Discrete-Time Delay Dynamics of Boundedly Rational Monopoly*

Akio Matsumoto
Chuo University

Ferenc Szidarovszky
Science University of Pecs

November 29, 2011

Abstract

This paper discusses delay dynamics of monopoly with discrete time scales. It is assumed that a monopoly has delayed and limited information on demand. Such a boundedly rational monopoly forms the demand expectations using past observed data and adopts the gradient of the marginal profit to determine its output adjustments. In the case of one-step delay in which the expectation is equal to output produced in the previous period, it is shown that the steady state undergoes a period-doubling cascade to chaos. In the cases of two and three delays where data at one, two and three previous time periods are available, it is shown that the steady state undergoes to complex dynamics through either a period-doubling or a Neimark-Sacker bifurcation, depending on the specified values of the parameters. Numerical examples illustrate the theoretical results. Finally the case of geometric delay is also analyzed to show the birth of the period-doubling bifurcation.

Key words: Bounded rationality, Delay with discrete-time scales, Gradient dynamics, Hopf bifurcation, Neimark-Sacker bifurcation

*The authors highly appreciate financial supports from the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (C) 21530172), Chuo University (Joint Research Grant 0981) and the Japan Economic Research Foundation. The paper was prepared when the second author visited the Department of Economics, Chuo University. He appreciated its hospitality over his stay. The usual disclaimer applies.

1 Introduction

Most dynamic models examined in the mathematical economics literature assume the existence of simultaneous information about the markets as well as about the competitors. However in real economies there is always a time-lag because finding optimal decisions and implementing them always result in time delays. In many cases the firms are willing to react to an average of past data instead of following sudden market changes and fluctuations. If continuous time scales are assumed, then time lags can be modeled with fixed delays or by assuming continuously distributed delays. A comprehensive summary of the relevant mathematical methodology can be found, for example, in Bellman and Cooke (1956) and Cushing (1977). Continuous-time dynamic monopoly with a single or two fixed delays is examined in Matsumoto and Szidarovszky (2011a), the cases of continuously distributed delays are discussed in Matsumoto and Szidarovszky (2011b).

In the cases of discrete time scales it is usually assumed that the delays are integer multiples of the unit step, so delayed models can be rewritten as higher order discrete systems. This is the approach that is followed in this paper, which can be considered as the discrete time counter part of our earlier papers mentioned above.

This paper is a straightforward extension of the delay monopoly model with bounded rationality considered by Agiza *et al.* (2001) and reconsidered later by Sun (2011). When demand information is uncertain or incomplete, the monopolistic firm bases its expectations on past observed data. It is customary to assume naively expected demand that is equal to the output actually produced in the previous period. Since naive expectation formed at the current period is based on the past data obtained at one period before, we say that there is a one-step delay in the naive expectation formation. In Agiza's model, the one-step delay is replaced with a two-step delay where expectation is a weighted average of the past outputs at one period and two periods before. Then it is shown that the two-step delay increases stability in comparison to the one-step delay in a sense that it enlarges the stability region of a monopoly equilibrium. Sun's model examines stability of Agiza's model by introducing a three-step delay defined in the same way and unveils that a multi-delay destabilizes the monopoly equilibrium more than a single-delay model. These results are also numerically demonstrated. In addition, it is numerically verified that complicated dynamics can be born in a single-delay as well as in multi-delay models when the monopoly equilibrium loses stability. Adopting the delay structure in these monopoly models, we extend their results in the following way:

(1) We analytically re-examine stability of the delay monopoly model with multiple delays and present complete stability analysis.

(2) We show the existence of two different routes to complex dynamics involving chaos, one via a period-doubling bifurcation and the other via a Neimark-Sacker bifurcation.

This paper is organized as follows. In Section 2, we construct a basic model with one-step delay (i.e., naive expectation) and show that the dynamic equation is conjugate to the logistic map when the gradient method is assumed. In Section 3, we study monopoly dynamics with two-step delay and deduce a double routes to chaos. In Section 4, we proceed to monopoly dynamics with three-step delay and demonstrate that it can generate a wide spectrum of dynamics ranging from simple periodic cycles to chaotic oscillations.

2 Dynamic Monopoly with One-Step Delay

We construct a basic delay monopoly model with bounded rationality, following Agiza *et al.* (2001). Let q be the output of a firm. The price function is assumed to be linear,

$$p = a - bq$$

and the production cost function to be quadratic,

$$C(q) = cq^2$$

where a, b and c are positive constants. It is assumed that a monopolistic firm has only partial knowledge of the demand function and determines its output decision on basis of a gradient of the marginal expected profit. That is, the firm increases or decreases output according to whether the marginal profit is positive or negative. Taking expected demand q^e given, the expected marginal profit is assumed to have the form,

$$\frac{d\pi(q^e)}{dq^e} = a - 2(b+c)q^e.$$

The level of output at period $t+1$ is determined in such a way that the output growth rate is proportional to the expected marginal profit

$$\frac{q(t+1) - q(t)}{q(t)} = \alpha \frac{d\pi(q^e(t+1))}{dq^e(t+1)}$$

where $\alpha > 0$ is an adjustment coefficient. The output adjustment process is rewritten in the form,

$$q(t+1) = q(t) + \alpha q(t) \left\{ a - 2(b+c)q^e(t+1) \right\}. \tag{1}$$

The time evolution of the monopolistic firm depends on the formation of the expected demand. We start dynamic analysis with a benchmark case where the expected demand at time $t+1$ is equal to the quantity actually realized in the previous period,

$$q^e(t+1) = q(t). \tag{2}$$

This formation is usually called a naive expectation. Since there is one-step delay in the naive expectation, we call it *one-step delay*. Substituting (2) into (1) yields a first-order difference equation,

$$q(t+1) = q(t) + \alpha q(t) \left[a - 2(b+c)q(t) \right]. \tag{3}$$

It has two steady states; a trivial (zero) solution and a non-trivial (positive) solution. The trivial solution is eliminated from further considerations. The non-trivial solution, which will be called a monopoly equilibrium, is given by

$$q_M = \frac{a}{2(b+c)}.$$

The monopoly equilibrium is locally asymptotically stable if the slope of the dynamic equation (3) evaluated at q_M is less than unity in absolute value,

$$\left|\frac{dq(t+1)}{dq(t)}\right| = |1 - a\alpha| < 1.$$

With $a\alpha > 0$, the derivative is always less than unity. If $a\alpha^*$ denotes the threshold value for which the derivative is -1, then the stability condition is

$$a\alpha < a\alpha^* = 2.$$

It can be shown that the dynamic equation (3) is conjugate to the logistic map[1] and the confinement condition is $a\alpha \leq 3$. Therefore if $a\alpha$ increases from 2 to 3, then the monopoly equilibrium is destabilized through a period-doubling (PD henceforth) bifurcation in which stability is replaced with a 2-periodic cycle which is then further replaced by a 4-periodic cycle, and so on. We summarize this result as follows:

Proposition 1 *If the quantity expectation is formed with one-step delay, then the monopoly equilibrium q_M is locally asymptotically stable for $a\alpha < a\alpha^*$, loses stability for $a\alpha = a\alpha^*$ and undergoes a PD cascade to chaos when $a\alpha$ increases from $a\alpha^*$ to 3 where $a\alpha^* = 2$.*

3 Two-Step Delay

We now proceed to a *two-step delay* case where the expected demand for period $t+1$ is a weighted average of the outputs at periods t and $t-1$,

$$q^e(t+1) = \omega_0 q(t) + \omega_1 q(t-1) \tag{4}$$

where $\omega_i \geq 0$ for $i = 0, 1$ and $\omega_0 + \omega_1 = 1$. The coefficient ω_0 is the weight of the one-step delay and ω_1 is the weight of the two-step delay in the expectation formation. Inserting (4) into the dynamic equation (1) yields a second-order difference equation

$$q(t+1) = q(t) + \alpha q(t)\left\{a - 2(b+c)\left[\omega_0 q(t) + \omega_1 q(t-1)\right]\right\} \tag{5}$$

which is rewritten as a two-dimensional first-order system in the form

$$x(t+1) = q(t),$$

$$q(t+1) = q(t) + \alpha q(t)\left\{a - 2(b+c)[\omega_0 q(t) + (1-\omega_0)x(t)]\right\}.$$

[1] Transformation is possible by a variable change, $x(t) = q(t)/\bar{q}$ with $\bar{q} = (1+a\alpha)/2\alpha(b+c)$.

