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Toichiro Asada
Chuo University

Akio Matsumoto
Chuo University

Ferenc Szidarovszky
University of Pécs

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INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

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Toichiro Asada,† Akio Matsumoto‡ and Ferenc Szidarovszky§

Abstract

In this study we demonstrate the possibility of persistent oscillations of national income in ala Marxian growth model with consumption and investment sectors. Three main results are shown. First, the existence of the steady state is confirmed. Second, the discrete-time dynamic model augmented with pollution effects generates simple as well as complex dynamics. Finally, the continuous-time dynamic model with production delay gives rise to regular as well as erratic dynamics. These findings indicate that the Marxian growth model with nonlinearity and production delay may explain various dynamic phenomena of economic variables.

Keywords: Marxian growth model, Pollution effect, Production delay, Complex dynamics, Discrete-time dynamics, Continuous-time dynamics

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†Professor, Department of Economics, Senior Researcher, International Center for further Development of Dynamic Economic Research, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, asada@tamacc.chuo-u.ac.jp

‡Professor, Department of Economics, Senior Researcher, International Center for further Development of Dynamic Economic Research, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, akiom@tamacc.chuo-u.ac.jp

§Professor, Department of Applied Mathematics, University of Pécs, Ifjúság u. 6., H-7624, Pécs, Hungary, szidarka@gmail.com

1 Introduction

Applying the recent development of nonlinear dynamic theory to a Marxian two-sector growth model, we aim to reconsider the provocative result, "the inevitable breakdown of capitalism," that scientific socialism advocates. We first demonstrate convergence to a steady state of the growth model and then the possible birth of persistent oscillations that is inevitable phenomenon observed in an actual economy.

It has been well known that the two underpinnings of Marxian economics are the theory of surplus value and the theory of historical materialism. The former measures worker exploitation by capitalism that might be a source of the instability of the capitalist system. The latter describes the law of society that social phenomena obey as natural phenomena obey the law of nature. Accordingly social evolution to the socialist society from the capitalist society through the end of capitalism is historical inevitability. It is often mentioned that Japanese Marxian economist, Nobuo Okishio, formulated the theory of surplus value and proved many Marxian theorems. Among others, Okishio (1955) confirmed that the exploitation of surplus labor is the necessary condition for the existence for positive profit, which was later called "Marxian Fundamental Theorem" by Japanese mathematical economist, Michio Morishima. Recently, mathematical formalization of the historical materialism has begun (see Yamashita and Ohnishi (2002), Ohnishi (2014) and , for example). Using the basic structure of the neoclassical optimal growth model, a two-sector growth model with labor being the primary factor of production is constructed. It shows historical transition from the beginning of the capital accumulation caused by the Industrial Revolution to the end of the capitalism at which the capital accumulation arrives at the saturation point and thus stops further development. In the existing literature of Marxian economics, however, not much has yet been revealed with respect to the circumstances under which capital accumulation exhibits persistence cyclic oscillations that are frequently observed in actual economic data. In particular, many researchers have been involved to explain why an economy fluctuates since the pioneering works by Kalecki (1935), Samuelson (1939), Kaldor (1940), Hicks (1950), Goodwin (1951), to name only a few. The main purpose of this study is to demonstrate the possibility of persistent fluctuations of national income as well as capital accumulation under Marxian circumstances when both negative effects on production caused by the capital accumulation and production delays are taken into account.

The rest of the paper is organized as follows. In the next section, we reconstruct a simplified Marxian two-sector growth model and confirm saturation of the capital accumulation. In Section 3, we examine dynamics in discrete-time scales after introducing time-to-build technology and pollution effect of the capital. In Section 4 the same dynamic model is cast into a continuous-time framework focusing on a destabilizing role of the production delay. Section 5 offers concluding remarks.

