_2 = 0$.

Let us write the (4×4) Jacobian matrix of this system *at the equilibrium point* as J. Then, we can write the characteristic equation of this system as

$$\Delta(\lambda) \equiv |\lambda I - J| = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{9}$$

$$a_1 = -traceJ, \quad a_2 = M_2, \quad a_3 = -M_3, \quad a_4 = \det J, \tag{10}$$

where M_j is the sum of all principal j-th order minors of $J (j = 2,3)$.[4]

Dockner and Feichtinger(1991) proved that the following relationships are satisfied in case of this particular Jacobian matrix J.

$$traceJ = 2r, \quad -M_3 + rM_2 - r^3 = 0 \tag{11}$$

Following Dockner and Feichtinger(1991), let us write

$$K \equiv M_2 - r^2. \tag{12}$$

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[3] We assume that the second order conditions are also satisfied.
[4] See Mathematical Appendix of Asada, Chiarella, Flaschel and Franke(2003).

Then, we can rewrite Eq. (11) as

$$traceJ = 2r, \quad -M_3 + rK = 0. \tag{13}$$

Then, we have the following expression substituting equations (12) and (13) into a set of relationships (10).

$$a_1 = -traceJ = -2r < 0, \quad a_2 = r^2 + K, \quad a_3 = -rK, \quad a_4 = \det J \tag{14}$$

It is worth to note that we have

$$traceJ = \sum_{j=1}^{4} \lambda_j = 2r > 0, \tag{15}$$

where $\lambda_j (j = 1,2,3,4)$ are the characteristic roots of Eq. (9). Therefore, this system has at least one root with positive real part.

Furthermore, Dockner and Feichtinger(1991) proved that the following set of conditions (DF) is equivalent to the condition (H2) in Theorem 1 in this paper.

$$\det J > (K/2)^2, \quad (K/2)^2 + r^2(K/2) - \det J = 0 \tag{DF}$$

More accurately, they proved the following quite useful theorem.

Theorem 2. (Dockner and Feichtinger 1991)

The characteristic equation $\Delta(\lambda) \equiv |\lambda I - J| = 0$ of the particular Jacobian matrix J of the system (8) has the following properties (i) – (iv).

(i) The characteristic equation has two positive real roots and two negative real roots *if and only if*

$$K < 0, \quad 0 < \det J \leqq (K/2)^2. \tag{16}$$

(ii) The characteristic equation has a pair of complex roots with positive real part and a pair of complex roots with negative real part *if and only if*

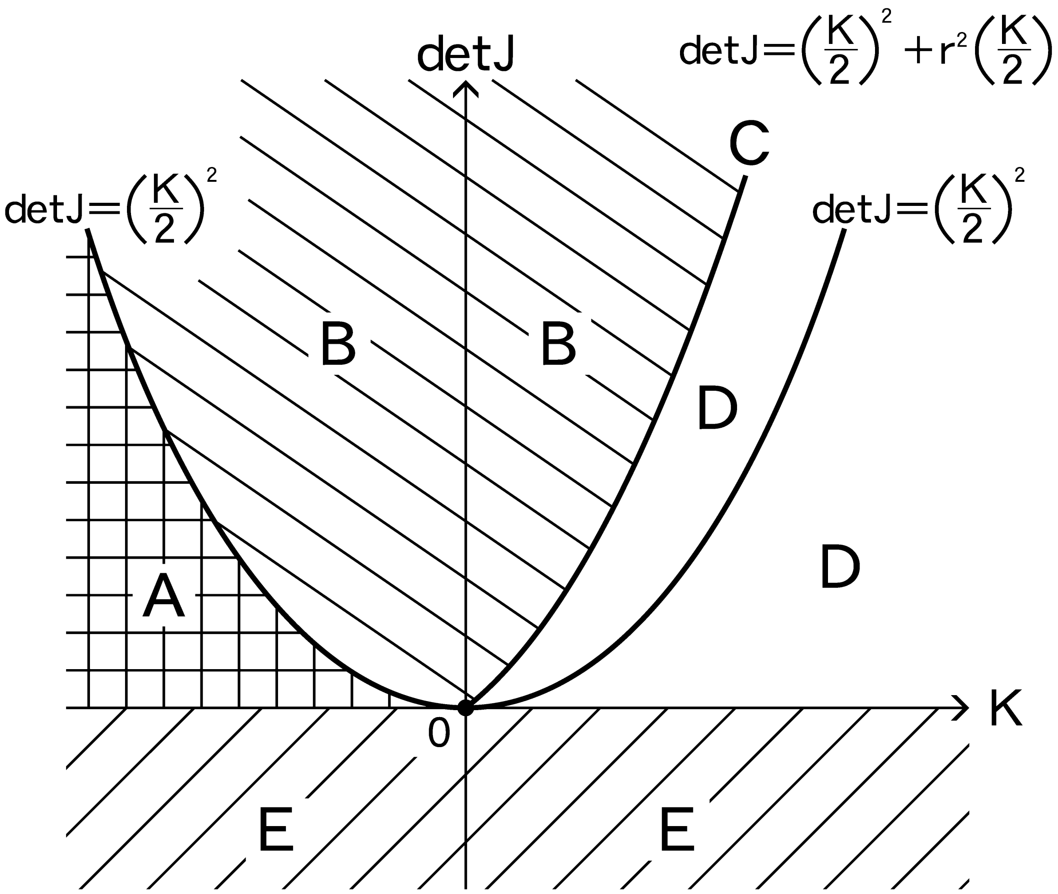
$$\det J > (K/2)^2, \quad \det J - (K/2)^2 - r^2(K/2) > 0. \tag{17}$$

(iii) The characteristic equation has one positive real root and three roots with negative real parts, *or* it has three roots with positive real parts and one negative real roots *if and only if*

$$\det J < 0. \tag{18}$$

(iv) The characteristic equation has a pair of complex roots with positive real part and a pair of pure imaginary roots *if and only if* the condition (DF) is satisfied.

Dockner and Feichtinger(1991) expressed the result of this theorem visually by using Figure 1.



(A) Two positive real roots and two negative real roots (real roots type saddle point)

(B) A pair of complex roots with positive real part and a pair of complex roots with negative real part (complex roots type saddle point)

(C) A pair of complex roots with positive real part and a pair of pure imaginary roots (Hopf Bifurcation curve)

(D) Four roots with positive real parts (totally unstable)

(E) Three roots with positive real parts and one negative real root

Source : Dockner and Feichtinger(1991) p. 36 and Feichtinger, Novak and Wirl(1994) p.356

Figure 1. Classification of the nature of the roots of characteristic equation (9)

Next, let us turn to the investigation of the conditions for the occurrence of the Hopf Bifurcation in a general system of nonlinear differential equations without restricting to the particular dynamic optimization model. It is worth noting that the following 'Liu's theorem' provides us very powerful result that is applicable to general n-dimensional system of differential equations.