Notice that the monopoly equilibrium q_M is the positive stationary state of this system. The coefficient matrix of the linearized system is

$$\begin{pmatrix} 0 & 1 \\ -a\alpha(1-\omega_0) & 1-a\alpha\omega_0 \end{pmatrix}$$

and its characteristic equation is transformed into a quadratic equation

$$\lambda^2 + a_1\lambda + a_2 = 0$$

where $a_1 = -(1-a\alpha\omega_0)$ and $a_2 = a\alpha(1-\omega_0)$. The sufficient and necessary conditions that a quadratic equation has roots inside the unit cycle are given by relations[2]

$$\begin{aligned} &1 + a_1 + a_2 = a\alpha > 0, \\ &1 - a_1 + a_2 = 2 - a\alpha(2\omega_0 - 1) > 0, \\ &1 - a_2 = 1 - a\alpha(1-\omega_0) > 0. \end{aligned} \tag{6}$$

The inequality of the first condition always holds implying that the characteristic equation does not have a unit root. The directions of the other two conditions depend on the parameter values to be specified and thus are ambiguous.

To consider the effects caused by the two-step time delay, we first examine two boundary cases. In one case with $\omega_0 = 1$ and $\omega_1 = 0$, the two-step delay is reduced to the one-step delay, which is already considered in the previous section. In the other boundary case with $\omega_0 = 0$ and $\omega_1 = 1$, the expected demand at time $t+1$ is equal to the output realized at period $t-1$,

$$q^e(t+1) = q(t-1).$$

This special expectation formation is called an *extreme two-step delay* with which the dynamic equation (5) is reduced to a second-order difference equation

$$q(t+1) = q(t) + \alpha q(t)\left\{a - 2(b+c)q(t-1)\right\}. \tag{7}$$

The second condition of (6) is satisfied under $\omega_0 = 0$. Destabilization occurs only by violating the third condition from which we obtain the following threshold value denoted by $a\alpha^{**}$,

$$a\alpha^{**} = 1.$$

When $a\alpha = a\alpha^{**}$, the monopoly equilibrium changes stability through a pair of complex conjugate roots. As $a\alpha$ becomes larger, stability is replaced with a periodic cycle, which is then replaced with a quasi-periodic cycle. Such a stability change is called a Neimark-Sacker (NS henceforth) bifurcation. Since $a\alpha^* < a\alpha^{**}$, this special two-step delay has a stronger destabilizing effect than the one-step delay. Our second result of delay dynamics is the following:

[2] See, for example, Okuguchi and Szidarovszky (1999) and Bischi, *et al.* (2010).

Proposition 2 *If the quantity expectation is formed with an extreme two-step delay, then the monopoly equilibrium is locally asymptotically stable if* $a\alpha < a\alpha^{**}$, *loses stability if* $a\alpha = a\alpha^{**}$ *and is locally unstable through a NS bifurcation if* $a\alpha > a\alpha^{**}$ *where* $a\alpha^{**} = 1$.

We now turn attention to a *general two-step delay* with $\omega_0 > 0$ and $\omega_1 > 0$. The second condition of (6) is satisfied when $\omega_0 \leq 1/2$ and leads to a PD boundary defined for $\omega_0 > 1/2$,

$$a\alpha^I_{PD} = \frac{2}{2\omega_0 - 1}. \tag{8}$$

A PD bifurcation occurs when the monopoly equilibrium crosses this boundary on which one of the eigenvalues is equal to -1. Since $\omega_0 < 1$, the third condition of (6) is not necessarily true either. Replacing the inequality with the equality and solving it for $a\alpha$ yields the NS boundary defined by

$$a\alpha^I_{NS} = \frac{1}{1 - \omega_0} > 1. \tag{9}$$

On this curve the characteristic equation has purely imaginary roots. A NS bifurcation arises when the monopoly equilibrium crosses this boundary. The partition curve defined by

$$a\alpha = \min[a\alpha^I_{NS}, a\alpha^I_{PD}]$$

divides the $(\omega_0, a\alpha)$ plane into two parts: the monopoly equilibrium is locally asymptotically stable in the region below this curve and locally unstable in the region above. A two-parameter bifurcation diagram for $0 < \omega_0 < 1$ and $0 < a\alpha < 4$ is illustrated in Figure 1 where the upward-sloping convex curve is the NS boundary, the downward-sloping convex curve is the PD boundary and the two curves intercept at $\omega_0 = 3/4$. The monopoly equilibrium is locally stable in the red region. Different color indicates different period of cycles up to 16. In the gray region periodic cycles have periods longer than 16 or aperiodic cycles are born. The solution becomes infeasible for the parameter values selected from the white region. The following two observations can be made:

(i) The general two-step delay leads to lesser stability in the monopoly equilibrium than the one-step delay for $0 < \omega_0 < 1/2$ as $a\alpha^I_{NS} < a\alpha^*$.

(ii) The general two-step delay leads to greater stability in the monopoly equilibrium than the one-step delay for $1/2 < \omega_0 < 1$ as $\min[a\alpha^I_{NS}, a\alpha^I_{PD}] > a\alpha^*$.

Notice also that the stability intervals in the two boundary cases can be found in Figure 1, namely, $(0,\ a\alpha^*)$ at $\omega_0 = 1$ and $(0,\ a\alpha^{**})$ at $\omega_0 = 0$. The third results generated by the delay dynamics can be mentioned as follows:

Proposition 3 *If the quantity expectation is formed with a general two-step delay, then the* $(\omega_0, a\alpha)$ *plane is divided by the partition curve*

$$a\alpha = \min[a\alpha^I_{NS}, a\alpha^I_{PD}]$$

and the stability of the monopoly equilibrium is replaced with a periodic or aperiodic oscillation through a NS bifurcation if it loses stability at $a\alpha = a\alpha^I_{NS}$ *for* $0 < \omega_0 \leq 3/4$ *whereas it goes to chaos via a PD bifurcation if it loses stability at* $a\alpha = a\alpha^I_{PD}$ *for* $3/4 < \omega_0 < 1$.

