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Abstract

This study sheds light on statistical properties of chaotic economic dynamics. To this end, it builds a simple Cournot dynamic model in which reactions functions are nonlinear and goods are complements. When nonlinearities get strong enough, the output adjustment process generates ergodic chaos. It is analytically as well as numerically demonstrated that for *both* firms, a long-run average profit taken along a chaotic trajectory can be higher than a profit taken at a stationary point. This result implies that chaotic dynamics can be beneficial from the long-run point of view.

keywords: statistical dynamics, long-run average profit, density function, nonlinear Cournot model, chaotic dynamics

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1 Introduction

In the last two decades, the research in nonlinear economic dynamics has remarkably advanced its role in analyzing complex phenomena. See, for example, Rosser [12] and Majumdar, Mitra and Nishimura [8] for overview of the development of nonlinear dynamic theory in economics. In the existing literature, much effort has been devoted to show the existence of chaotic dynamics in various economic models. Further, Boldrin and Montrucchio [2] reveal a qualitative aspect of chaotic dynamics, namely, they show the efficiency of chaotic fluctuations in a discounted dynamic optimization model. In consequence, it is confirmed that deterministic nonlinear dynamic models may explain various erratic fluctuations observed in many economic variables. However, only limited effort has been devoted to discover the statistical properties of such chaotic fluctuations in the long run, and many results obtained are verified in a one-dimensional dynamic model.

The main purpose of this study is to explore the possibility that chaotic dynamics may be profitable from the long-run perspective even in a two-dimensional dynamic model. For this purpose we use a Cournot dynamic model in which firms produce complementary goods and their reaction functions are nonlinear. We demonstrate analytically as well as numerically that chaotic fluctuations may be favorable phenomena in the long-run although the existing studies have not settled this question once and for all.[1]

Rand [11] has shown that a Cournot duopoly model with unimodal reaction functions can give rise to chaotic dynamics of output. Since then a Cournot dynamics is extended in various directions. Dana and Montrucchio [3] generalize Rand's model to infinite horizon game models and shows that firms' intertemporal optimal strategies (i.e., Markov-perfect equilibria) can be chaotic. Puu [10] and Kopel [7] present two possible but distinct microeconomic foundations which support unimodal reaction functions. In the former study, it is shown that the profit maximization with the hyperbolic market demand and the linear cost function lead to the unimodal reaction function. In the latter, it is shown that the profit maximization with linear market demand and the nonlinear cost function involving production externalities of the rival's production activities also result in the unimodal reaction function.

In both analyses, goods involved are assumed to be perfect substitutes. It is, however, often observed in real economy that two goods are connected and affect each other but are not perfect substitutes. The complementary relationship between software and hardware in the computer industry is a typical example. Therefore, in our study, in order to move one step forward and to extend their studies, we consider the case in which goods are *complements*. In short, we investigate statistical properties of chaotic fluctuations in a two-dimensional Cournot dynamic process in which goods

[1] Huang [5] shows that perpetual fluctuations in a simple cobweb model may be preferable to a stationary equilibrium. On the other hand, Kopel [6] shows the inferiority of chaotic dynamics to equilibrium in a simple model of evolutionary dynamics. Further, Matsumoto [9] shows that in a simple exchange model with two agents and two goods, the long-run average profit taken along a chaotic trajectory can be more than the equilibrium profit for one agent but less for the other agent.

involved are complementary and microeconomic underpinnings for chaotic fluctuations are provided.

This paper is organized as follows. Section 2 constructs an inter-market model of complementary goods. Section 3 examines the existence and stability of stationary points. Section 4 considers the long-run behavior of firms to highlight asymptotic features of the market interaction. Section 5 gives concluding remarks.

2 Model

Consider a two-market economy with complementary goods, x and y. On the demand side of the economy, inverse demands in two markets are given by

$$\begin{aligned} p_1(x,y) &= \alpha_1^2 - \beta_1 x + (\gamma_1 y)^2, \\ p_2(x,y) &= \alpha_2^2 - \beta_2 y + (\gamma_2 x)^2, \end{aligned} \tag{1}$$

where p_1 and p_2 are the market prices of x and y, respectively, and α_i, β_i and γ_i $(i = 1, 2)$ are non-negative constant. It can be seen that the inverse demand is downward sloping with respect to its own product and upward sloping with respect to the other firm's product because x and y are assumed to be complements. Since the sales possibilities of one firm are positively influenced by the production choice of the other firm, it can be said that a *positive sales externality* arises in terms of the market demand. Further it should be noted that the cross effect on the market price p_1 caused by a change in expected production y^e is not necessarily equal to the cross effect on p_2 (i.e., $\gamma_1 \neq \gamma_2$). This is because we consider an asymmetric case while the traditional arguments are limited to the symmetric case in which $\gamma_1 = \gamma_2$.

On the supply side, there are two monopolistic firms; firm 1 produces goods x in the first market, and firm 2 produces goods y in the second market. To make its decision, each firm forecasts the other firm's choice and faces production externalities; it is assumed that the production possibilities of one firm is influenced by the choice of the production level by the other firm in terms of the cost function.[2] Although there are various ways to introduce production externalities, we confine our analysis to a simple case in which the production cost linearly depends on not only its own output but also the other firm's output in the following way,

$$C_1(x,y) = c_1 yx \text{ and } C_2(y,x) = c_2 xy, \tag{2}$$

where c_i $(i = 1, 2)$ are nonnegative. Since the marginal cost of each firm increases with the level of the other firm's output, it can be said that each firm has a *negative production externality*.