2 Marxian Growth Theory

Recapitulating the basic elements of the Marxian growth model examined by Takahashi (2011), we consider an economy with two sectors in which capital and consumption goods are produced in the capital goods sector and in the consumption goods sector, respectively. It is assumed that both sectors are characterized by Cobb-Douglas production functions. Denoting production of capital goods (i.e., investment) by Y_1 and production of consumption goods by Y_2,

$$Y_1 = A_1 K_1^{\beta} L_1^{1-\beta},\ 0 < \beta < 1 \tag{1}$$

and

$$Y_2 = A_2 K_2^{\alpha} L_2^{1-\alpha},\ 0 < \alpha < 1 \tag{2}$$

where A_i denotes the production technology level, K_i and L_i are capital and labor in sector i $(i = 1, 2)$ satisfying

$$A_i > 0,\ K_1 + K_2 = K > 0 \text{ and } L_1 + L_2 = L > 0.$$

K and L are the total amount of capital goods and total population. L is assumed to be constant only for the sake of analytical simplicity whereas the capital goods is changing according to the capital accumulation equation,

$$\dot{K}(t) = Y_1(t) - \delta K(t) \tag{3}$$

where the dot over a variable means time derivative and $\delta > 0$ is the depreciation rate of capital. Equilibrium in the consumption goods market is described by

$$Y_2(t) = wL$$

where w is the wage rate. It is further assumed that production factors are constant proportions of the existing total amounts,

$$K_1 = \gamma K\ ,\ K_2 = (1-\gamma)K \text{ with } 0 < \gamma < 1 \tag{4}$$

and

$$L_1 = \varepsilon L,\ L_2 = (1-\varepsilon)L \text{ with } 0 < \varepsilon < 1. \tag{5}$$

The next task is to construct a dynamic equation of the model by substituting (4), (5) into (3),

$$\dot{K}(t) = A_1 \left[\gamma K(t)\right]^{\beta} (\varepsilon L)^{1-\beta} - \delta K(t).$$

Dividing both sides of the last equation by L and introducing the new variable, $k = K/L$, present an accumulation equation of the percapita capital,

$$\dot{k}(t) = A_1 \left[\gamma k(t)\right]^{\beta} \varepsilon^{1-\beta} - \delta k(t)$$

that is simplified as

$$\dot{k}(t) = a k(t)^{\beta} - \delta k(t) \text{ with } a = A_1 \gamma^{\beta} \varepsilon^{1-\beta} > 0. \tag{6}$$

This is the *fundamental equation* of the Marxian growth model. The steady state of the percapita capital is

$$k^* = \left(\frac{a}{\delta}\right)^{\frac{1}{1-\beta}}.$$

Under the well-behaved Cobb-Douglas production function, the stability of k^* is confirmed, that is, $k(t)$ approaches k^* monotonically as $t \to \infty$, regardless of the initial value of $k(0) > 0$ and $k(0) \neq k^*$. We summarize the results obtained:

Theorem 1 *The steady state values are obtained as*

$$K^* = k^* L,\ Y_1^* = a_1 L \text{ and } Y_2^* = a_2 L$$

where

$$a_1 = A_1 \gamma^\beta \varepsilon^{1-\beta} (k^*)^\beta > 0 \text{ and } a_2 = A_2 (1-\gamma)^\alpha (1-\varepsilon)^{1-\alpha} (k^*)^\alpha > 0$$

and each variable monotonically converges to its corresponding equilibrium value as $t \to \infty$.

Since Y_2^* can be rewritten as

$$Y_2^* = a_2'(1-\varepsilon)^{1-\alpha}\varepsilon^\alpha \tag{7}$$

with

$$a_2' = A_2(1-\gamma)^\alpha \left(\frac{A_1\gamma^\beta}{\delta}\right)^{\frac{a}{1-\beta}} L,$$

differentiating equation (7) with respect to ε, equating it to zero and solving the resultant equation yields the optimal labor allocation ratio that maximizes consumption Y_2^*,

$$\varepsilon^* = \alpha.$$

Notice the resemblance between the Marxian and neoclassical (i.e., Solow-Swan) growth models. The optimal labor allocation that maximizes consumption is determined in the Marxian model where as the optimal saving rate that maximizes consumption is determined in the neoclassical model.

3 Pollution Effect and Production Delay

We have seen that all trajectories of $k(t)$ monotonically approach the steady state in the continuous-time framework. In a real economy, we often observe first that production process may require gestation (i.e., time-to-build technology) and/or information lag and second that production process could be described by strong nonlinear production functions. Considering these points, we modify the model (6) by augmenting sufficient nonlinearities and a time delay. Concerning nonlinearities, we consider a "pollution effect" caused by increasing concentration of capital according to a multiplicative term, $(m-k)^\eta$ with $\eta > 0$.