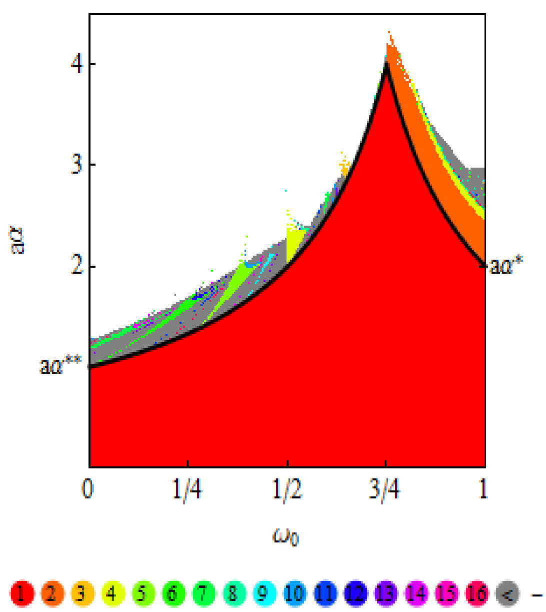


Figure 1. Bifurcation diagram with a general two-step delay

4 Three-Step Delay

In this section, we extend our analysis to a *three-step delay* where the expectation at time $t+1$ is formed as a weighted average of the past outputs at three different periods of time, t, $t-1$ and $t-2$,

$$q^e(t+1) = \omega_0 q(t) + \omega_1 q(t-1) + \omega_2 q(t-2) \tag{10}$$

where $\omega_i \geq 0$ and $\omega_0 + \omega_1 + \omega_2 = 1$. There are seven cases depending on the number of positive ω_i values:

$$(1)\ \omega_0 > 0,\ \omega_1 > 0 \text{ and } \omega_2 > 0,$$

$$(2\text{-}k)\ \omega_k = 0 \text{ and } \omega_i > 0 \text{ for } i \neq k\ (k = 0, 1, 2)$$

and

$$(3\text{-}k)\ \omega_k = 1 \text{ and } \omega_i = 0 \text{ for } i \neq k\ (k = 0, 1, 2).$$

We call case (1) a *general three-step delay* and confine our attention to this case. The following three cases with $\omega_2 = 0$ are already considered: (2-2) $\omega_0 > 0$, $\omega_1 > 0$ and $\omega_2 = 0$ (i.e., the general two-step delay), (3-0) $\omega_0 = 1$ and $\omega_1 = \omega_2 = 0$ (i.e., the one-step delay) and (3-1) $\omega_1 = 1$ and $\omega_0 = \omega_2 = 0$ (i.e., the extreme two-step delay). In Figure 2, points $\mathbb{A}$ and $\mathbb{B}$ correspond to (3-0) and (3-1). A line connecting points $\mathbb{A}$ and $\mathbb{B}$ corresponds to (2-2). Notice that the one-step delay is a boundary case of the two-step delay and the two-step delay is a boundary case of the three-step delay. The remaining three cases, (2-0), (2-1) and (3-2), are also boundary cases where ω_2 is always positive. Since they can be discussed similarly to those given in this paper, we omit the further considerations to avoid repeating similar discussions.

Substituting (10) into the output dynamic equation (1) yields a third-order difference equation

$$q(t+1) = q(t) + \alpha q(t) \left\{a - 2(b+c)\left[\omega_0 q(t) + \omega_1 q(t-1) + \omega_2 q(t-2)\right]\right\}. \quad (11)$$

This can be written as a three-dimensional first-order dynamic system in the form,

$$x(t+1) = y(t),$$

$$y(t+1) = q(t),$$

$$q(t+1) = q(t) + \alpha q(t) \left\{a - 2(b+c)\left[\omega_0 q(t) + \omega_1 y(t) + (1 - \omega_0 - \omega_1) x(t)\right]\right\},$$

where q_M is the positive stationary point of this system. The coefficient matrix of the linearized dynamic system evaluated at the stationary point is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a\alpha(1-\omega_0-\omega_1) & -a\alpha\omega_1 & 1 - a\alpha\omega_0 \end{pmatrix}.$$

Its characteristic equation is reduced to a cubic equation,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \quad (12)$$

where

$$a_1 = a\alpha\omega_0 - 1,$$

$$a_2 = a\alpha\omega_1,$$

$$a_3 = a\alpha(1 - \omega_0 - \omega_1).$$

The monopoly equilibrium is locally asymptotically stable if all eigenvalues of (12) are less than unity in absolute value. As it has been proved in Farebrother (1973) and Okuguchi and Irie (1990), the most simplified form of the sufficient and necessary conditions for the cubic equation to have roots only inside the unit cycle is

$$
\begin{aligned}
1 + a_1 + a_2 + a_3 &= a\alpha > 0, \\
1 - a_1 + a_2 - a_3 &= 2 - a\alpha(1 - 2\omega_1) > 0, \\
1 - a_2 + a_1 a_3 - a_3^2 &= \varphi(\omega_0, \omega_1, a\alpha) > 0, \\
a_2 &= a\alpha\omega_1 < 3,
\end{aligned}
\tag{13}
$$

where

$$
\varphi(\omega_0, \omega_1, a\alpha) = 1 - a\alpha(1 - \omega_0) - (a\alpha)^2 \left(2\omega_0^2 - 3\omega_0(1 - \omega_1) + (1 - \omega_1)^2\right). \tag{14}
$$

The first condition of (13) always holds as $a > 0$ and $\alpha > 0$ are assumed. The other three conditions depend on the specification of the parameter values.

4.1 General Three-Step Delay

Under the conditions $\omega_i > 0$ for $i = 0, 1, 2$, the second condition of (13) is always satisfied for $\omega_1 \geq 1/2$ and leads to a PD boundary for $\omega_1 < 1/2$,

$$
a\alpha_{PD}^{II} = \frac{2}{1 - 2\omega_1} > 2. \tag{15}
$$

From the third condition of (13), a NS boundary is defined by $\varphi(\omega_0, \omega_1, a\alpha) = 0$ and its shape depends on the sign of the multiplier of $(a\alpha)^2$ in equation (14). So let $A = 2\omega_0^2 - 3\omega_0(1 - \omega_1) + (1 - \omega_1)^2$, which can be rewritten as

$$
A = 2(\omega_0 - (1 - \omega_1)) \left(\omega_0 - \frac{1}{2}(1 - \omega_1)\right). \tag{16}
$$

We first consider the case of $A = 0$ in which either of the following two relations of ω_0 and ω_1 holds,

$$
\omega_0 = 1 - \omega_1
$$

or

$$
\omega_0 = \frac{1}{2}(1 - \omega_1).
$$

The first relation implies $\omega_2 = 0$. As already mentioned, the boundary cases with $\omega_2 = 0$ are already examined. Therefore, we can assume the second case only when $\omega_0 < 1/2$ and the third condition of (13) has the special form

$$
1 - a\alpha(1 - \omega_0) > 0,
$$

or

$$
a\alpha < a\alpha_{NS}^{I}.
$$

An alternative form of the NS boundary in terms of ω_0 can be defined over the interval $(1/4,\ 1/2)$ by substituting the relation of the second case into equation (15),

$$a\alpha_{PD}^{II} = \frac{2}{4\omega_0 - 1}.$$

The last condition of (13) holds if

$$a\alpha < \frac{3}{\omega_1} = \frac{3}{1 - 2\omega_0}.$$

Notice that

$$a\alpha_{PD}^{II} - a\alpha_{NS}^{I} = \frac{3(1 - 2\omega_0)}{(4\omega_0 - 1)(1 - \omega_0)} > 0,$$

furthermore

$$\frac{3}{\omega_1} = \frac{3}{1 - 2\omega_0} > a\alpha_{NS}^{I} \text{ for } \omega_0 < \frac{1}{2}.$$

Hence the $a\alpha = a\alpha_{NS}^{I}$ boundary separates the stability region from the instability region.