We thus consider a situation involving double externalities: positive externality via the market demand and negative externality via the cost function. As a result, the

[2] We follow the spirit of Kopel [6] in this respect.

profit of one firm depends not only on its own output but also on the other firm's output. The profit functions of the two firms are accordingly,

$$\begin{aligned} \Pi_1(x,y) &= p_1(x,y)x - c_1 yx \\ \Pi_2(x,y) &= p_2(x,y)y - c_2 xy. \end{aligned} \tag{3}$$

Firm 1 maximizes $\Pi_1(x,y)$ with respect to x, and so does firm 2 $\Pi_2(x,y)$ with respect to y. The condition for the profit maximization is the equality between the marginal revenue and the marginal cost. Solving the condition for its own output, each firm derives its reaction function. Let r_i $(i = 1, 2)$ be the solution for firm i,

$$\begin{aligned} r_1(y) &\equiv \arg\max_x \Pi_1(x,y), \\ r_2(x) &\equiv \arg\max_y \Pi_2(x,y), \end{aligned} \tag{4}$$

where r_i is a reaction function of firm i,

$$\begin{aligned} r_1(y) &= \frac{\alpha_1^2 - c_1 y + (\gamma_1 y)^2}{2\beta_1}, \\ r_2(x) &= \frac{\alpha_2^2 - c_2 x + (\gamma_2 x)^2}{2\beta_2}. \end{aligned} \tag{5}$$

As can be seen, the reaction function takes on various shapes depending on the relative magnitude between the sales externality and the production externality. Roughly speaking, when the sales externality is smaller (i.e., γ_i is much smaller than c_i), the reaction curve becomes linearly downward sloping due to the dominance of the production externality. When on the other hand, the production externality is smaller (i.e., c_i is much smaller than γ_i), it becomes quadratically upward sloping due to the dominance of the sales externality. For the medium values of c_i and γ_i, the reaction function takes on a U-shaped profile because the sales externality is dominate for smaller values of y^e, and the production externality is dominate for larger values.

The U-shaped reaction curve shows a sharp contrast to not only the traditional reaction curve, either downward sloping or upward sloping but also the mound-shaped reaction curve that is often considered in the nonlinear oligopoly setting, see Puu [10] and Kopel [7]. What is the source of such a sharp contrast ? Due to Bulow, Geanakoplos and Klemperer [1], firms strategies are said to be *strategic substitutes* or *strategic complements* according to whether their reaction curves are downward sloping or upward sloping. Thus the mound-shaped reaction curve indicates the case in which the firms changes their strategic profile from being strategic complements to strategic substitutes. In this study, we extend the analysis by considering the case in which the firms changes their strategic profile in the reversed order that is, from being strategic substitutes to strategic complements.

Maintaining the characteristics of the convexity of the reaction functions, we set the values for the parameters as followings,

$$\begin{aligned} c_i &= 2\alpha_i\gamma_i, & (6)\\ \gamma_1 &= \alpha \text{ and } \gamma_2 = \beta, & (7)\\ \alpha_1 &= \alpha - 1,\ \alpha_2 = 1 \text{ and } \beta_1 = \beta_2 = \frac{1}{2}. & (8) \end{aligned}$$

From (6)-(8), the reaction functions, (9) and (10), now have simple forms,

$$\begin{aligned} r_1(y) &= (\alpha y - \alpha + 1)^2, & (9)\\ r_2(x) &= (\beta x - 1)^2, & (10) \end{aligned}$$

both are U-shaped and have critical values for which the best response is zero output. It can be checked that (6) makes the form of the reaction function to be perfect square, the convenient form for analytical considerations; (7) implies the asymmetric sales externalities; (8) is set only for the analytical simplicity. Under these parametric specifications, the marginal costs of production are $(\alpha-1)\alpha y$ for firm 1 and βx for firm 2. Then, it is further required to be $\alpha \geq 1$ and $\beta > 0$ for nonnegative marginal costs.

We are interested in studying dynamic interactions between the two firms. To consider the dynamic process, we lag the variables, assuming the naive expectation formation, that is, $x_t^e = x_{t-1}$ and $y_t^e = y_{t-1}$ where the superscript, "e", means the expected value. At each period, one firm expects that the other firm is going to continue to keep its output at the level of production in the previous period and decides the best response to the expectation. The other firm can reason the same way and decides its best output. In consequence, the dynamic process is governed by the iterations of the following two dimensional inverted logistic map.

$$H(x_t, y_t) : \begin{cases} x_{t+1} = (\alpha y_t - \alpha + 1)^2, \\ \\ y_{t+1} = (\beta x_t - 1)^2. \end{cases} \qquad (11)$$

3 Stationary Points

In this section, we verify the existence conditions of the stationary points and then the stability conditions. Before proceeding, we restrict the domain of the parameters in order to eliminate economically meaningless cases. (9) indicates that the reaction curve of firm 1 is U-shaped with respect to y^e; it takes the value $(\alpha-1)^2$ for $y^e = 0$, zero for $y^e = \frac{\alpha-1}{\alpha}$ and unity for $y^e = 1$. It maps the unit interval into itself if $\alpha \leq 2$. (10) indicates that the reaction curve of firm 2 is also U-shaped with respect to x^e; it starts at unity, produces zero output at $x^e = \frac{1}{\beta}$ and increases to $(\beta-1)^2$ for $x^e = 1$. It also maps the unit interval into itself if $\beta \leq 2$. Therefore, when the combination of parameters is restricted to the set,

$$A \equiv \{(\alpha, \beta) \mid 1 \leq \alpha \leq 2 \text{ and } 0 \leq \beta \leq 2\}, \qquad (12)$$

H is a two-dimensional transformation of the unit set, $[0,1]\times[0,1]$. It will be shown shortly that such parameter restriction prevents a trajectory generated by H from being divergent.