The production technology is modified such that per capita output is produced according to the function,

$$y = ak^{\beta}(m-k)^{\eta} \tag{8}$$

For a smaller k, the first factor is small and this term does not make any essential contribution to per capita production. On the other hand, for a larger k close to m, productivity is largely reduced due to a negative (i.e., pollutant) effect on per capita production. For the sake of analytical simplicity, we assume $\beta = 1$, $\eta = 1$ and $m = 1$ with which the production function becomes

$$y = ak(1-k). \tag{9}$$

This is considered to be the AK production function taking the influence of pollution into account as a first approximation. Concerning the time delay, we reformulate the model in discrete time in which economic activities and decisions are made at discrete time intervals. The aim of this section is to verify whether the dynamic emerging in the fundamental equation in continuous time are robust with respect to these modifications. Replacing $\dot{k}(t)$ with $k_{t+1} - k_t$ reduces model (6) to

$$k_{t+1} = ak_t(1-k_t) + (1-\delta)k_t. \tag{10}$$

Introducing the new variable

$$x = \frac{a}{a+(1-\delta)}k$$

reduces equation (10) to the logistic equation

$$x_{t+1} = \theta x_t(1-x_t) \text{ with } \theta = a + (1-\delta). \tag{11}$$

A positive fixed point is obtained for $\theta > 1$,

$$x^* = \frac{\theta - 1}{\theta}. \tag{12}$$

The dynamics of k_t generated by the fundamental equation in discrete time is equivalent to the dynamics of x_t controlled by the logistic equation (11). The dynamic structure of the logistic equation has been extensively studies so far and it is now well-known that it can generate a wide variety of dynamics ranging from monotonic convergence to chaotic fluctuations, depending on the value of the parameter θ. For $0 < \theta \leq 1$, the fixed point is zero, which is eliminated for further investigation since it is economically unrealistic. We will perform numerical simulations under different values of $\theta > 1$.

In the first simulation illustrated in Figure 1, we start with an initial condition $x_0 = 0.8$ and θ is taken to be 1.5. The percapita capital quickly and monotonically converges to the positive fixed point.

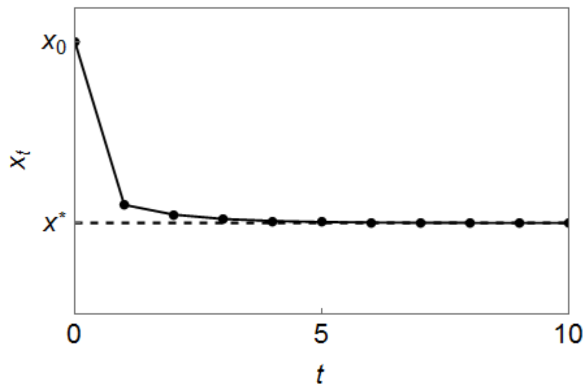

Figure 1. Monotonic convergence to x^*

The second simulation illustrated in Figure 2 is performed taking the same initial value of x_0 and $\theta = 2.9$. The time trajectory shows dumping oscillations approaching to the positive fixed point.

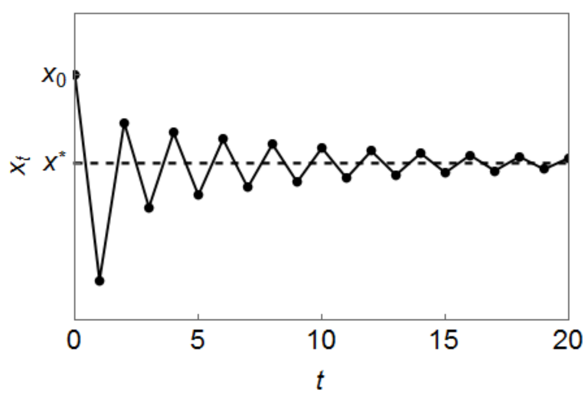

Figure 2. Oscillatory convergence to x^*

We still take the same initial value of x but increase the value of θ to 3.5. The resulting third simulation is illustrated in Figure 3 in which the time trajectory shows persistent oscillations (i.e., a period-4 cycle). The fixed point is locally unstable since the derivative of the logistic equation evaluated at the fixed point is greater than unity in absolute value,

$$\left| \frac{dx_{t+1}}{dx_t} \right| > 1.$$

However the nonlinearity of the logistic equation prevents unstable trajectories from diverging. In consequence the time trajectory neither converges to the fixed point nor diverges in the large.