Proposition 4 *If* $0 < \omega_1 < 1$ *and* $\omega_0 = \frac{1}{2}(1 - \omega_1)$*, then* $a\alpha = a\alpha_{NS}^{I}$ *is a partition curve below which the monopoly equilibrium is locally asymptotically stable and above which it is destabilized through a NS bifurcation where*

$$a\alpha_{NS}^{I} = \frac{1}{1 - \omega_0} \textit{ for } 0 < \omega_0 < \frac{1}{2}.$$

.

Suppose next that $A \neq 0$. Solving $\varphi(\omega_0, \omega_1, a\alpha) = 0$ for $a\alpha$ gives explicit forms of the two NS boundaries

$$a\alpha_{(+)} = \frac{-(1 - \omega_0) + \sqrt{D}}{2A} \tag{17}$$

and

$$a\alpha_{(-)} = \frac{-(1 - \omega_0) - \sqrt{D}}{2A} \tag{18}$$

where D is the discriminant defined by

$$D = 5 + 9\omega_0^2 - 4\omega_1(2 - \omega_1) + 2\omega_0(6\omega_1 - 7).$$

Solving $D = 0$ yields two threshold values of ω_0,

$$\omega_0^{(+)} = \frac{7 - 6\omega_1 + 2\sqrt{1 - 3\omega_1}}{9}$$

and

$$\omega_0^{(-)} = \frac{7 - 6\omega_1 - 2\sqrt{1 - 3\omega_1}}{9}$$

$\omega_0^{(+)}$ and $\omega_0^{(-)}$ are distinct for $\omega_1 < 1/3$ (i.e., $1/2 \leq \omega_0^{(-)} < 5/9 < \omega_0^{(+)} < 1$), identical for $\omega_1 = 1/3$ (i.e., $\omega_0^{(+)} = \omega_0^{(-)} = 5/9$) and $D = 0$ has no real solution if $\omega_1 > 1/3$.

The divisions of the (ω_1, ω_0) plane are given in Figure 2. The unit square is divided by the $\omega_0 = 1-\omega_1$ line into the gray region and the white region which is eliminated from further considerations because $\omega_2 < 0$ there. The former region is further divided by the $\omega_0 = (1-\omega_1)/2$ line into two smaller triangles. It is clear that $A < 0$ in the upper triangle shaded with light-gray and $A > 0$ in the lower triangle with medium-gray. $D < 0$ is inside the half-oval region shaded with dark-gray in the upper triangle. It can be checked that the lower part of the $D = 0$ curve (i.e., $\omega_0 = \omega_0^{(-)}$) is downward-sloping for $0 < \omega_1 < 1/4$, upward-sloping for $1/4 < \omega_1 < 1/3$ and takes the minimum value $1/2$ at $\omega_1 = 1/4$. The upper part of the $D = 0$ curve (i.e., $\omega_0 = \omega_0^{(+)}$) is downward sloping and below the $\omega_0 = 1 - \omega_1$ line. The discriminant is always positive (i.e., $D > 0$) for $\omega_1 > 1/3$ and the second condition of (13) is satisfied for $\omega_1 > 1/2$.

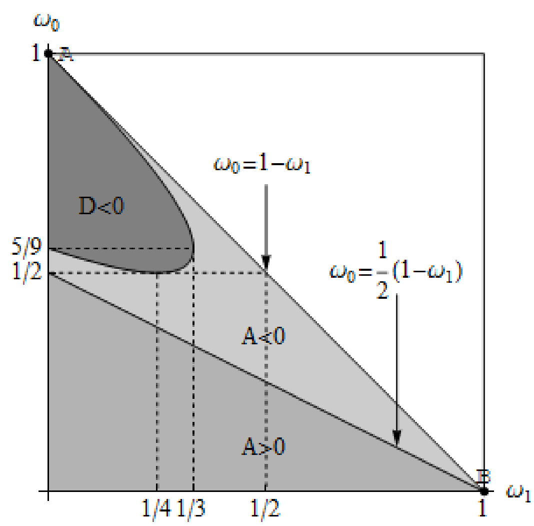


Figure 2. Division of the (ω_1, ω_0) plane

We first examine stability of q^M in the lower triangle where $A > 0$, $D > 0$ and $\omega_0 < 1/2$. Both denominators of $a\alpha_{(+)}$ and $a\alpha_{(-)}$ are positive for $A > 0$. Concerning the sign of the numerators, it is able to check that

$$\begin{aligned}(1-\omega_0)^2 - D &= \left\{(1-\omega_0) + \sqrt{D}\right\}\left\{(1-\omega_0) - \sqrt{D}\right\} \\ &= 8(1-\omega_0-\omega_1)\left(\omega_0 - \frac{1}{2}(1-\omega_1)\right) < 0.\end{aligned} \tag{19}$$

The factorization in the first line of (19) and the inequality in the second line lead to that $-(1-\omega_0)+\sqrt{D}>0$. As a direct result, the numerator of $a\alpha_{(+)}$ is positive while that of $a\alpha_{(-)}$ is negative. In consequence, a NS boundary is given by the $a\alpha = a\alpha_{(+)}$ curve. On the other hand, the PD boundary is already defined in equation (15) and is shown to be greater than 2. Since it can be confirmed that $2 > a\alpha_{(+)}$ for $\omega_0 < 1/2$,[3] we have the following relation in this case with $A > 0$:

$$a\alpha_{(+)} > a\alpha_{PD}^{II}.$$

In addition

$$a\alpha_{(+)} < 2 < \frac{3}{\omega_1}$$

So, we have the following:

Proposition 5 *If $\omega_0 < \frac{1}{2}(1-\omega_1)$ and $\omega_1 < 1$, then $a\alpha = a\alpha_{(+)}$ is a partition line below which the monopoly equilibrium is locally asymptotically stable and above which it is destabilized through a NS bifurcation.*

We next devote a little more space to examining the upper triangle with $A < 0$. The PD boundary is the same as the one given in (15). Although the NS boundaries are given in (17) and (18), their shapes in the $(\omega_0, a\alpha)$ plane are sensitive to a choice of ω_1. We need to look more closely at them. Due to the definition of the upper triangle, the feasible interval of ω_0 has the lower bound ω_0^L and the upper bound ω_0^U,

$$\omega_0^L = \frac{1-\omega_1}{2} \text{ and } \omega_0^U = 1-\omega_1.$$

When $\omega_1 < 1/3$, $\omega_0^{(-)}$ and $\omega_0^{(+)}$ are well-defined. Substituting $\omega_0^{(-)}$ and $\omega_0^{(+)}$, respectively, into $a\alpha_{(+)}$ (or $a\alpha_{(-)}$) yields two threshold values of $a\alpha$,

$$a\alpha_M = \frac{9(1+3\omega_1+\sqrt{1-3\omega_1})}{2+3\omega_1+9\omega_1^2+2\sqrt{1-3\omega_1}(1+3\omega_1)},$$

$$a\alpha_m = \frac{9(1+3\omega_1-\sqrt{1-3\omega_1})}{2+3\omega_1+9\omega_1^2-2\sqrt{1-3\omega_1}(1+3\omega_1)}$$

and

$$a\alpha_m - a\alpha_M = -\frac{2\sqrt{1-3\omega_1}}{\omega_1(1+\omega_1)} < 0.$$

[3] By definition of $a\alpha_{(+)}$,

$$2 - a\alpha_{(+)} = \frac{1}{2A}\left\{4A - \sqrt{D} + (1-\omega_0)\right\}.$$

By (16) and (19), we have $4A = D - (1-\omega_0)^2$ which is substituted into the expression in the parentheses to obtain

$$\left\{\sqrt{D} - (1-\omega_0)\right\}\left\{\sqrt{D} - \omega_0\right\} > 0$$

where the inequality is due to $\sqrt{D} > (1-\omega_0)$ and $1-\omega_0 > \omega_0$ if $\omega_0 < 1/2$.