3.1 Existence of Stationary Point

A fixed point of H is a stationary point of the dynamical process (11). Graphically it is the intersection of the reaction functions, $r_1(y)$ and $r_2(x)$.[3] It can be also constructed through a fixed point of a one-dimensional map by combining one reaction function with the other. Let F be a combined one-dimensional map from the unit interval into itself,

$$F(x) \equiv r_1 \circ r_2(x) = (1+\alpha\beta x(\beta x - 2))^2. \tag{13}$$

Since F is a fourth-order polynomial, it possesses at most four roots (i.e., fixed points), $x_j^* = F(x_j^*)$ for $j = 1,2,3,4$. Accordingly, H has four fixed points,

$$S_j = (x_j^*, y_j^*) \text{ such that } x_j^* = F(x_j^*) \text{ and } y_j^* = r_2(x_j^*). \tag{14}$$

For convenience, we assume $x_1^* < x_2^* < x_3^* < x_4^*$ when roots are real and distinct. As illustrated in Figure 1, the number of real roots changes from two to four depending on the specified values of the parameters.[4]

The general outline of F is W-shaped for $\alpha > 1$. F generates two real roots when its center hump is below the diagonal, as illustrated in Figure 1(a), and four when it is above, as in Figure 1(c) and Figure 1(d). Consequently there is a boundary case, as illustrated in Figure 1(b), in which the center hump just gets tangential to the diagonal. There, F has a multiple root, $x^* = x_2^* = x_3^*$. Since $F(x^*)$ lies on the diagonal, it must satisfy the two conditions at x^*, namely $F(x^*) = x^*$ and $F'(x^*) = 1$. To find such a point, we differentiate F and set the resultant expression to unity which we solve for x,

$$4\alpha\beta(\beta x - 1)(\alpha(\beta x - 1)^2 - (\alpha - 1)) = 1. \tag{15}$$

We denote a solution of (15) with the negative second derivative (i.e., $F'' < 0$) by $\hat{x}(\alpha,\beta)$.[5] Using the solution, we can divide the parameter set A into following three subsets:

$$\begin{aligned}
A_1 &\equiv A \setminus (A_2 \cup A_3), \\
A_2 &\equiv \{(\alpha,\beta) \in A \mid F(\hat{x}(\alpha,\beta)) = \hat{x}(\alpha,\beta) \text{ and } F'(\hat{x}(\alpha,\beta)) = 1\}, \\
A_3 &\equiv \{(\alpha,\beta) \in A \mid F(\hat{x}(\alpha,\beta)) > \hat{x}(\alpha,\beta) \text{ and } F'(\hat{x}(\alpha,\beta)) = 1\}.
\end{aligned} \tag{16}$$

[3] We suppress the time subscript "t" for a while only for notational simplicity.

[4] We can show that given $\alpha \geq 1$, $F(x) = x$ always has two distinct real roots, x_1^* and x_4^*, such that $0 < x_1^* < 1$, and $x_4^* \gtreqless 1$ according to $\beta \lesseqgtr 2$. In Figure 1(a), 1(b) and 1(c), x_4^* is greater than unity so that it is not depicted.

[5] (15) is a cubic equation that has at most three roots. The shape of F indicates that one root has a negative second derivative while the other two have positive second derivatives. Since the graphs of F gets tangential to the diagonal from below, two roots with positive second derivative can not satisfy the two conditions just mentioned.

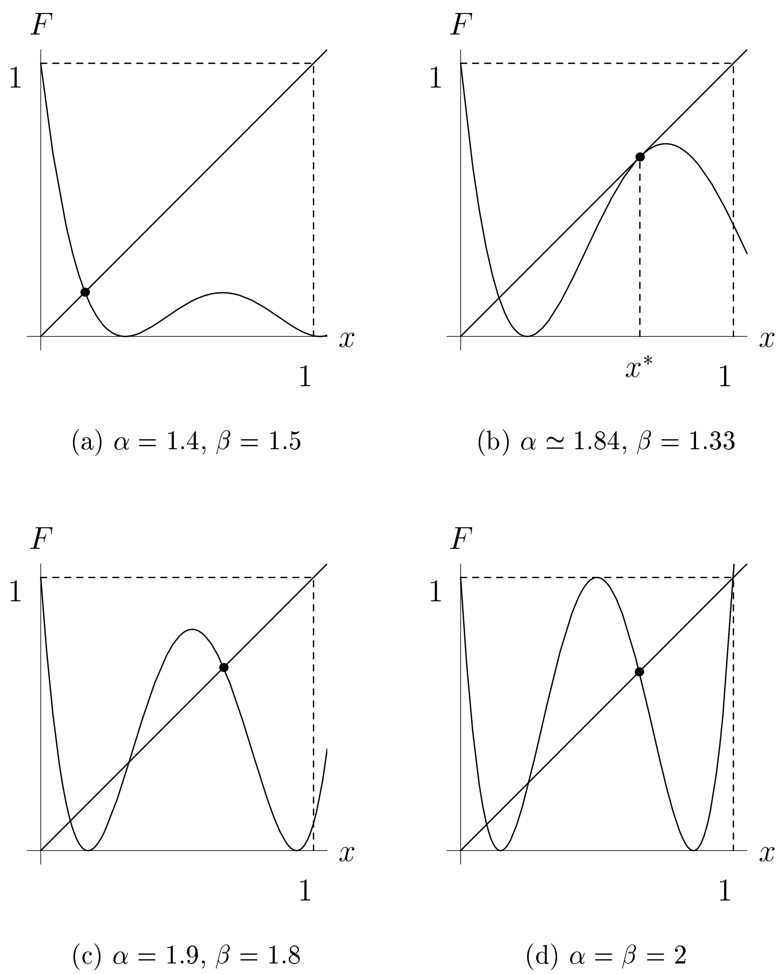


Figure 1: Determinations of fixed points of $F(x)$.

We summarize these results in the following theorem.