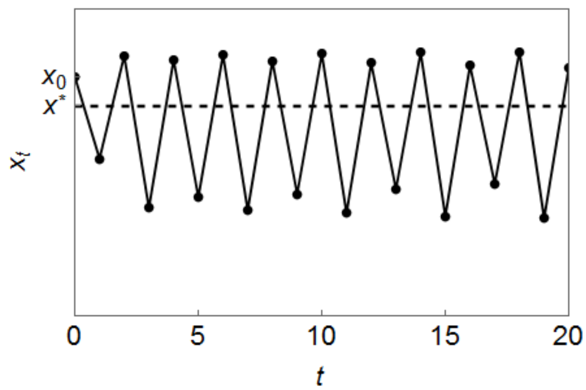

Figure 3. Periodic oscillations around x^*

The fourth and last simulation was done with the initial value $x_0 = 0.45$ and $\theta = 4$.[1] As is seen in Figure 4, the time trajectory exhibits persistent and irregular (i.e., chaotic) oscillations.

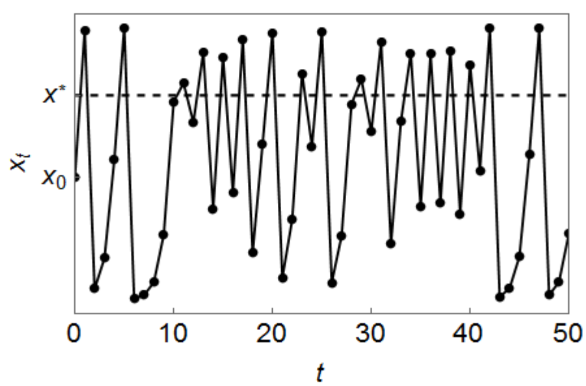

Figure 4. Chaotic oscillations around x^*

We can summarize the simulations results as follows:

Theorem 2 *The steady state of k is defined by*

$$k^* = \frac{\theta - 1}{\theta - 1 + \delta} \text{ for } \theta > 1$$

[1] For $\theta > 4$, the dynamic equation (10) generates time trajectories going to negative infinity. Such dynamics is economically meaningless and thus is also eliminated from further investigation.

and follows the dynamics described as

> *(1) monotonic convergence to zero for* $0 < \theta \leq 1$,
> *(2) monotonic convergence to* k^* *for* $1 < \theta \leq 2$,
> *(3) oscillatory convergence to* k^* *for* $2 < \theta \leq 3$,
> *(4) periodic oscillations around* k^* *for* θ *being close to* 3,
> *(5) aperiodic oscillations around* k^* *for* θ *being close to* 4.

4 Hopf Bifurcation of Capital Accumulation

In this section we focus on the case in which economic activities and decisions run continuously and then investigate the effects caused by the production delay in the continuous-time scales.[2] To this end, we return to equation (6) with a different type of production function that possesses essentially the similar nonlinear property,

$$y = ak^{\beta}e^{-\eta k}$$

with $\eta > 0$. This function takes a mound-shaped profile that has zero value at $k = 0$, increases for $k < \beta/\eta$, decreases for $k > \beta/\eta$ and converges to zero as $k \to \infty$. It has the maximum value at $k = \beta/\eta$, implying that the negative pollutant effect caused by increasing concentration of capital is defined for any $k > \beta/\eta$. With this modified function including production delay and assumption $\beta = 1$,[3] the dynamic equation becomes

$$\dot{k}(t) = ak(t-\tau)e^{-\eta k(t-\tau)} - \delta k(t) \tag{13}$$

Multiplying both sides of the above equation by η and introducing the new variable $x(t) = \eta k(t)$, simplify the expression,

$$\dot{x}(t) = ax(t-\tau)e^{-x(t-\tau)} - \delta x(t). \tag{14}$$

The steady state is defined by $\dot{x}(t) = 0$ and solves the following equation,

$$x\left(ae^{-x} - \delta\right) = 0.$$

The trivial steady state is $x = 0$ and the non-trivial steady state denoted by x^* is

$$x^* = \log\left[\frac{a}{\delta}\right] \tag{15}$$

where, to ensure a positive value of the nontrivial steady state, it is assumed throughout the paper that

$$a > \delta.$$

This inequality assumption is natural since the production technology a contributes to increase the production level (i.e., $a > 0$) and the depreciation rate δ is usually not greater than unity (i.e., $\delta \leq 1$).