Notice that $a\alpha_{(+)} = a\alpha_{(-)} = a\alpha_m$ at $\omega_0 = \omega_0^{(-)}$ and $a\alpha_{(+)} = a\alpha_{(-)} = a\alpha_M$ at $\omega_0 = \omega_0^{(+)}$. As seen in Figure 3, it is further possible to show that

$$a\alpha_{PD}^{II} < a\alpha_m < a\alpha_M \text{ if } \omega_1 < 1/4,$$

$$a\alpha_m < a\alpha_{PD}^{II} < a\alpha_M \text{ if } 1/4 < \omega_1 < 5/16,$$

$$a\alpha_m < a\alpha_M < a\alpha_{PD}^{II} \text{ if } 5/16 < \omega_1 < 1/3.$$

It is also clear that the curve $3/\omega_1$ is above the curves $a\alpha_M$, $a\alpha_m$ and $a\alpha_{PD}^{II}$ if $\omega_1 \leq 1/3$. If $1/3 < \omega_1 < 3/8$, then the PD boundary is below the $3/\omega_1$ line, and if $\omega_1 > 3/8$, then the opposite holds. They intercept at $\omega_1 = 3/8$.

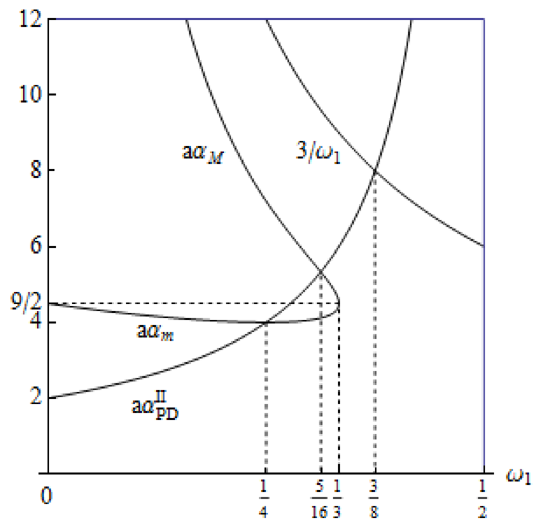


Figure 3

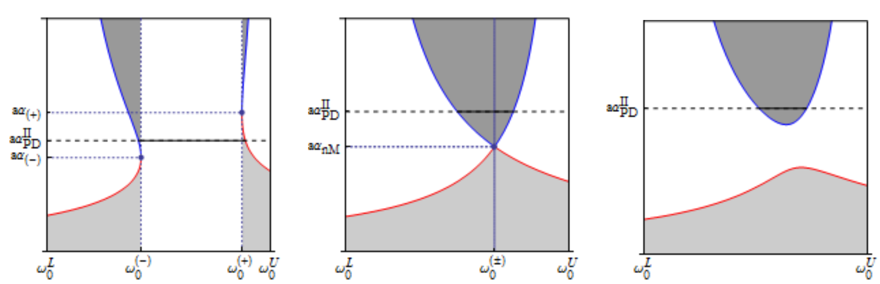
Three boundaries with three different values of ω_1 are illustrated in Figure 4 in which the $a\alpha = a\alpha_{(+)}$ curve is colored in red, the $a\alpha = a\alpha_{(-)}$ curve is colored in blue and the PD boundary is a dotted horizontal line. The function $\varphi(\omega_0, \omega_1, a\alpha)$ is defined and the third stability condition of (13) with positive discriminant is satisfied in the (darker or lighter) gray-colored regions and so is the stability condition with negative discriminant in the white region. The second stability condition of (13) is satisfied below the PD boundary. The last condition is satisfied if $a\alpha < 3/\omega_1$, where the PD boundary is below the $3/\omega_1$ line. In Figure 4(A) where we take $\omega_1 = 0.29 < 1/3$, there are two distinct real roots, $\omega_0^{(-)} < \omega_0^{(+)}$. Since the specified value of ω_1 satisfies the inequalities $1/4 < \omega_1 < 5/16 (= 0.315)$, it first implies that $a\alpha_m < a\alpha_{PD}^{II} < a\alpha_M$ and second that the red curve is connected with the blue curve at point $(\omega_0^{(-)}, a\alpha_m)$

or $(\omega_0^{(+)}, a\alpha_M)$. That is, the two curves consist of one NS boundary in interval $(\omega_0^L,\ \omega_0^{(-)}]$ and the other NS boundary in interval $[\omega_0^{(+)},\ \omega_0^U)$. For ω_0 such that $\omega_0^{(-)} < \omega_0 < \omega_0^{(+)}$, we have $D < 0$ so that neither of the NS boundaries is defined in this interval where stability is determined by the PD boundary. In Figure 4(B) where we take $\omega_1 = 1/3$, $D = 0$ has a single root, $\omega_0^{(-)} = \omega_0^{(+)} = 5/9 = \omega_0^{(\pm)}$ and so $a\alpha_m = a\alpha_M = 9/2 = a\alpha_{mM} < a\alpha_{PD}^{II}$ as seen in Figure 3. The $a\alpha = a\alpha_{(+)}$ curve is depicted in red, convex-increasing for $\omega_0 < \omega_0^{(\pm)}$, convex-decreasing for $\omega_0 > \omega_0^{(\pm)}$ and has a kink at $\omega_0 = \omega_0^{(\pm)}$. The stable region below the red-NS boundary is colored in light-gray. The $a\alpha = a\alpha_{(-)}$ curve is depicted in blue, convex-decreasing for $\omega_0 < \omega_0^{(\pm)}$, convex-increasing for $\omega_0 > \omega_0^{(\pm)}$ and has a kink at $\omega_0 = \omega_0^{(\pm)}$. The PD boundary crosses this NS boundary at two points. The stable region is above the blue-NS boundary and below the PD boundary. The two stability regions have one common point $(\omega_0^{(\pm)}, a\alpha_{mM})$. In Figure 4(C), we take $\omega_1 = 0.34 > 1/3$. Since $D > 0$ and $A < 0$ lead to that $a\alpha_{(-)} > a\alpha_{(+)}$. The $a\alpha = a\alpha_{(+)}$ curve is mound-shaped and colored in red while the $a\alpha = a\alpha_{(-)}$ curve is U-shaped and colored in blue. The two gray regions are separated each other. As in Figure 4(B), the real PD boundary divides the darker-gray region into two subregions. If $\omega_1 > 3/8$, then the $a\alpha_{PD}^{II}$ line has to be replaced by the $a\alpha = 3/\omega_1$ line. The results of our investigation are summarized as follows;