Theorem 1 *In the unit interval, $F(x)$ has one real root, x_1^*, for $(\alpha, \beta) \in A_1$, two real roots, x_1^* and $x_2^* = x_3^*$, for $(\alpha, \beta) \in A_2$, three real roots, x_1^*, x_2^* and x_3^* for $(\alpha, \beta) \in A_3$ but $\beta < 2$ and four real roots, x_1^*, x_2^*, x_3^* and x_4^* for $\alpha = \beta = 2$.*

3.2 Stability and Bifurcation of Stationary Points

To investigate the stability of each stationary point, we return to the two dimensional map, H, and linearize it to obtain its Jacobi matrix, J,

$$J = \begin{pmatrix} 0 & \dfrac{\partial r_1^*}{\partial y} \\ \dfrac{\partial r_2^*}{\partial x} & 0 \end{pmatrix}, \tag{17}$$

where an asterisk is attached to a derivative to imply that it is evaluated at the stationary point. Eigenvalues λ_1 and λ_2 of the characteristic equation satisfy

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0, \\ \lambda_1 \lambda_2 &= -\frac{\partial r_1^*}{\partial y}\frac{\partial r_2^*}{\partial x}. \end{aligned} \tag{18}$$

Let $\lambda = \lambda_1 = -\lambda_2$. Then a stationary point is locally stable if eigenvalues are less than unity in absolute value,

$$|\lambda^2| = \left| \frac{\partial r_1^*}{\partial y}\frac{\partial r_2^*}{\partial x} \right| < 1. \tag{19}$$

By definition of F, $F' = \frac{\partial r_1^*}{\partial y}\frac{\partial r_2^*}{\partial x}$. The stability condition (19) indicates that if the derivative of F is less than (greater than) unity in absolute value, the fixed point is stable (unstable). We can see that S_2 as well as S_4 is unstable for all $(\alpha, \beta) \in A$ because the slope of F at stationary point x_2^* as well as x_4^* is steeper than unity as illustrated in Figure 1(c) as well as Figure 1(d). On the other hand, the slope at x_1^* as well as x_3^* can be either greater or less than unity, so that S_1 as well as S_3 can be either of stable or unstable depending on the parameter combination of (α, β).

We first construct the sets of (α, β) for which S_1 is stable. By Theorem 1, for $(\alpha, \beta) \in A_1$, only S_1 exists. We can divide A_1 into following two subsets according whether the slope of F is greater than unity in absolute value or not,

$$\begin{aligned} B_1 &\equiv \{(\alpha, \beta) \in A_1 \mid |F'(x_1^*)| < 1\}, \\ B_2 &\equiv \{(\alpha, \beta) \in A_1 \mid |F'(x_1^*)| > 1\}, \end{aligned} \tag{20}$$

where S_1 is stable for $(\alpha, \beta) \in B_1$ and unstable for $(\alpha, \beta) \in B_2$. Since S_3 emerges for $(\alpha, \beta) \in A_3$, it is possible to divide A_3 into two according to, again, whether the same condition holds or not, but the derivative is evaluated at x_3^*.

$$B_3 \equiv \{(\alpha,\beta) \in A_3 \mid |F'(x_3^*)| < 1\},$$
$$B_4 \equiv \{(\alpha,\beta) \in A_3 \mid |F'(x_3^*)| > 1\}, \tag{21}$$

where S_3 is stable for $(\alpha,\beta) \in B_3$ and unstable for $(\alpha,\beta) \in B_2$. Then the above discussions are summarized in

Theorem 2 *(i) If $(\alpha,\beta) \in B_1$, S_1 is stable.*

(ii) If $(\alpha,\beta) \in B_2$, S_1 is unstable.

(iii) If $(\alpha,\beta) \in B_3$, S_1 is unstable and S_3 is stable.

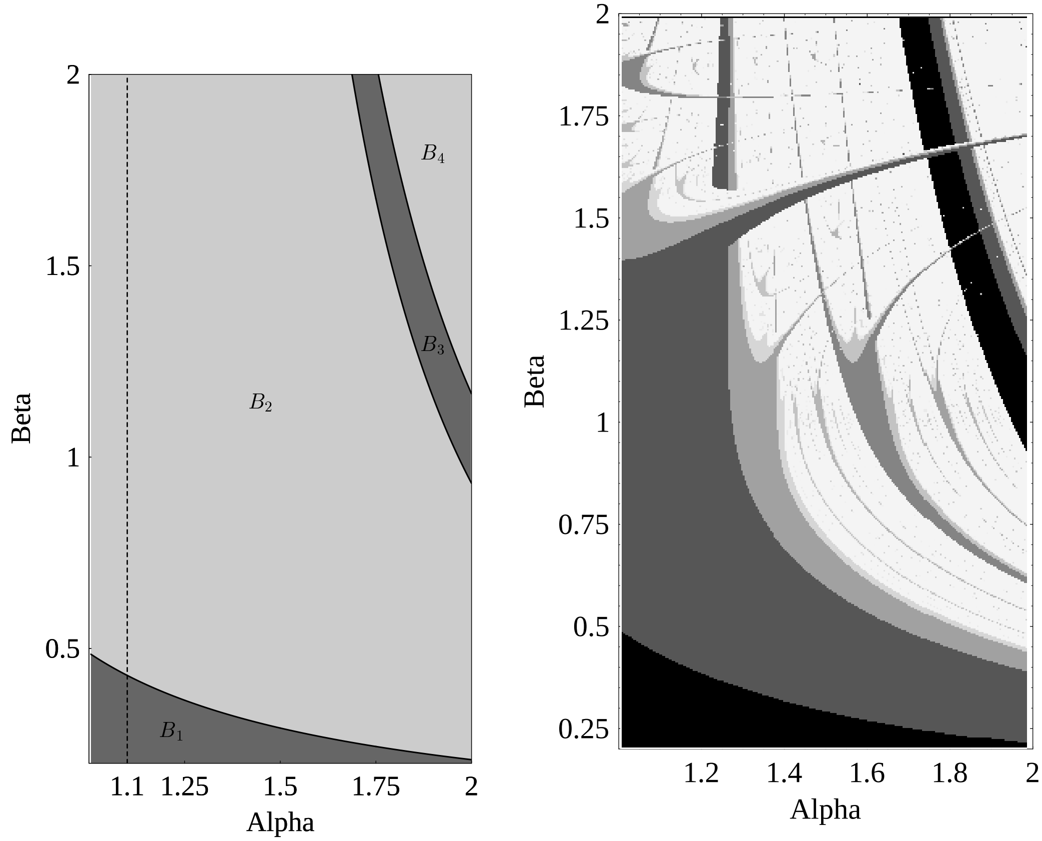
(iv) If $(\alpha,\beta) \in B_4$, neither S_1 nor S_3 is stable.