Theprem 3. (Liu 1994)

Consider the following characteristic equation with $n \geqq 3$:

$$\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n = 0. \tag{19}$$

This characteristic equation has a pair of pure imaginary roots and $(n-2)$ roots with negative real parts *if and only if* the following set of conditions are satisfied :

$$A_j > 0 \text{ for all } j \in \{1,2,\cdots,n-2\}, \quad A_{n-1} = 0, \quad b_n > 0, \tag{20}$$

where $A_j (j = 1,2,\cdots,n-1)$ are Routh-Hurwitz terms defined as

$$A_1 = b_1, \quad A_2 = \begin{vmatrix} b_1 & b_3 \\ 1 & b_2 \end{vmatrix}, \quad A_3 = \begin{vmatrix} b_1 & b_3 & b_5 \\ 1 & b_2 & b_4 \\ 0 & b_1 & b_3 \end{vmatrix}, \quad \cdots\cdots,$$

$$A_{n-1} = \begin{vmatrix} b_1 & b_3 & b_5 & b_7 & \cdots & 0 & 0 \\ 1 & b_2 & b_4 & b_6 & \cdots & 0 & 0 \\ 0 & b_1 & b_3 & b_5 & \cdots & 0 & 0 \\ 0 & 1 & b_2 & b_4 & \cdots & 0 & 0 \\ 0 & 0 & b_1 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & b_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_n \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & b_{n-1} \end{vmatrix}. \tag{21}$$

Although this 'Liu's theorem' is quite useful in the sense that it can be applicable to the general n-dimensional system of differential equations, it has the following deficiency. The Hopf Bifurcation in which all the characteristic roots *except* a pair of purely imaginary ones have *negative* real parts is called the 'simple' Hopf Bifurcation.

Liu's theorem is applicable only to the case of 'simple' Hopf Bifurcation. But, in the typical dynamic optimization model, usually there exists at least one characteristic root that has positive real part. This means that Liu's theorem is inapplicable to the typical dynamic optimization model. On the other hand, Asada and Yoshida(2003) provided the following complete mathematical characterization of the criteria for the occurrence of the Hopf Bifurcation including the 'non simple' as well as the 'simple' case, although their analysis is restricted to four-dimensional system.

Theorem 4. (Asada and Yoshida 2003)

(1) Consider the characteristic equation

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0. \tag{22}$$

(i) The characteristic equation (22) has a pair of pure imaginary roots and two roots with non-zero real parts *if and only if* either of the following set of conditions (A) or (B) is satisfied.

$$b_1 b_3 > 0,\ \ b_4 \neq 0,\ \ \Phi \equiv b_1 b_2 b_3 - b_1{}^2 b_4 - b_3{}^2 = 0. \tag{A}$$

$$b_1 = b_3 = 0,\ \ b_4 < 0. \tag{B}$$

(ii) The characteristic equation (22) has a pair of pure imaginary roots and two roots with negative real parts *if and only if* the following condition (C) is satisfied.

$$b_1 > 0,\ \ b_3 > 0,\ \ b_4 > 0,\ \ \Phi \equiv b_1 b_2 b_3 - b_1{}^2 b_4 - b_3{}^2 = 0. \tag{C}$$

(2) Consider the characteristic equation

$$\lambda^4 + b_1(\varepsilon)\lambda^3 + b_2(\varepsilon)\lambda^2 + b_3(\varepsilon)\lambda + b_4(\varepsilon) = 0, \tag{23}$$

where it is assumed that the coefficients $b_j\,(j = 1,2,3,4)$ are the continuously differentiable functions of a parameter ε. Then, we have the following properties (i) and (ii).

(i) Suppose that we have $b_1(\varepsilon_0)b_3(\varepsilon_0) > 0,\ b_4(\varepsilon_0) \neq 0,$ and

$\Phi(\varepsilon_0) \equiv b_1(\varepsilon_0)b_2(\varepsilon_0)b_3(\varepsilon_0) - b_1(\varepsilon_0)^2 b_4(\varepsilon_0) - b_3(\varepsilon_0)^2 = 0$ at the point $\varepsilon = \varepsilon_0$.

Then, the condition (H3) in Theorem 1 is equivalent to the following condition (D).

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=\varepsilon_0} \neq 0 \tag{D}$$

(ii) Suppose that we have $b_1(\varepsilon_0)=0$, $b_3(\varepsilon_0)=0$, and $b_4(\varepsilon_0)<0$ at the point $\varepsilon=\varepsilon_0$. Then, the condition (H3) in Theorem 1 is equivalent to the following condition (E).

$$[b_2(\varepsilon_0)+\sqrt{b_2(\varepsilon_0)^2-4b_4(\varepsilon_0)}]b_1{}'(\varepsilon_0)-2b_3{}'(\varepsilon_0)\neq 0 \tag{E}$$

Asada and Yoshida(2003) proved the following proposition by applying Theorem 4 (1) (i) to the particular characteristic equation (9).[5]

Proposition 1. (Asada and Yoshida 2003)

(i) The characteristic equation (9) of the particular system of differential equations (8) has a set of pure imaginary roots and two roots with non-zero real parts *if and only if* the following set of conditions (AY) is satisfied.

$$K>0,\ \ (K/2)^2+r^2(K/2)-\det J=0 \tag{AY}$$

(ii) A set of conditions (AY) is equivalent to a set of conditions (DF) by Dockner and Feichtinger(1991).

[Proof.]

(i) First, it follows from the relationships (14) that

$$\Phi \equiv a_1a_2a_3-a_1{}^2a_4-a_3{}^2=4r^2[(K/2)^2+r^2(K/2)-\det J]. \tag{24}$$

Second, a set of conditions (A) in Theorem 4 (1) (i) is equivalent to the following set of conditions in case of the particular characteristic equation (9).

$$a_3<0,\ \ a_4\neq 0,\ \ \Phi=0 \tag{25}$$

We can see from the relationships (14) that the condition $a_3<0$ is equivalent to the condition $K>0$. Furthermore, the condition $\Phi=0$ is equivalent to the condition $(K/2)^2+r^2(K/2)-\det J=0$. If these two conditions are satisfied, we also have $a_4\neq 0$ because of the fact that $a_4=\det J=(K/2)^2+r^2(K/2)>0$.

(ii) First, let us suppose that a set of conditions (DF) is satisfied. In this case, we have

$$\det J=(K/2)^2+r^2(K/2)>(K/2)^2, \tag{26}$$

which means that $K>0$. This proves the causality (DF)$\Rightarrow$(AY).

Next, let us suppose that a set of conditions (AY) is satisfied. Also in this case, we

[5] We reproduce the proof here. The method of proof is quite simple and straightforward.

have the relationship (26), which means that a set of conditions (DF) is satisfied. This proves the causality (AY)$\Rightarrow$(DF). □

Remark 1.