Proposition 6 *If* $\frac{1}{2}(1-\omega_1) < \omega_0 < 1-\omega_1$ *and* $0 < \omega_1 < 1$, *the stability region is defined by*

$$S = \left\{ (\omega_0, a\alpha) \mid \omega_0^L < \omega_0 \le \omega_0^U,\ a\alpha < \min\left\{ \frac{2}{1-2\omega_1}, \frac{3}{\omega_1} \right\} \text{ and } \varphi(\omega_0, \omega_1, a\alpha) > 0 \right\}$$

and its shape depends on a specified value of ω_1



(A) $\omega_1 < 1/3$ (B) $\omega_1 = 1/3$ (C) $\omega_1 > 1/3$

Figure 4. NS boundaries with three different values of ω_1

4.2 Numerical Examples

We perform numerical simulations to confirm the analytical results obtained in Propositions 5 and 6. Assume first that $D < 0$. Then $A < 0$ as well, so

$\varphi(\omega_0, \omega_1, a\alpha)$ is a convex parabola, and therefore the third condition of (13) holds for all $a\alpha$. If $D = 0$, then this condition holds for all $a\alpha$, except for

$$a\alpha_{(+)} = a\alpha_{(-)} = -\frac{1-\omega_0}{2A}.$$

Solving $\varphi(\omega_0, \omega_1, a\alpha) = 0$ with $a\alpha = 2/(1-2\omega_1)$ for ω_0 yields two points of intersection at which the PD boundary crosses the NS boundary,

$$\omega_0^a = \frac{1}{2} \text{ and } \omega_0^b = \frac{5}{4} - 2\omega_1.$$

It can be shown that for $\omega_1 < 1/3$,

$$\omega_0^a < \omega_0^{(-)} \text{ and } \omega_0^{(+)} < \omega_0^b$$

since

$$\omega_0^{(-)} - \omega_0^a = \frac{1}{18}(2\sqrt{1-3\omega_1} - 1)^2 > 0$$

and

$$\omega_0^b - \omega_0^{(+)} = \frac{1}{36}(4\sqrt{1-3\omega_1} - 1)^2 > 0.$$

It is also able to check that ω_0^b is larger than ω_0^U if $\omega_1 < 1/4$ and smaller if the inequality is reversed. In order to detect the destabilizing effect caused by the general three-step delay, it is convenient to divide the general three-step case into the following four distinct subcases depending on the value of ω_1:

(A) $0 < \omega_1 \leq 1/4$ (where the second inequality is written as $\omega_0^U \leq \omega_0^b$)

(B) $1/4 < \omega_1 \leq 1/3$ (where the first inequality is written as $\omega_0^b < \omega_0^U$)

(C) $1/3 < \omega_1 < 1/2$ ($a\alpha_{PD}^{II}$ is defined)

(D) $1/2 < \omega_1 < 1$ ($a\alpha_{PD}^{II}$ is not defined)

Case A $0 < \omega_1 \leq \frac{1}{4}$

When $\omega_1 < 1/4$, as shown in Figure 3, we have

$$a\alpha_{PD}^{II} < a\alpha_m < a\alpha_M.$$

These inequalities imply that the PD boundary crosses the $a\alpha = a\alpha_{(+)}$ curve at two points, ω_0^a and ω_0^b. Since $\omega^U < \omega_0^b$ for $\omega_1 < 1/4$, the domain of ω_0 is an interval $(0, 1-\omega_1)$. Thus the stability region is bounded by the NS and PD boundaries defined over this interval;

$$a\alpha = a\alpha_{(+)} \text{ if } 0 < \omega_0 \leq \omega_0^a$$

$$a\alpha = a\alpha_{PD}^{II} \text{ if } \omega_0^a < \omega_0 < 1 - \omega_1.$$

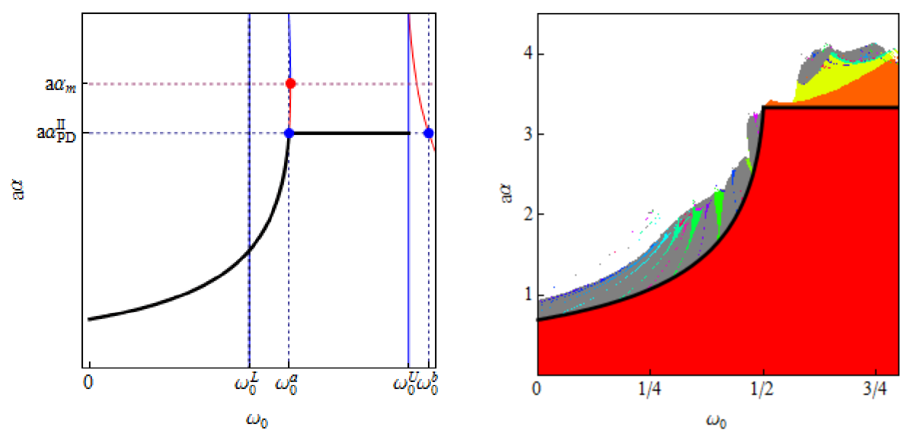
Taking $\omega_1 = 1/5$, we illustrate the piecewise continuous partition curve by the thick black curve in Figure 5(A) where the blue points are the intersections of the NS and PD boundaries and $a\alpha_{(-)} = a\alpha_{(+)}$ holds at the red points.[4] The monopoly equilibrium undergoes either PD or NS bifurcation according to whether it crosses the PD or the NS boundary. The two-parameter bifurcation diagram is presented in Figure 5(B) where the color has the same meaning as in Figure 1. The partition curve shifts upward and ω_0^U decreases as ω_1 becomes larger. Proposition 5 describes dynamics for $\omega_0 < \omega_0^L$ and so does Proposition 6 for $\omega_0 > \omega_0^L$.

Proposition 7 *For $0 \leq \omega_1 < 1/4$, the stability region of the monopoly equilibrium is given as*

$$S_I^A = \left\{(\omega_0, a\alpha) \mid 0 < \omega_0 \leq 1 - \omega_1 \text{ and } a\alpha < \min\left[a\alpha_{(+)}, a\alpha_{PD}^{II}\right]\right\} \text{ if } D > 0,$$

and

$$S_{II}^A = \left\{(\omega_0, a\alpha) \mid 0 < \omega_0 \leq 1 - \omega_1 \text{ and } a\alpha < a\alpha_{PD}^{II}\right\} \text{ if } D \leq 0.$$



(A) Partition curve (B) Bifurcation diagram

Figure 5 Dynamics for $0 < \omega_1 \leq 1/4$

Case B $\dfrac{1}{4} < \omega_1 \leq \dfrac{1}{3}$

Increasing ω_1 from $1/4$ leads to $\omega_0^U > \omega_0^b$. As seen in Figure 3, the inequality relation $a\alpha_m < a\alpha_{PD}^{II}$ holds always but the ordering between $a\alpha_M$ and $a\alpha_{PD}^{II}$ depends on the value of ω_1,

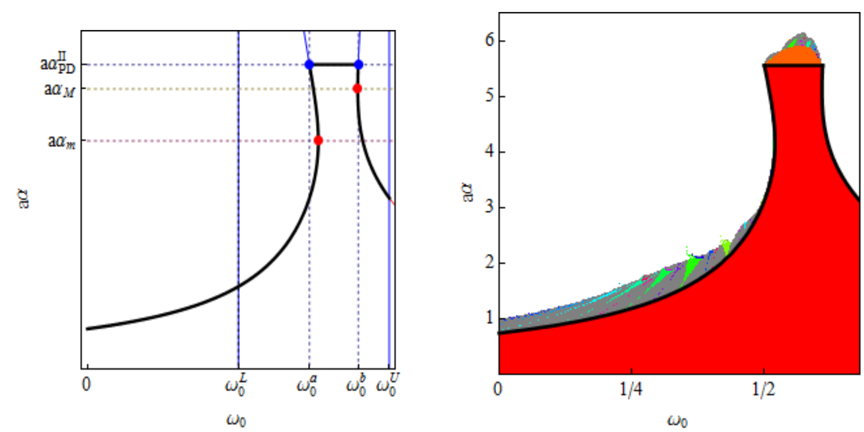
$$a\alpha_m < a\alpha_{PD}^{II} \leq a\alpha_M \text{ if } \tfrac{1}{4} < \omega_1 \leq \tfrac{5}{16},$$

$$a\alpha_m < a\alpha_M < a\alpha_{PD}^{II} \text{ if } \tfrac{5}{16} < \omega_1 \leq \tfrac{1}{3}.$$

[4] $a\alpha_M$ is not depicted in Figure 4 as it is larger than the upper bound of the figure.