Even if a stationary point loses its local stability, the trajectories in the unit interval are always meaningful, being always bounded, that is, divergence cannot occur in the chosen parameter range. Figure 2(a) illustrates the parameter set A which is divided into four subsets, B_k $(k = 1,2,3,4)$.[6] Figure 2(b) presents a two-parameter bifurcation diagram that is a picture of bifurcation response of the adjustment process to changes in the parameters.[7] It is produced using a 250×450 grid of α and β where $1 \le \alpha \le 2$ and $0.2 \le \beta \le 2$. Different colors indicate different regions of stable, periodic or aperiodic behavior. In the black-colored region, either S_1 or S_3 is stable. That is, the black triangular-like region in the lower-left, and the band-like region in the upper-right part of Figure 2(b) correspond to B_1 and B_3 of Figure 2(a) respectively. Complex dynamics involving chaos occurs for (α,β) in the white region. To see how dynamics change, suppose first that the parameters are such that S_1 is stable (i.e., $(\alpha,\beta) \in B_1$). Then the bifurcation diagram implies that if either α or β increases enough, S_1 becomes unstable and chaotic dynamics occur after a sequence of period-doubling bifurcations. When the pair (α,β) is in the black band-like regions in B_3, the stationary point S_3 becomes stable and attracts trajectories. For other parameter values in B_3, it can be seen that there are regions of bistability. The periodicity tongues entering region B_3 from B_2 demonstrate that for such parameter values there are at least two attractors: the stationary point and a stable cycle. In fact, the initial condition used to perform the Figure 2(b) converges to the cycle, and we know from the stability analysis that in that region, S_3 is stable.[8] If the parameters increase further, a trajectory starts oscillating again and goes through a regime of period doublings into the chaotic state.

[6] Ignore the vertical dotted line for now.

[7] To draw the bifurcation diagram, we set the initial condition on firm 2's reaction curve such as $x_0 = \underline{x}$ and $y_0 = r_2(\underline{x})$ where $\underline{x}$ is the local minimum of F.

[8] Each attractor has initial-condition dependency. if an initial condition is such as $x_0 = \bar{x}$ and $y_0 = r_2(\bar{x})$ where $\bar{x}$ is the local maxmimum of F, it converges to S_3.



(a) stability condition

(b) bifurcation diagram

Figure 2: Stability and bifurcation of stationary points

4 Long-run Average Profit

In this section, we investigate the asymptotic behavior of the market interactions in the long-run. We calculate the profit at a stationary point as a reference point, and then the long-run average profit taken along a chaotic trajectory. We compare one with the other to find whether a firm can benefit from chaotic fluctuations or not. In Section 4.1 and Section 4.2, we focus on the case with $\alpha = \beta = 2$ in which we can analytically calculate the long-run average profit. In Section 4.3 we extend our investigation to other cases in which $\alpha \neq 2$ and / or $\beta \neq 2$. We perform numerical simulations to evaluate the long-run average profits of the two firms.

4.1 Stationary Profit

We calculate the profit of each firm at a stationary point (henceforth, stationary profit) when $\alpha = \beta = 2$. The reaction functions for this case are reduced to

$$r_1(y) = 4y(y-1) + 1 \text{ and } r_2(x) = 4x(x-1) + 1. \tag{22}$$

These intersect at four distinct points as illustrated in Figure 3,

$$S_1 = (s_1, s_3),\ S_2 = (s_2, s_2),\ S_3 = (s_3, s_1) \text{ and } S_4 = (s_4, s_4) \tag{23}$$

where

$$s_1 = \frac{3-\sqrt{5}}{8},\ s_2 = \frac{1}{4},\ s_3 = \frac{3+\sqrt{5}}{8} \text{ and } s_4 = 1. \tag{24}$$

It can be checked that $s_1 = x_1^* = y_3^*$, $s_2 = x_2^* = y_2^*$, $s_3 = x_3^* = y_1^*$ and $s_4 = x_4^* = y_4^*$. Substituting these stationary values into the profit functions (3), we obtain the

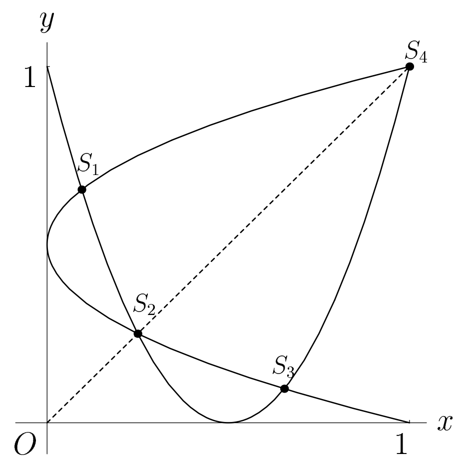


Figure 3: Reaction functions with $\alpha = \beta = 2$.

stationary profits,

$$\begin{aligned}
\Pi_1^1 &= \frac{7-3\sqrt{5}}{64} \text{ and } \Pi_2^1 = \frac{7+3\sqrt{5}}{64} \text{ at } S_1, \\
\Pi_1^2 &= \Pi_2^2 = \frac{1}{32} \text{ at } S_2, \\
\Pi_1^3 &= \frac{7+3\sqrt{5}}{64} \text{ and } \Pi_2^3 = \frac{7-3\sqrt{5}}{64} \text{ at } S_3, \\
\Pi_1^4 &= \Pi_2^4 = \frac{1}{2} \text{ at } S_4,
\end{aligned} \tag{25}$$

where Π_i^j indicates the profit for firm i at the stationary point S_j ($i = 1, 2$ and $j = 1, 2, 3, 4$). It is observed in Figure 3 that S_2 and S_4 are located on the diagonal while