Comparing Theorem 2 (iv) and Proposition 1, we can see that the particular characteristic equation (9) has a pair of pure imaginary roots and two complex roots with positive real parts if a set of conditions (AY) is satisfied. In this case, the condition (D) in Theorem 4 (2) (i) is equivalent to the condition

$$\frac{d}{d\varepsilon}[(K/2)^2 + r^2(K/2) - \det J]\bigg|_{\varepsilon=\varepsilon_0} \neq 0. \tag{27}$$

In the next section, we shall apply the analytical results which were surveyed in this section to an extended version of Gaskins' (1971) model of dynamic limit pricing.

3. An application to dynamic limit pricing

3 – 1. Gaskins' prototype model of dynamic limit pricing

First, let us summarize the prototype model of dynamic limit pricing that was originated by Gaskins(1971). We consider a partial equilibrium model of an industry in which one dominant large firm and many small fringe firms exist. The demand function is expressed by the following linear decreasing function：

$$q = a - bp \ ; \ a > 0, \ b > 0, \tag{28}$$

where q is the demand for the product of this industry, p is the price of this product., and a, b are two parameters of the demand function.[6]

The dominant large firm acts as the price leader (the price setter) subject to the threat of entry by the fringe firms. Fringe firms behave as price takers and the entry dynamics of the fringe firms are expressed by the differential equation

$$\dot{x} = \alpha(p - \bar{p}) \ ; \ \alpha > 0, \ \bar{p} > 0, \tag{29}$$

where x is the total output of fringe firms and α, $\bar{p}$ are parameters of the entry dynamics. It is assumed that the dominant large firm selects its output level corresponding to $(q - x)$, and the average cost of the dominant large firm (c) is constant. Then, the discounted present value of the dominant large firm becomes

[6] Gaskins(1971) used more general demand function that is not necessarily linear, but we use the linear demand function for simplicity of the analysis following Dixit(1990) Chap. 10.

$$W = \int_0^\infty (p-c)(a-bp-x)e^{-rt}dt, \tag{30}$$

where r is the rate of discount, which is a positive parameter.

The dominant large firm is supposed to select the dynamic path of price (p) that maximizes W subject to the dynamic constraint (29) and given initial value $x(0)$. Although this is a typical dynamic optimization problem of single agent with one state variable (x), we can interpret that this is implicitly a kind of Stackerberg differential game in which the dominant large firm acts as the leader and fringe firms act as followers (cf. Asada and Semmler 2004).[7]

The current value Hamiltonian of this dynamic optimization problem can be written as

$$H = (p-c)(a-bp-x) + \mu\alpha(p-\bar{p}), \tag{31}$$

where μ is the costate variable corresponding to the dynamic constraint (29). A set of necessary conditions for optimality becomes as

(i) $\dot{x} = \partial H / \partial \mu,$

(ii) $\dot{\mu} = r\mu - \partial H / \partial x,$

(iii) $\underset{p}{Max}\, H,$

(iv) $\lim_{t\to\infty} x\mu e^{-rt} = 0.$ (32)

Solving Eq. (32) (iii) with respect to μ, we have $\mu = \mu(p)$. Substituting this relationship into equations (i) and (ii) in (32), we obtain the following two dimensional system of differential equations with single transversality condition, where the initial value of the state variable $x(0)$ is pre-determined, but the initial value of the control variable $p(0)$ is *not* pre-determined.

(i) $\dot{x} = F_1(x, p)$

(ii) $\dot{p} = F_2(x, p)$

(iii) $\lim_{t\to\infty} x\mu(p)e^{-rt} = 0$ (33)

Gaskins(1971) proved that the economically meaningful equilibrium point such that $\dot{x} = \dot{p} = 0$ exists under some reasonable conditions, and it becomes a saddle point, namely, the (2×2) Jacobian matrix of this system at the equilibrium point has one positive real root and one negative real root. This means that there exists only one initial value $p(0)$ that ensures the convergence to the equilibrium point corresponding

[7] As for the exhaustive exposition of the theory of differential game, see Dockner, Jorgensen, Van Long and Sorger(2000).

to the given initial value $x(0)$. Only the convergent path satisfies the transversality condition (33) (iii).

In sum, in Gaskins' prototype model cyclical fluctuations do not occur but only the monotonic convergence to the equilibrium point occurs.

3 – 2. Cyclical fluctuations in an extended Gaskins model of dynamic limit pricing

It is possible to extend and develop Gaskins' prototype model in several ways. For example, Judd and Petersen(1986) and Asada and Semmler(2004) extended Gaskins model by introducing the investment behaviors of firms. In particular, Asada and Semmler(2004) provided an example of the occurrence of cyclical fluctuations in such an extended model by means of numerical simulations. In this subsection, we shall present another simple extension of Gaskins model that can produce cyclical fluctuations, which is an example of the direct application of the analytical results summarized in section 2 of this paper.

Instead of the dynamic constraint (29), let us adopt the following new formulation.

$$\dot{x} = \alpha(p^e - \overline{p}) \ ; \ \alpha > 0, \ \overline{p} > 0, \tag{34}$$

$$\dot{p}^e = \beta(p - p^e) \ ; \ \beta > 0, \tag{35}$$

where p^e is the *expected* price, which is the price expected by fringe firms. Eq. (35) means that the dynamic of expected price is governed by a formula of adaptive expectation hypothesis, and β is the speed of adaptation that can be interpreted as the *reciprocal* of the average time lag of expectation adaptation.[8]

The dynamic optimization problem of the dominant large firm is to select the dynamic path of price (p) that maximizes W in Eq. (30) subject to two dynamic constraints (34), (35) with given initial values of two state variables $x(0)$ and $p^e(0)$. In this case, the current value Hamiltonian becomes

$$H = (p-c)(a-bp-x) + \mu_1\alpha(p^e - \overline{p}) + \mu_2\beta(p-p^e), \tag{36}$$

where μ_1 and μ_2 are two costate variables which correspond to two state variables x and p^e respectively.

A set of necessary conditions for optimality becomes

(i) $\dot{x} = \partial H / \partial \mu_1 = \alpha(p^e - \overline{p})$,

(ii) $\dot{p}^e = \partial H / \partial \mu_2 = \beta(p - p^e)$,

(iii) $\dot{\mu}_1 = r\mu_1 - \partial H / \partial x = r\mu_1 + p - c$,

(iv) $\dot{\mu}_2 = r\mu_2 - \partial H / \partial p^e = (r+\beta)\mu_2 - \mu_1\alpha$,

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[8] In the appendix of this paper, we reinterpret this equation by means of a continuously distributed lag model of expectation formation.

(v) $\underset{p}{Max}\, H,$

(vi) $\lim_{t\to\infty} x\mu_1 e^{-rt} = 0,\ \ \lim_{t\to\infty} p^e \mu_2 e^{-rt} = 0.$ (37)

Now, let us turn to the condition (37) (v). The first order condition for the maximization of H with respect to p becomes[9]

$$\partial H / \partial p = -2bp + a - x + bc + \mu_2\beta = 0. \quad (38)$$

Solving this equation with respect to p, we have

$$p = \frac{1}{2b}(a - x + bc + \mu_2\beta). \quad (39)$$

Substituting Eq. (39) into Eq. (37) (i) – (iv), we obtain the following four-dimensional system of linear differential equations.