As a result, the shape of stability region is affected in two ways: in one way, the $\varphi(\omega_0, \omega_1, a\alpha) = 0$ curve defined over the interval $[\omega_0^{(+)}, \omega_0^U)$ can be the NS boundary; in the other way, the NS boundary defined over $(\omega_0^L, \omega_0^{(-)}]$ becomes bent backwardly. We take $\omega_1 = 0.32 > 5/16$ $(= 0.3125)$ in Figure 6(A)[5] where the thick curve is the partition curve which is piecewise continuous having three parts with the two kinks at the blue points. The two parameter bifurcation diagram is presented in Figure 6(B). Let C_L and C_R denote the union of the $a\alpha_{(+)}$ and $a\alpha_{(-)}$ curves for $\omega_0 \leq \omega_0^{(-)}$ and for $\omega_0 \geq \omega_0^{(+)}$ respectively. Then the third condition of (13) holds if the point in the $(\omega_0, a\alpha)$ space is between these curves.

Proposition 8 *For $1/4 < \omega_1 < 1/3$, the stability region of the monopoly equilibrium is bounded by the curves C_L, C_R, and the horizontal line $a\alpha = a\alpha_{PD}^{II}$ if $D < 0$ and is below the $a\alpha = a\alpha_{NS}$ curve if $D > 0$.*



(A) Partition curve (B) Bifurcation diagram

Figure 6 Dynamics for $1/4 < \omega_1 \leq 1/3$

Case C $\dfrac{1}{3} \leq \omega_1 < \dfrac{1}{2}$

When $\omega_1 = 1/3$, the $a\alpha = a\alpha_{(+)}$ curve becomes continuous and so does the $a\alpha = a\alpha_{(-)}$ curve, as shown in Figure 4(B). Further the two curves have one common point $(\omega_0^{(\pm)}, a\alpha_{mM})$. Notice that $D > 0$ and none of $a\alpha_m$, $a\alpha_M$, $\omega_0^{(-)}$ and $\omega_0^{(+)}$ is defined for $\omega_1 > 1/3$. Increasing ω_1 from 1/3 shifts the $a\alpha = a\alpha_{(+)}$ curve downward and the $a\alpha = a\alpha_{(-)}$ curve upward and then forms two separate regions satisfying the stability conditions of (13). One is

$$S_I^C = \{(\omega_0, a\alpha) \mid 0 \leq \omega_0 \leq \omega_0^U,\ a\alpha < a\alpha_{(+)}\}$$

[5] Even in the case of $\omega_1 < 5/16$, the partition curve has essentially the same shape.

and the other is

$$S_{II}^C = \{(\omega_0, a\alpha) \mid \omega_0^a \le \omega_0 \le \omega_0^b,\ a\alpha_{(-)} < a\alpha < a\alpha_{PD}^{II}\}.$$

We first show that the steady state is stable in S_I^C. If $A > 0$, then S_I^C has been shown to be a stability region in Proposition 5. $A < 0$ is assumed henceforth. Using (17) and (18), $\varphi(\omega_0, \omega_1, a\alpha)$ can be factored,

$$\varphi(\omega_0, \omega_1, a\alpha) = -A(a\alpha - a\alpha_{(+)})(a\alpha - a\alpha_{(-)}).$$

It can be verified by subtracting (17) from (18) that $0 < a\alpha_{(+)} < a\alpha_{(-)}$ if $A < 0$. In consequence

$$\varphi(\omega_0, \omega_1, a\alpha) > 0 \text{ for } 0 < a\alpha < a\alpha_{(+)} \tag{20}$$

which means that the third condition of (13) is satisfied. Subtracting (15) from (17) yields

$$a\alpha_{(+)} - a\alpha_{PD}^{II} = \frac{(1-2\omega_1)\sqrt{D} - (1-\omega_0)(1-2\omega_1) - 4A}{2A(1-2\omega_1)} \tag{21}$$

where the denominator is negative. From (16) and (19), we have

$$(1-\omega_0)^2 - D = -4A > 0 \tag{22}$$

which leads to $1 - \omega_0 - \sqrt{D} > 0$. After substituting the left hand side of (22) for $-4A$ of (21), the numerator of (21) can be factored,

$$\begin{aligned}&(1-2\omega_1)\sqrt{D} - (1-\omega_0)(1-2\omega_1) + (1-\omega_0)^2 - D \\ &= \left(2\omega_1 - \omega_0 + \sqrt{D}\right)\left(1 - \omega_0 - \sqrt{D}\right) > 0\end{aligned}$$

where the inequality is due to $2\omega_1 - \omega_0 > 0$ and $1 - \omega_0 - \sqrt{D} > 0$ under $A < 0$. Therefore $a\alpha_{(+)} < a\alpha_{PD}^{II}$ implying that the second condition of (13) is also satisfied for any $a\alpha < a\alpha_{(+)}$. Lastly subtracting $3/\omega_1$ from (17) yields

$$a\alpha_{(+)} - \frac{3}{\omega_1} = \frac{\omega_1\sqrt{D} - (1-\omega_0)\omega_1 - 6A}{2A\omega_1} \tag{23}$$

where the denominator is negative. Further substituting the left hand side of (22) into the numerator of (23) gives

$$\begin{aligned}&\omega_1\sqrt{D} - (1-\omega_0)\omega_1 + \frac{3}{2}\left\{(1-\omega_0)^2 - D\right\} \\ &= \frac{3}{2}\left(1 - \omega_0 - \frac{2}{3}\omega_1 + \sqrt{D}\right)\left(1 - \omega_0 - \sqrt{D}\right) > 0\end{aligned}$$

where the inequality is due to $1 - \omega_0 - 2\omega_1/3 > 0$ and $1 - \omega_0 - \sqrt{D} > 0$ under $A < 0$. Therefore $a\alpha_{(+)} < 3/\omega_1$, consequently the fourth condition of (13) is also satisfied for any $a\alpha < a\alpha_{(+)}$. We have shown that all conditions of (13) are satisfied for $(\omega_0, a\alpha) \in S_I^C$.