S_1 and S_3 are off the diagonal but symmetric with respect to it. Accordingly, the stationary profits are the same for both firms at S_2 as well as S_4. The stationary profit at S_3 is larger than the one at S_1 for firm 1 while the ordering is reversed for firm 2. It can be seen that a firm makes larger profit when it produces larger output,

$$\begin{aligned} &\Pi_1^4 > \Pi_1^3 > \Pi_1^2 > \Pi_1^1, \\ &\Pi_2^4 > \Pi_2^1 > \Pi_2^2 > \Pi_2^3. \end{aligned} \tag{26}$$

4.2 Ergodic Chaos

To consider the long-run behavior, we first briefly review the basic properties of ergodic chaos in the one-dimensional logistic map, $\theta(x) = 4x(1-x)$, defined on the unit interval and introduce the mean ergodic theorem to investigate the statistical properties of ergodic chaos. Then, based on the results obtained in the one-dimensional map, we proceed to our analysis of the two-dimensional map, H.

It has been demonstrated that the logistic map, θ, exhibits chaos. Chaotic dynamics has two salient features, irregularity of trajectories and extreme sensitivity to the initial conditions. The former implies that a trajectory fluctuates so erratically that it is difficult to distinguish chaotic behavior of a deterministic process from truly random behavior of a stochastic process. The latter implies that even a slightly different choice of initial conditions can drastically alter the whole future (particularly, long-term) behavior of trajectories. Each chaotic trajectory moves in such a complicated way that there is no simple way to characterize its behavior. However, the frequencies of a trajectory $\{x_t\}_{t=0}^{\infty}$ generated by the logistic map can converge to a unique stable density function, $\varphi(x)$,

$$\varphi(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \tag{27}$$

which has the same properties as that of a stochastic process, $\varphi(x) \geq 0$ for all x and $\int_I \varphi(x)dx = 1$. Once the explicit form of a density function is constructed, it is possible to calculate the long-run average taken along the chaotic trajectory using the following mean ergodic theorem.[9]

Theorem 3 *Since the dynamical system $\theta(x_t) = 4x_t(1-x_t)$ is chaotic and ergodic, the time average of a function $f(x)$ associated with a chaotic trajectory, $\{x_t\}_{t=0,1,2...}$, equals to the space average,*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(\theta^t(x_0)) = \int_I f(x)\varphi(x)dx$$

where x_0 is an initial point, $\theta^t = \theta^{t-1}\cdot\theta$, θ^0 is an identical map, $f \in C^1$, and I is the support of density function φ.

[9]See for example, Chapter 8 of Day [4] for derivation of the explicit form of density function $\varphi(x)$ and an assertion of the following Theorem 3.

The dynamic process $H(x_t, y_t)$ is a two-dimensional map while the equality between the time average and the space average in Theorem 3 is confirmed only in a one-dimensional map. In spite of this, we can apply Theorem 3 to analyze the long-run dynamics generated by H. Since optimal behavior of each firm can be described by iterating the composite one-dimensional map, $r_i \circ r_j$, each firm behaves as if it moves one after the other in a iteration process. In particular, suppose that a trajectory of the output adjustment starts on firm 1's reaction curve (i.e., $(x_0, y_0) = (r_1(y_0), y_0)$) in period zero. The point is mapped to a point, $(x_1, y_1) = (x_0, r_2(x_0))$, on firm 2's reaction function in period one. In period two, the point is mapped again on the firm 1's reaction curve and then the iteration process repeats. Along the trajectory, the iterated point is on firm 1's reaction curve at every even period and on firm 2's reaction function at every odd period. That is, $(x_{2k}, y_{2k}) = (r_1(y_{2k}), y_{2k})$ and $(x_{2k+1}, y_{2k+1}) = (x_{2k+1}), r_2(x_{2k+1}))$ for $k = 0, 1, \ldots$. Thus the average profit of firm i over $M = 2N$ periods is

$$\begin{aligned}
\frac{1}{M}\sum_{t=0}^{M-1}\Pi_i(x_t, y_t) &= \frac{1}{2N}\{[\Pi_i(x_0, y_0) + \Pi_i(x_2, y_2) + \cdots + \Pi_i(x_{M-2}, y_{M-2})] \\
&\quad + [(\Pi_i(x_1, y_1) + \Pi_i(x_3, y_3) + \cdots + \Pi_i(x_{M-1}, y_{M-1})]\} \\
&= \frac{1}{2}\left\{\frac{1}{N}\sum_{k=0}^{N-1}\Pi_i(r_1(y_{2k}), y_{2k}) + \frac{1}{N}\sum_{k=0}^{N-1}\Pi_i(x_{2k+1}, r_2(x_{2k+1}))\right\} \\
&= \frac{1}{2}\left\{\frac{1}{N}\sum_{k=0}^{N-1}\Pi_i(x_{2k}, r_2(x_{2k})) + \frac{1}{N}\sum_{k=0}^{N-1}\Pi_i(r_1(y_{2k+1}), y_{2k+1})\right\}.
\end{aligned}$$

Transition from the first line to the second is carried out in the following way. Assume now that the trajectory starts on firm 2's reaction curve. Then at every even period, it is on firm 2's reaction curve and firm i's profit is written as $\Pi_i(x_{2k}, r_2(x_{2k}))$. On the other hand, at every odd period, the trajectory is on firm 1's reaction function and thus firm i's profit is written as $\Pi_i(r_1(y_{2k+1}), y_{2k+1}))$. The same reasoning applies to the transition from the first to the third. Therefore, as shown above, the average profit over M periods is the average of the sum of the average profits taken along these reaction curves.