(i) $\dot{x} = \alpha(p^e - \overline{p}) \equiv G_1(p^e;\alpha)$

(ii) $\dot{p}^e = \beta\{\frac{1}{2b}(a - x - bc + \mu_2\beta) - p^e\} \equiv G_2(x, p^e, \mu_2;\beta)$

(iii) $\dot{\mu}_1 = r\mu_1 + \frac{1}{2b}(a - x + bc + \mu_2\beta) - c \equiv G_3(x, \mu_1, \mu_2; r, \beta)$

(iv) $\dot{\mu}_2 = (r + \beta)\mu_2 - \mu_1\alpha \equiv G_4(\mu_1, \mu_2; r, \alpha, \beta)$ (40)

Next, we shall consider the nature of the equilibrium solution $(x^*, p^e{}^*, p^*, \mu_1{}^*, \mu_2{}^*)$ that satisfies $\dot{x} = \dot{p}^e = \dot{\mu}_1 = \dot{\mu}_2 = 0.$ It is easy to see that we have

$$p^e{}^* = p^* = \overline{p} > 0. \quad (41)$$

Other three equilibrium values are determined by the following linear system of equations.

$$\begin{bmatrix} -1 & 0 & \beta \\ -1 & 2br & \beta \\ 0 & -\alpha & r+\beta \end{bmatrix} \begin{bmatrix} x \\ \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 2b\overline{p} + bc - a \\ bc - a \\ 0 \end{bmatrix} \quad (42)$$

It is easy to see that the solution of this system of equations becomes

$$x^* = \frac{(a - 2b\overline{p} - bc)r(r+\beta) - \overline{p}\alpha\beta}{r(r+\beta)} = (a - 2b\overline{p} - bc) - \frac{\overline{p}\alpha\beta}{r(r+\beta)}$$
$$< a - 2b\overline{p} - bc, \quad (43)$$

$$\mu_1{}^* = \frac{-\overline{p}}{r} < 0, \quad (44)$$

[9] Since $\partial^2 H / \partial p^2 = -2b < 0$, the second order condition is always satisfied.

$$\mu_2{}^* = \frac{-\alpha\bar{p}}{r(r+\beta)} = \frac{\alpha\mu_1{}^*}{r+\beta} < 0. \quad (45)$$

Proposition 2.

We have $x^* > 0$ for all $\beta > 0$ if the parameter a (upper limit of demand) is fixed at sufficiently large positive value and the parameter α (adjustment speed of entry) is fixed at sufficiently small positive value.

[Proof.]

It is easy to see that we have $x^* > 0$ *if and only if* the inequality

$$Z(\beta) \equiv (a - 2b\bar{p} - bc)r(r+\beta) - \bar{p}\alpha\beta > 0 \quad (46)$$

is satisfied. Incidentally, we have

$$Z(0) = (a - 2b\bar{p} - bc)r^2, \quad (47)$$

$$Z'(\beta) = (a - 2b\bar{p} - bc)r - \bar{p}\alpha. \quad (48)$$

Therefore, we have $Z(0) > 0$ and $Z'(\beta) > 0$ if a is sufficiently large and α is sufficiently small. In this case, we obtain $Z(\beta) > 0$ for all $\beta > 0$, which means that we have $x^* > 0$ for all $\beta > 0$. □

Now, let us study the dynamic property of this model by assuming as follows.

Assumption 1.

The combination of the parameter values (a, α) is at the level such that $x^* > 0$ for all $\beta > 0$.

The Jacobian matrix of this system becomes

$$J = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\frac{\beta}{2b} & -\beta & 0 & \frac{\beta^2}{2b} \\ -\frac{1}{2b} & 0 & r & \frac{\beta}{2b} \\ 0 & 0 & -\alpha & r+\beta \end{bmatrix}. \quad (49)$$

We can write the characteristic equation of this system as

$$\Delta(\lambda) \equiv |\lambda I - J| = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \quad (50)$$

where

$$a_1 = -traceJ = -2r < 0, \quad (51)$$

$a_2 = M_2 =$ sum of all principal second-order minors of J

$$= \begin{vmatrix} 0 & \alpha \\ -\dfrac{\beta}{2b} & -\beta \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ -\dfrac{1}{2b} & r \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & r+\beta \end{vmatrix} + \begin{vmatrix} -\beta & 0 \\ 0 & r \end{vmatrix} + \begin{vmatrix} -\beta & \dfrac{\beta^2}{2b} \\ 0 & r+b \end{vmatrix} + \begin{vmatrix} r & \dfrac{\beta}{2b} \\ -\alpha & r+\beta \end{vmatrix}$$

$$= r^2 + \beta(\frac{\alpha}{b} - r - \beta), \tag{52}$$

$$a_3 = -M_3 = -(\text{sum of all principal third-order minors of } J), \tag{53}$$

$$a_4 = \det J = \frac{\alpha\beta r(r+\beta)}{2b} \equiv \det J(\beta) > 0. \tag{54}$$

Since this dynamic optimization model with two state variables is only a particular case of the model that was explained in section 2, we can apply Theorem 2 in section 2 to this model. To this purpose, let us consider the following three relationships.

$$K \equiv M_2 - r^2 = -\beta^2 + (\frac{\alpha}{b} - r)\beta \equiv K(\beta), \tag{55}$$

$$\Omega(\beta) \equiv (K/2)^2 - \det J$$
$$= \frac{\beta}{2}[\frac{1}{2}\beta^3 + (r - \frac{\alpha}{b})\beta^2 + \{\frac{1}{2}(r - \frac{\alpha}{b})^2 - \frac{\alpha r}{b}\}\beta - \frac{\alpha r^2}{b}], \tag{56}$$

$$\Psi(\beta) \equiv (K/2)^2 + r^2(K/2) - \det J$$
$$= \beta[\beta^3 + \frac{1}{2}(r - \frac{\alpha}{b})\beta^2 + \frac{\alpha}{b}(\frac{\alpha}{b} - 4r)\beta - r^3]. \tag{57}$$

Now, we can prove the following important results by applying Dockner and Feichtinger's theorem (Theorem 2 in section 2).

Proposition 3.

Suppose that $0 < r < \frac{\alpha}{b}$. Then, we have the following properties (i) – (ii).

(i) The characteristic equation (50) has a pair of complex roots with positive real part and a pair of complex roots with negative real part for all sufficiently small values of $\beta > 0$.

(ii) Eq. (50) has two positive real roots and two negative real roots for sufficiently large values of $\beta > 0$.

[Proof.]

Suppose that $0 < r < \frac{\alpha}{b}$. Then, the function $K(\beta)$ becomes a differentiable function that has the following property (P_1) because of Eq. (55).