[2] This section depends on Matsumoto and Szidarovszky (2011, 2013).
[3] See Matsumoto and Szidarovszky (2013) for the case with $\beta \neq 1$.

Set $z(t) = x(t) - x^*$, then $z(t)$ satisfies

$$\dot{z}(t) = -\delta z(t) - \delta x^* \left(1 - e^{-z(t-\tau)}\right) + \delta z(t-\tau)e^{-z(t-\tau)}.$$

The linearization of the last equation at $z(t) = 0$ is

$$\dot{z}(t) = -\delta z(t) - \delta\left(x^* - 1\right) z(t-\tau)$$

and its characteristic equation with a possible solution $z(t) = e^{-\lambda t}u$ is

$$\lambda + \delta + \delta\left(x^* - 1\right) e^{-\lambda\tau} = 0. \tag{16}$$

It is confirmed that the nontrivial state is locally asymptotically stable for $\tau = 0$ since $\delta > 0$ implies $\lambda = -\delta x^* < 0$. For $\tau > 0$, it is clear that $\lambda = 0$ does not solve the characteristic equation. At a stability switch, we suppose that $\lambda = i\omega$ with $\omega > 0$ is a solution,

$$i\omega + \delta + \delta(x^* - 1)\left(\cos\tau\omega - i\sin\tau\omega\right) = 0.$$

Separating the real and imaginary parts, we obtain two equations,

$$\delta(x^* - 1)\cos\tau\omega = -\delta \tag{17}$$

and

$$\delta(x^* - 1)\sin\tau\omega = \omega \tag{18}$$

Adding the squares of these equations yields

$$\omega^2 = \delta^2 x^*(x^* - 2).$$

ω can be positive if and only if $x^* > 2$ or equivalently if $a > \delta e^2$. Let the positive root of ω be denoted by ω_0,

$$\omega_0 = \delta\sqrt{x^*(x^* - 2)} > 0$$

and solving the real part equation for τ yields

$$\tau_m = \frac{1}{\omega_0}\left[\cos^{-1}\left(-\frac{1}{x^* - 1}\right) + 2m\pi\right] \text{ for } m = 0, 1, 2, ... \tag{19}$$

Solving the imaginary part equation yields the same value of the delay in a different form.

Differentiating the characteristic equation with respect to τ presents

$$\left[1 - \delta(x^* - 1)e^{-\lambda\tau}\tau\right]\frac{d\lambda}{d\tau} - \delta(x^* - 1)\lambda e^{-\lambda\tau} = 0$$

which is solved for $(d\lambda/d\tau)^{-1}$

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{1 - \delta(x^* - 1)e^{-\lambda\tau}\tau}{\delta(x^* - 1)\lambda e^{-\lambda\tau}} \\ &= \frac{1}{\delta(x^* - 1)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}.\end{aligned}$$

Using the characteristic equation,

$$\delta(x^* - 1)e^{-\lambda\tau} = -(\lambda + \delta),$$

substituting $\lambda = i\omega$ and taking the real part yield

$$\begin{aligned}
\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega}\right] &= \mathrm{Re}\left[\left.\frac{1}{-\lambda(\lambda+\delta)}\right|_{\lambda=i\omega}\right] \\
&= \mathrm{Re}\left[\frac{1}{\omega^2 - i\delta\omega}\right] \\
&= \frac{1}{\omega^2 + \delta^2} > 0.
\end{aligned}$$

So at the smallest critical value τ_0 stability is lost and cannot be regained later.
We then have the following results:

Theorem 3 *For the capital accumulation equation, the following hold:*

(1) *If $\delta < a \leq \delta e^2$, then $k^* = x^*/\eta$ is locally asymptotically stable.*
(2) *If $a > \delta e^2$, then $k^* = x^*/\eta$ is asymptotically stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$.*
(3) *For $a > \delta e^2$, $k^* = x^*/\eta$ loses stability and undergoes Hopf bifurcation at $\tau = \tau_0$,*

where τ_0 is defined as

$$\tau_0 = \frac{\cos^{-1}\left(-\frac{1}{\eta k^* - 1}\right)}{\delta\sqrt{\eta k^*(\eta k^* - 2)}}.$$

We now perform simulations to confirm the theoretical results. We take $\alpha = 5$, $\eta = 1$ and $\delta = 0.2$ for which the positive steady state is ensured. Further the critical value of the length of the delay is

$$\tau_0 = \frac{\cos^{-1}\left(-\frac{1}{k^* - 1}\right)}{\delta\sqrt{k^*(k^* - 2)}} \simeq 5.145.$$

In the first simulation, we take $\tau = 3 < \tau_0$ and thus have convergence to the steady state k^* as shown in Figure 5.