When a trajectory is chaotic, the average profit is the limiting value of the finite average,

$$\begin{aligned}
&\lim_{M\to\infty}\frac{1}{M}\sum_{t=0}^{M-1}\Pi_i(x_t, y_t) \\
&= \frac{1}{2}\left\{\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\Pi_i(r_1(y_{2k}), y_{2k})) + \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\Pi_i(x_{2k+1}, r_2(x_{2k+1}))\right\}. \quad (28)
\end{aligned}$$

The reaction functions defined in (22) can be transformed to logistic maps, $4z(1-z)$ by applying a shift map $z = 1 - x$ or $z = 1 - y$. According to Theorem 3, each term in the brace in (28) can converge to its space average which is the limiting value of the finite average.

$$
\begin{aligned}
\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \Pi_i(r_1(y_{2k}), y_{2k}) &= \int_0^1 \Pi_i(r_1(y), y)\varphi(y)dy, \\
\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \Pi_i(x_{2k+1}, r_2(x_{2k+1})) &= \int_0^1 \Pi_i(x, r_2(x))\varphi(x)dx
\end{aligned} \tag{29}
$$

where $\varphi(\cdot)$ is the invariant density function defined in (27). Summing up our findings, we have

Theorem 4 *For the profit function $\Pi_i(x, y)$ $(i = 1, 2)$, the average profit of firm i taken along a trajectories equals to the average of the sum of the average profit taken along each reaction curve,*

$$
\lim_{M\to\infty} \frac{1}{M}\sum_{t=0}^{M-1} \Pi_i(x_t, y_t) = \frac{1}{2}\left\{\int_0^1 \Pi_i(r_1(y), y)\varphi(y)dy + \int_0^1 \Pi_i(x, r_2(x))\varphi(x)dx\right\}
$$

where $r_1(y) = 4y(y-1)+1$, $r_2(x) = 4x(x-1)+1$ and $\varphi(u) = \frac{1}{\pi\sqrt{u(1-u)}}$ $(u = x, y)$.

Applying Theorem 4, we can calculate the average profit of firm 1 as well as firm 2. For $\alpha = \beta = 2$, the long-run average profits of both firms are the same and given by

$$
\begin{aligned}
\bar{\Pi}_i &\equiv \lim_{M\to\infty} \frac{1}{M}\sum_{t=0}^{M-1} \Pi_i(x_t, y_t) \\
&= \frac{1}{2}\left\{\int_0^1 \frac{(1-2u)^4}{2\pi\sqrt{u(1-u)}}du + \int_0^1 \frac{u(2(8u^2-8u+1)^2-u)}{2\pi\sqrt{u(1-u)}}du\right\} \\
&= \frac{1}{8}, \quad i = 1, 2.
\end{aligned}
$$

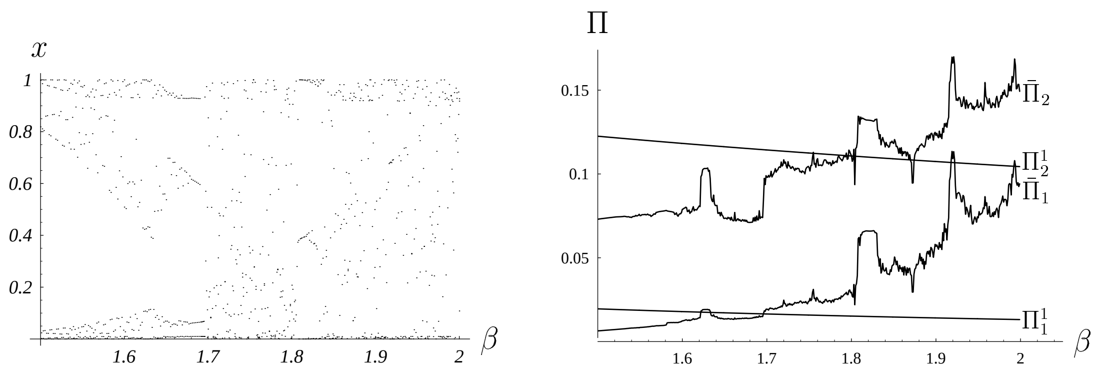
By (26), the orderings of the stationary profits and the long-run average profit for firms 1 and 2 are

$$
\begin{aligned}
\Pi_1^4 > \Pi_1^3 > \bar{\Pi}_1 > \Pi_1^2 > \Pi_1^1, \\
\Pi_2^4 > \Pi_2^1 > \bar{\Pi}_2 > \Pi_2^2 > \Pi_2^3.
\end{aligned} \tag{30}
$$

Since stationary point S_4 is special in the sense that it can emerge if and only if $\alpha = \beta = 2$, we eliminate this point for the time being. Then (30) indicates that stationary point S_3 is the best for firm 1 but the worst for firm 2 while stationary point S_1 is the best for firm 2 but the worst for firm 1. The long-run average profit is the second-best for both firms. A question which we naturally raise is whether this ordering can be held when parameters are not equal to 2. We are able to construct an explicit form of chaotic density only in the case of $\alpha = \beta = 2$ and not for any other cases. In consequence, the mean ergodic theorem is not applicable for any other cases. In order to answer this question, we perform numerical simulations in the following.

4.3 Numerical Simulations

In this subsection we consider the long-run average behavior in cases in which neither α nor β is equal to 2. Figure 4 shows the results of numerical simulations along the dotted line in Figure 2(a). In these simulations, we fix $\alpha = 1.1$ and increase β in steps of 0.01 from 1.5 to 2.0. Figure 4(a) illustrates the one parameter bifurcation diagram and Figure 4(b) illustrates the loci of the long-run average profits of firm 1 (i.e., $\bar{\Pi}_1$) and firm 2 (i.e., $\bar{\Pi}_2$) as well as the stationary profits (i.e., Π_1^1 and Π_2^1).[10] The loci of the stationary profits are slightly downward sloping while the loci of the long-run average profit exhibit upward tendency with many ups and downs. As a result, the locus of the long-run average profit crosses the locus of the stationary profit for each firm. As confirmed in Theorem 1, only stationary point S_1 can emerge along the dotted line. We observe in Figure 4(b) that given $\alpha = 1.1$, the long-run average profits of firm 1 is higher than the corresponding stationary profit when the value of β is larger than about 1.7 and the long-run average profit of firm 2 is higher when β is larger than about 1.8.