$K(0)=0,\ \ K'(\beta)>0$ for all $\beta\in[0,\frac{\alpha/b-r}{2}),\ \ K'(\frac{\alpha/b-r}{2})=0,\ \ K'(\beta)<0$ for all

$$\beta\in(\frac{\alpha/b-r}{2},\infty),\ \ K(\frac{\alpha}{b}-r)=0,\ \ \lim_{\beta\to\infty}K(\beta)=-\infty. \qquad (P_1)$$

On the other hand, the functions $\det J(\beta)$, $\Omega(\beta)$ and $\Psi(\beta)$ become the differential functions which have the following properties $(P_2)-(P_4)$ because of the equations (54), (56) and (57).

$$\det J(0)=0,\ \ \det J'(\beta)>0 \text{ for all } \beta\in[0,\infty),\ \ \lim_{\beta\to\infty}\det J(\beta)=\infty. \qquad (P_2)$$

$$\Omega(0)=0,\ \ \Omega'(0)=-\frac{\alpha r^2}{2b}<0,\ \ \lim_{\beta\to\infty}\frac{\Omega(\beta)}{\beta^4}=\frac{1}{4}>0. \qquad (P_3)$$

$$\Psi(0)=0,\ \ \Psi'(0)=-r^3<0. \qquad (P_4)$$

These properties $(P_1)-(P_4)$ imply the following results.

The combination $(K,\det J)$ is located at the origin of Figure 1 when $\beta=0$. As β increases from $\beta=0$, this combination moves to the north-east direction continuously until it reach the point $\beta=\frac{\alpha/b-r}{2}$, and the property (P_4) implies that this combination is located at the region B of Figure 1 for all sufficiently small values of $\beta>0$. After the point $\beta=\frac{\alpha/b-r}{2}$, this combination moves to the north-west direction continuously and indefinitely according as the further increase of β. At the point $\beta=\frac{\alpha}{b}-r$ this combination is located at the vertical axis of Figure 1. On the other hand, $\lim_{\beta\to\infty}\frac{\Omega(\beta)}{\beta^4}>0$ implies that $\Omega(\beta)$ becomes positive for all sufficiently large values of $\beta>0$. This means that the combination is located at the region A of Figure 1 for all sufficiently large values of $\beta>0$. □

Proposition 4.

Suppose that $0<r<\frac{\alpha}{b}$ and r is sufficiently small. Then, there exist the parameter values $\beta_j\,(j=1,2,3,4)$ such that $0<\beta_1<\dfrac{\alpha/b-r}{2}<\beta_2<\dfrac{\alpha}{b}-r<\beta_3\leqq\beta_4<\infty$ which satisfy the following properties (i) – (iv).

(i) The characteristic equation (50) has a pair of complex roots with positive real part and a pair of complex roots with negative real part for all $\beta\in(0,\beta_1)\cup(\beta_2,\beta_3)$.

(ii) Eq. (50) has four roots with positive real parts for all $\beta\in(\beta_1,\beta_2)$.

(iii) Eq. (50) has a pair of complex roots with positive real part and a pair of pure imaginary roots at two points $\beta=\beta_1$ and $\beta=\beta_2$.

(iv) Eq. (50) has two positive real roots and two negative real roots for all $\beta\in[\beta_4,\infty)$.

[Proof.]

Suppose that $0<r<\dfrac{\alpha}{b}$. In this case, it follows from the method of the proof of Proposition 3 that there exist the parameter values $\beta_j\,(j=1,2,3)$ such that $0<\beta_1<\dfrac{\alpha/b-r}{2}<\beta_2<\dfrac{\alpha}{b}-r<\beta_3$ with the properties that (i) the trajectory of the combination $(K,\det J)$ is located at the region B in Figure 1 for all $\beta\in(0,\beta_1)\cup(\beta_2,\beta_3)$, (ii) it is located at the region D in Figure 1 for all $\beta\in(\beta_1,\beta_2)$, and (iii) it crosses the curve C at two points $\beta=\beta_1$ and $\beta=\beta_2$, *if and only if* the inequality $\Psi(\dfrac{\alpha/b-r}{2})>0$ is satisfied, where we have

$$\Psi(\frac{\alpha/b-r}{2})=(\frac{\alpha/b-r}{2})[(\frac{\alpha/b-r}{2})^3-\frac{1}{2}(\frac{\alpha}{b}-r)(\frac{\alpha/b-r}{2})^2+(\frac{\alpha}{b})(\frac{\alpha}{b}-4r)(\frac{\alpha/b-r}{2})-r^3] \tag{58}$$

from Eq. (57). It follows from Eq. (58) that

$$\lim_{r\to 0}\Psi(\frac{\alpha/b-r}{2})=\frac{9}{2}(\frac{\alpha}{2b})^4>0, \tag{59}$$

which means that we have $\Psi(\dfrac{\alpha/b-r}{2})>0$ for all sufficiently small values of $r>0$ by continuity. This proves (i) – (iii) of Proposition 4. Proposition 4 (iv) directly follows from Proposition 3. □

Proposition 5.

Suppose that $r \geqq \frac{\alpha}{b}$. Then, there exists a parameter value $\beta_0 \in (0,\infty)$ that satisfy the following properties (i) – (ii).

(i) The characteristic equation (50) has a pair of complex roots with positive real part and a pair of complex roots with negative real part for all $\beta \in (0, \beta_0)$.

(ii) Eq. (50) has two positive real roots and two negative real roots for all $\beta \in [\beta_0, \infty)$.

[Proof.]

Suppose that $r \geqq \frac{\alpha}{b}$. Then, the differentiable function $K(\beta)$ has the following property $(P_1)'$.

$$K(0)=0,\ \ K'(0)=0,\ \ K'(\beta)<0 \text{ for all } \beta>0,\ \lim_{\beta\to\infty} K(\beta)=-\infty \text{ if } r=\frac{\alpha}{b}, \text{ and}$$

$$K(0)=0,\ \ K'(\beta)<0 \text{ for all } \beta \geqq 0,\ \lim_{\beta\to\infty} K(\beta)=-\infty \text{ if } r>\frac{\alpha}{b}. \qquad (P_1)'$$

On the other hand, the properties (P_2) and (P_3) in the proof of Proposition 3 apply also in this case.