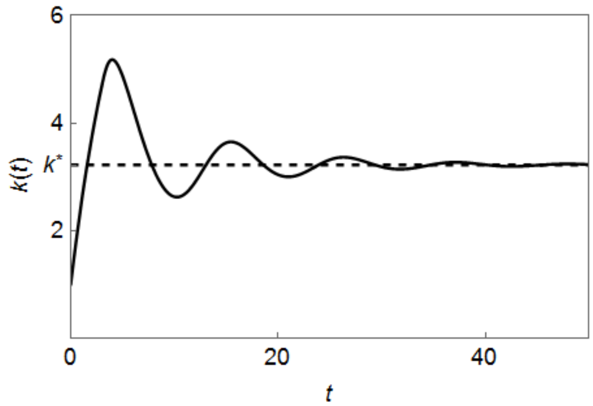

Figure 5. Oscillatory convergence

In the second simulation, we increase the value of τ to $6 > \tau_0$. which violates the stability condition. As a result, a limit cycle emerges via a Hopf bifurcation according to parts (2) and (3) of Theorem 2.

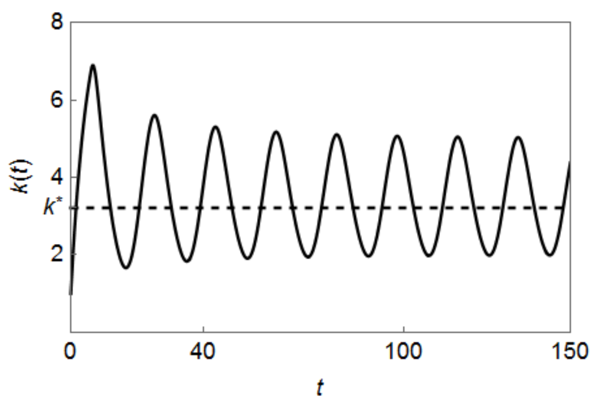

Figure 6. Emergence of limit cycle

In the third simulation, we further increase the value of τ to 30, with which the delay equation (13) gives rise to erratic behavior of $k(t)$.

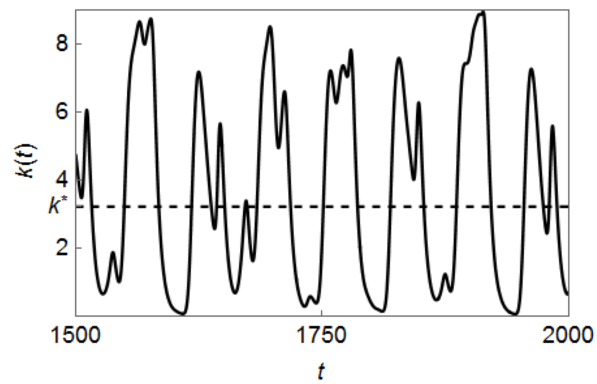

Figure 7. Chaotic oscillations

5 Concluding Remarks

In Section 2 of this paper, we have reconstructed a Marxian two sector growth model that is similar to the neoclassical growth model and then shown that any trajectory converges to the positive steady state. From the Marxian economic point of view, this convergence implies the end of capitalism in a sense that the economy ceases to grow. In Sections 3 and 4, we have demonstrated the existence of persistent cyclic fluctuations in the same model if we take into two essential elements, nonlinearity caused by the strong capital accumulation and a delay in the production process. In particular, dynamics in the discrete-time framework exhibits wide variety of dynamics ranging from monotonic convergence to complex dynamics involving chaos whereas dynamics in the continuous-time framework is characterized by the stability switching curve: the steady state is asymptotically stable if the length of the delay is below the curve, loses stability just on the curve and bifurcates to a limit cycle if above the curve. With these results the paper highlights the fact that nonlinearity and delay are essential ingredients for the emergence of persistent oscillations even in the simplified Marxian economic growth model.

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