(a) bifurcation cascade (b) variation of the average profits

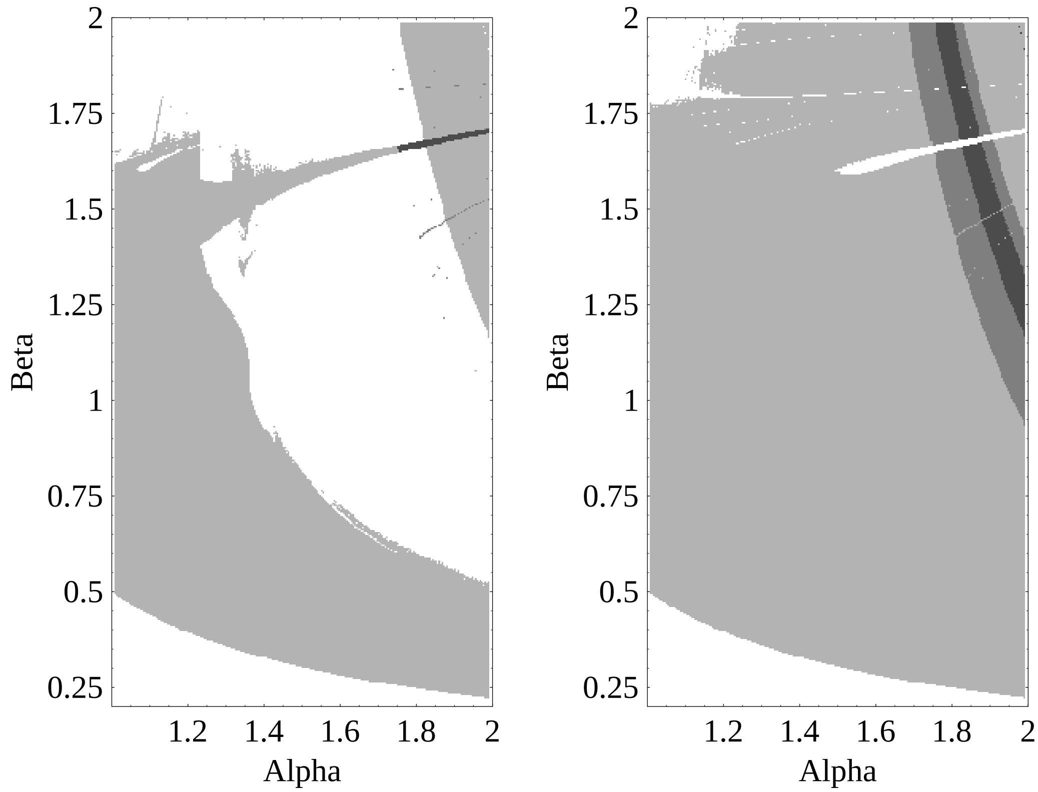
Figure 4: An example of beneficial chaos ($\alpha = 1.1$).

This numerical example indicates the possibility that the long-run average profits for *both* firms can be larger than the stationary profits for larger values of β. We perform further numerical simulations in order to confirm whether the finding is robust. The simulations proceed as follows. We first set $\alpha = 1.01$, then increase β from 0.2 to 2 in steps of 0.01 and calculate stationary profits at all possible stationary points as well as the long-run average profits for both firms,[11] and compare the stationary profits with the long-run profit and determine the ordering of those profits for each value of β. We increase α by 0.01 and repeat the same procedure until α became 2. The results are illustrated in Figure 5. White-colored regions corresponds to subsets of the parameter set A in which the average profit is larger than or equal to any possible stationary

[10] In these figures, we set $x_0 = 0.4$ and $y_0 = r_2(0.4)$.

[11] To calculate the long-run average profits, we use the same initial condition as used in Figure 4. We will have the same calculation result if the initial condition is on either of the two reaction curves.

profits, and gray-colored regions to subsets in which one of the stationary profits is higher than the long-run average profit. Comparing Figure 5(a) with Figure 5(b), we find the following:



(a) Firm 1

(b) Firm 2

Figure 5: Long-run average profits.

(i) In the triangular-like white regions at the bottom-left, the long-run average converges to the stationary value because stationary point S_1 is stable.

(ii) The white-colored region for firm 1 in the middle of Figure 5(a) is much larger than the one for firm 2 in the top-left corner of Figure 5(b).

(iii) In both Figure 5(a) and 5(b), there exists a common set of parameters for which the long-run average profit is strictly larger than the stationary profit.

The simulation result (iii) is particularly noteworthy and shows a sharp contrast with the results in the traditional economics where a chaotic fluctuation is considered to be an unfavorable phenomenon as they do not converge to a stationary point. Our results show the possibility that the long-run average profit of both firms can be higher than stationary profits. Returning to (1) and (6)-(8), we find that α is the cross effect on price of x caused by a change in output y, and β is the cross effect on price of y caused by a change in output x. Therefore, this finding implies that chaotic dynamics can be favorable from the long-run point of view when a strong asymmetry exists between the cross effects.

5 Concluding Remarks

In this paper, we have constructed a Cournot model in which goods are complementary and reaction functions are non-linear. We have demonstrated analytically as well as numerically the following two main results. The first result is that the double externalities, that is positive sales externality via the market demand and negative production externality via the cost function, can be sources of chaotic output fluctuations in complementary goods markets. The second is that the long-run average profit of each firm can be strictly higher than the stationary profit when the sales externality of firm 1 is small and of firm 2 is large (i.e., smaller α and larger β). These results imply that firms producing complementary goods and facing chaotic fluctuations of the output adjustment possibly have higher average profits in the long-run when asymmetry on the positive sales effect is strong.

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