The properties $(P_1)'$ and (P_2) mean that the combination $(K, \det J)$ is located at the origin of Figure 1 when $\beta = 0$, and this combination moves to the north-west direction continuously and indefinitely as β increases. The property (P_3) implies that this combination is located at the region B of Figure 1 for all sufficiently small values of $\beta > 0$, and it is located at the region A of Figure 1 for all sufficiently large values of $\beta > 0$. This means that there exists a parameter value $\beta_0 \in (0,\infty)$ that satisfy the property (i) of Proposition 5, and we have

$$\Omega(\beta_0)=0,\ \ \Omega'(\beta_0)=\frac{\beta_0}{2}\Big[\frac{3}{2}\beta_0{}^2+2\Big(r-\frac{\alpha}{b}\Big)\beta_0+\frac{1}{2}\Big(r-\frac{\alpha}{b}\Big)^2-\frac{\alpha r}{b}\Big]>0. \qquad (60)$$

In other words, the switching of the regions $B \to A$ (we call it ‘forward switching’) occurs at the point $\beta = \beta_0$. Next, let us consider whether the ‘backward switching’ (the switching of the regions $A \to B$) occurs according as the further increase of the parameter value β. For this purpose, let us suppose tentatively that there exists another switching point $\beta^* \in (\beta_0, \infty)$ such that

$$\Omega(\beta^*)=0,\ \ \Omega'(\beta^*)=\frac{\beta^*}{2}\Big[\frac{3}{2}\beta^{*2}+2\Big(r-\frac{\alpha}{b}\Big)\beta^*+\frac{1}{2}\Big(r-\frac{\alpha}{b}\Big)^2-\frac{\alpha r}{b}\Big]. \qquad (61)$$

Comparing equations (60) and (61), we can see that $\beta^* > \beta_0 > 0$ and $r \geqq \frac{\alpha}{b}$ imply

$$\Omega'(\beta^*) > \Omega'(\beta_0) > 0, \tag{62}$$

which contradicts that the point $\beta = \beta^*$ is a 'backward' switching point, because at the 'backward' switching point the inequality $\Omega'(\beta) < 0$ must be satisfied. This proves that the 'backward switching' cannot occur so that the property (ii) of Proposition 5 is satisfied in case of $r \geqq \frac{\alpha}{b}$. □

Figure 2 summarizes the results of Propositions 3 – 5. In the regions (B) and at the points (C) in this figure, the cyclical fluctuations occur. In the next section, we shall try to provide an interpretation of the analytical results obtained in this section.

4. An interpretation of the analytical results

Figure 2 provides us a convenient characterization of the solution of the extended dynamic limit pricing model that was presented in section 3 – 2. This figure shows that the characteristic equation of this system has two positive real roots and two negative real roots (regions (A) in this figure) irrespective of the value of the rate of discount $r > 0$ if the adjustment speed of adaptive expectation $\beta > 0$ is sufficiently large (if the time lag of the expectation adaptation $\tau = 1/\beta$ is sufficiently small). In this case, the equilibrium point of the system becomes a real roots type saddle point, and the number of the positive roots is equal to the number of the not-pre-determined costate variables in a system of four-dimensional linear differential equations (40). This means that the dominant firm can select the initial values of the costate variables which ensure the monotonic convergence to the equilibrium point. *If and only if* the convergent path is selected, the transversality conditions (37) (vi) are satisfied. This situation is illustrated in Figure 3.[10] It is worth noting that the solution path in Figure 3 is qualitatively the same as that of Gaskins' (1971) original model of dynamic limit pricing that was explained in section 3 – 1, which can be considered to be the limit case of $\beta \to \infty$ $(\tau \to 0)$.

Figure 2 also shows that the characteristic equation of this system has a pair of complex roots with positive real part and a pair of complex roots with negative real part

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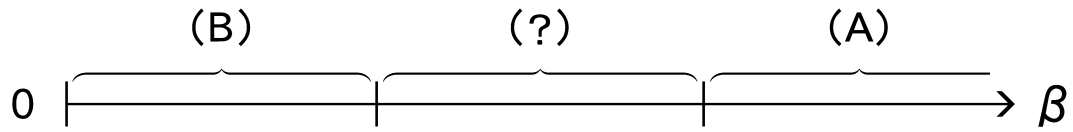
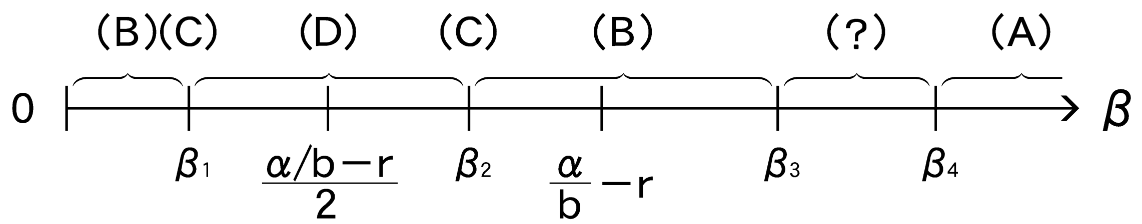
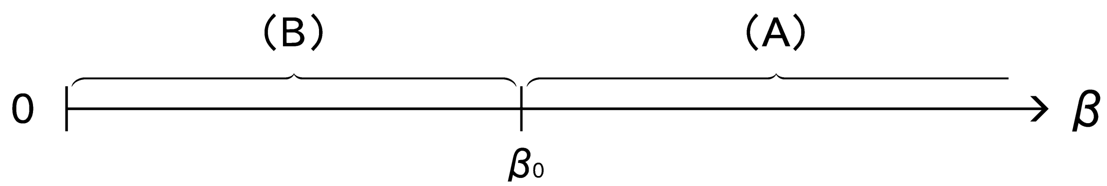
[10] Note that Eq. (39) means that the initial value of price $p(0)$ is determined if the initial value of a state variable $x(0)$ is given and the initial value of a costate variable $\mu_2(0)$ is selected.



Case 1a ： $0 < r < \frac{\alpha}{b}$ (Proposition 3)



Case 1b ： $0 < r < \frac{\alpha}{b}$, r is sufficiently small (Proposition 4)



Case 2 ： $r \geqq \frac{\alpha}{b}$ (Proposition 5)

Figure 2. Classification of the nature of the roots of characteristic equation (50)

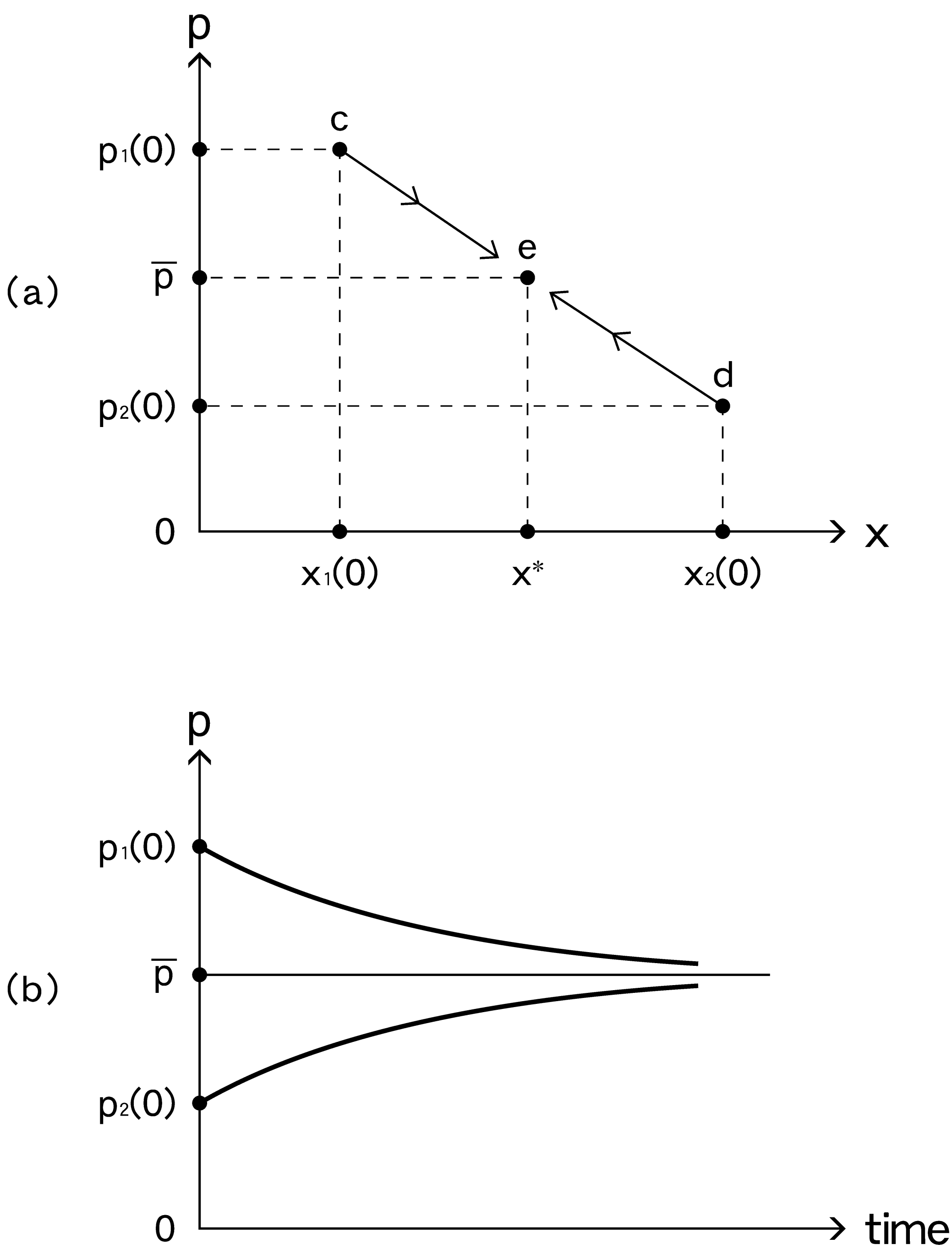


Figure 3. Monotonic convergence (Region (A) in Figure 2)

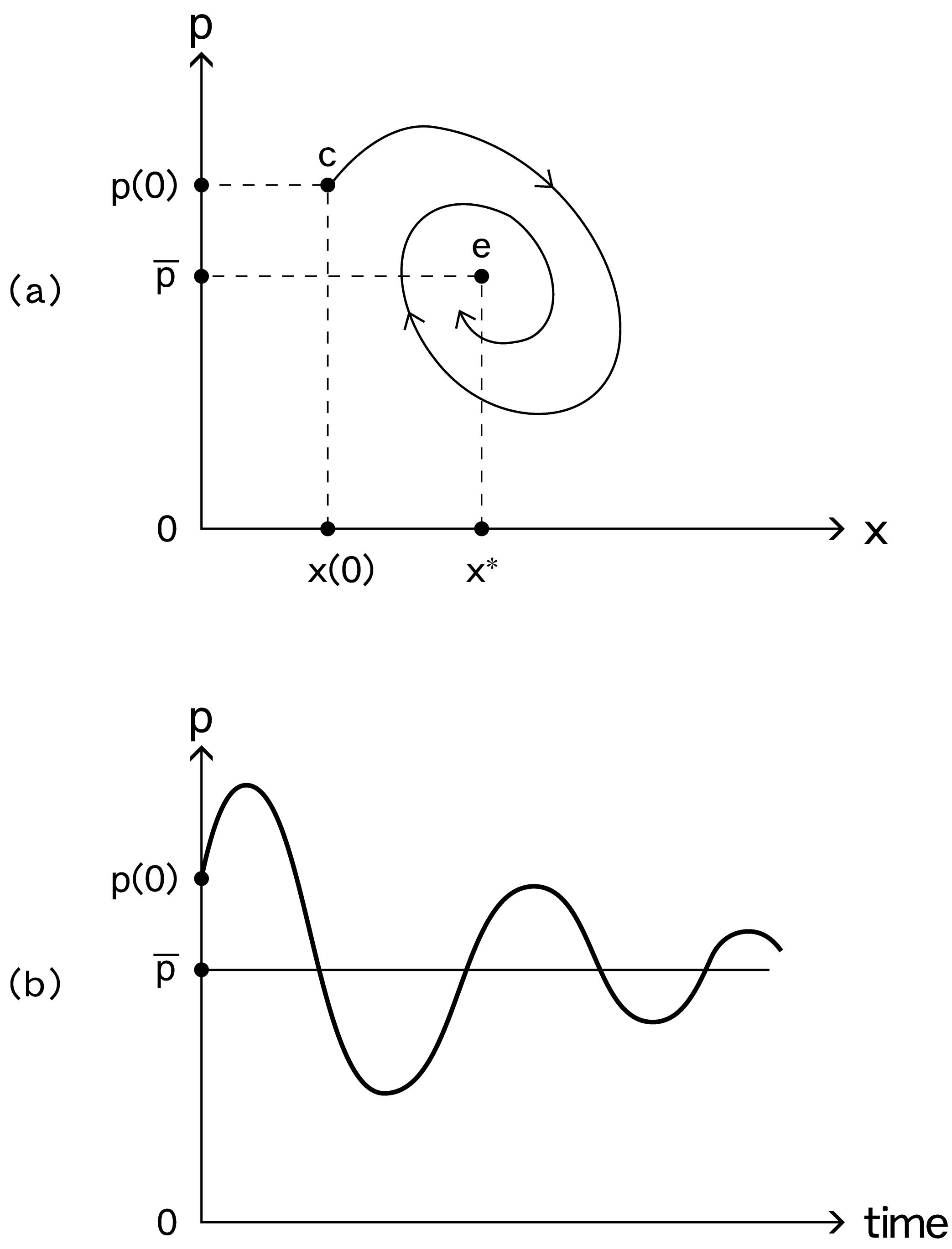


Figure 4. Cyclical convergence (Region (B) in Figure 2)

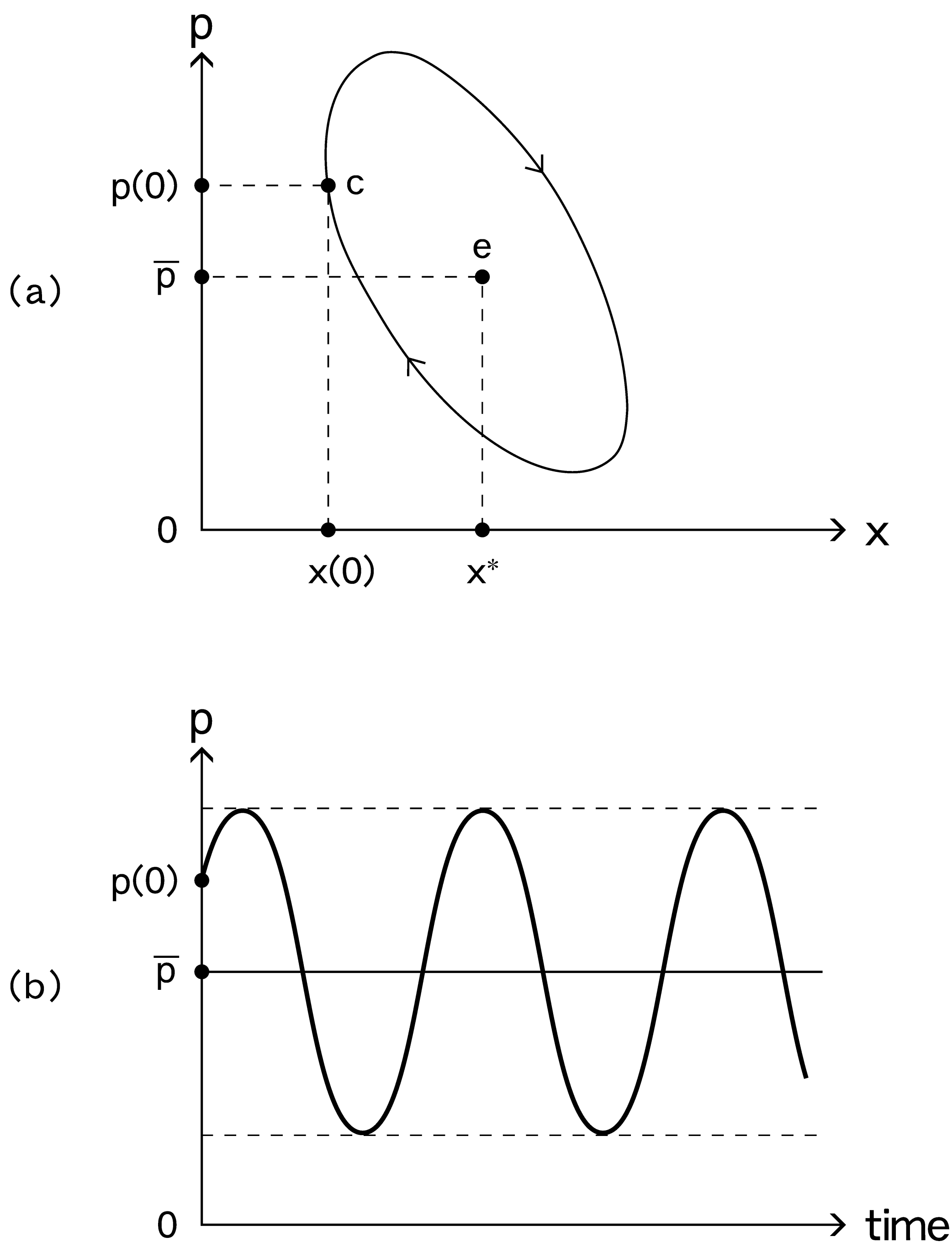


Figure 5 : Closed orbit (Points β_1 and β_2 in Figure 2)

(regions (B) in this figure) irrespective of the value of $r>0$ if $\beta>0$ is sufficiently small (if $\tau=1/\beta$ is sufficiently large). In this case, the equilibrium point becomes a complex roots type saddle point, and also in this case the number of the roots with positive real parts is equal to the number of the not-pre-determined costate variables. Therefore, also in this situation the dominant firm can select the convergent path, which satisfies the transversality conditions. In this case, however, the cyclical fluctuations occur even if the dominant firm selects the convergent path. This situation is illustrated in Figure 4.

Case 1b of Figure 2 provides us an additional important information in case of the sufficiently small values of the rate of discount $r>0$. In this case, the region of cyclical convergence (B) is interrupted by the region (D), at which the characteristic equation has four roots with positive real parts. If the parameter values are located at the region (D), it is impossible to satisfy the transversality conditions unless the initial values of two state variables are given at the equilibrium levels. In this case, a system of four-dimensional linear differential equations (40) fails to characterize the optimal solution.

Next, let us pay attention to two boundary points between the regions (B) and (D) in Case 1b of Figure 2, namely the points β_1 and β_2. At these points, the characteristic equation has a pair of complex roots with positive real parts and a pair of pure imaginary roots. These points correspond to the (degenerated) Hopf Bifurcation points in a system of linear differential equations. Also in this case, the number of the roots with positive real parts is equal to the number of the not-pre-determined costate variables. Hence, the dominant firm can select the non-divergent dynamic path. In this case, however, the non-divergent path does not converge to the equilibrium point but it becomes a closed orbit around the equilibrium point. The combination (p,x) continues to move along the closed orbit without becoming non-positive if the initial values of the state variables are not extremely far from the equilibrium point, and the dynamic path along the closed orbit satisfies the transversality conditions (37) (vi). This means that the closed orbit becomes the optimal path in this case. This situation is illustrated in Figure 5.

It must be noted that the Hopf Bifurcations in this model are 'degenerated' types because of the linearity of the dynamic system. This means that the probability of the occurrence of the closed orbit becomes 'measure zero' in the half line β in Case 1b of Figure 2. Nevertheless, the (converging) cyclical fluctuations occur at the wide range of the parameter value $\beta>0$ in this extended dynamic limit pricing model.

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Appendix

In this appendix, we reinterpret Eq. (35) in the text by means of a continuously distributed lag model of expectation formation following the procedure that was adopted by Shinkai(1970) and Yoshida and Asada(2007). Let us assume that the expected price is the weighted average of actual past prices, that is,

$$p^e(t) = \int_{-\infty}^{t} p(s)\omega(s)ds, \quad \text{(A1)}$$

where $\omega(s)$ is a weighting function such that

$$\omega(s) \geqq 0, \ \int_{-\infty}^{t} \omega(s)ds = 1. \quad \text{(A2)}$$

In particular, we assume that our model is described by means of the following 'simple exponential distributed lag'(cf. Shinkai 1970 Chap. 6 and Yoshida and Asada 2007).[11]

$$\omega(s) = (1/\tau)e^{-(1/\tau)(t-s)} \geqq 0 \ ; \ \tau > 0 \quad \text{(A3)}$$

Substituting (A3) into (A1), we obtain

$$p^e(t)e^{(1/\tau)t} = (1/\tau)\int_{-\infty}^{t} p(s)e^{(1/\tau)s}ds. \quad \text{(A4)}$$

Differentiating (A4) with respect to t, we obtain

$$\dot{p}^e(t) = (1/\tau)\{p(t) - p^e(t)\}, \quad \text{(A5)}$$

which is equivalent to Eq. (35) in the text if we write $\beta = 1/\tau$. We can interpret τ as the average time lag of expectation adaptation.

11 We have $\int_{-\infty}^{t} (1/\tau)e^{-(1/\tau)(t-s)}ds = (1/\tau)e^{-(1/\tau)t}\int_{-\infty}^{t} e^{(1/\tau)s}ds = e^{-(1/\tau)t}[e^{(1/\tau)s}]_{s=-\infty}^{s=t} = 1.$

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