We now draw attention to S_{II}^C. It has been shown that solving $a\alpha_{(-)} = 2/(1-2\omega_1)$ yields two solutions of ω_0, one is ω_0^a and the other is ω_0^b. They are equal when $\omega_1 = 3/8$ and

$$\omega_0^a < \omega_0^b \text{ if } \frac{1}{3} < \omega_1 < \frac{3}{8}$$

and

$$\omega_0^a > \omega_0^b \text{ if } \frac{3}{8} < \omega_1 < \frac{1}{2}.$$

It has been shown and illustrated in Figure 3 that

$$a\alpha_{PD}^{II} \lesseqqgtr \frac{3}{\omega_1} \text{ according to } \omega_1 \lesseqqgtr \frac{3}{8}.$$

Given $\omega_1 < 3/8$, for $(\omega_0, a\alpha) \in S_{II}^C$, we have $0 < a\alpha < a\alpha_{PD}^{II}$, $\varphi(\omega_0, \omega_1, a\alpha) > 0$ and $a\alpha < 3/\omega_1$. Since all conditions of (13) are satisfied, S_{II}^C is also a stability region if $\omega_1 < 3/8$. However, the following arguments indicate that S_{II}^C is not stability region if $\omega_1 > 3/8$.

Subtracting $3/\omega_1$ from (18) presents

$$a\alpha_{(-)} - \frac{3}{\omega_1} = -\frac{(1-\omega_0)\omega_1 + \omega_1\sqrt{D} + 6A}{2A\omega_1} \tag{24}$$

Since Proposition 5 deals with the case of $A > 0$, we confine attention to the case of $A < 0$ henceforth. Substituting the left hand side of (22) for A, the numerator of (24) can be rewritten and factored as follows:

$$\begin{aligned} &(1-\omega_0)\omega_1 + \omega_1\sqrt{D} - \frac{3}{2}\left\{(1-\omega_0)^2 - D\right\} \\ &= \frac{3}{2}\left(\sqrt{D} + 1 - \omega_0\right)\left(\sqrt{D} - \left(1 - \omega_0 - \frac{2}{3}\omega_1\right)\right) \end{aligned}$$

where the first factor is positive. The second factor can be further rewritten as

$$\begin{aligned} &\sqrt{D} - \left(1 - \omega_0 - \frac{2}{3}\omega_1\right) \\ &= \left\{\omega_0 - \frac{1}{12}\left(9 - 8\omega_1 - \sqrt{3}\sqrt{3-8\omega_1}\right)\right\}\left\{\omega_0 - \frac{1}{12}\left(9 - 8\omega_1 + \sqrt{3}\sqrt{3-8\omega_1}\right)\right\} \end{aligned}$$

which is positive if $\omega_1 > 3/8$. Since the denominator of (24) is negative and the numerator is positive, we have

$$a\alpha_{(-)} > \frac{3}{\omega_1} \text{ for } \omega_1 > \frac{3}{8}.$$

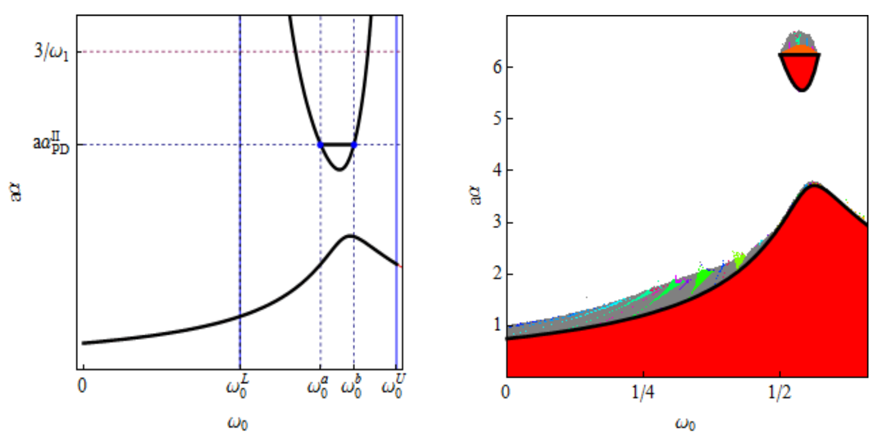
The first inequality implies that the third condition of (13) is satisfied (i.e., $\varphi(\omega_0, \omega_1, a\alpha) > 0$) for $a\alpha > a\alpha_{(-)}$ but the fourth condition is violated (i.e., $a\alpha > 3/\omega_1$) while the fourth condition is satisfied for $a\alpha < 3/\omega_1$ but the third condition is not. Therefore the stability conditions of (13) do not hold simultaneously if $\omega_1 > 3/8$. That is, S_{II}^C shrinks as ω_1 increases from $1/3$ and disappears when it arrives at $3/8$. When ω_1 is larger than $1/3$, then the two stability regions, S_I^C andS_{II}^C, become disjoint as depicted in Figure 7(B) where we take $\omega_1 = 0.34 < 3/8(= 0.375)$. The mound-shaped red region is S_I^C and the isolated (or flying) cup-shaped red region is S_{II}^C.

Proposition 9 *For* $1/3 < \omega_1 < 1/2$, *the stability region of the monopoly equilibrium is defined by*

$$S_I^C \cup S_{II}^C \text{ with } S_I^C \cap S_{II}^C = \varnothing \text{ if } \omega_1 < 3/8$$

and only

$$S_I^C \text{ if } \omega_1 > 3/8.$$



(A) Partition curve (B) Bifurcation diagram

Figure 7 Dynamics for $1/3 < \omega_1 \leq 1/2$

Case D $\dfrac{1}{2} < \omega_1 < 1$

In the same way as shown in case C, we can prove the following proposition. A numerical example is given in Figure 8.

Proposition 10 *For* $1/2 < \omega_1 < 1$, *the stability region of the monopoly equilibrium is defined by*

$$S^D = \{(\omega_0, a\alpha) \mid 0 \leq \omega_0 \leq \omega_0^U,\ a\alpha \leq a\alpha_{(+)}\}.$$

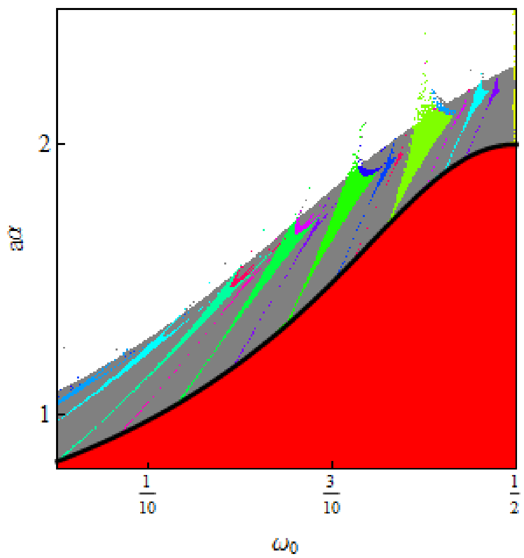


Figure 8. Bifurcation diagram with $\omega_1 = 0.5$

5 Geometric Delay

As a special case of distributed delay, we consider geometric delay with which the expected output is formed based on the whole sequence of the past output values,

$$q^e(t+1) = \sum_{\tau=0}^{\infty} \omega(1-\omega)^\tau q(t-\tau)$$

with the sum of the weighting coefficients being equal to unity

$$\sum_{\tau=0}^{\infty} \omega(1-\omega)^\tau = 1.$$

Delaying one unit period of the expected output and multiplying $(1-\omega)$, we take a difference between $q^e(t+1)$ and $(1-\omega)q^e(t)$ to have

$$q^e(t+1) = (1-\omega)q^e(t) + \omega q(t).$$

Substituting the right-hand side into the dynamic equation (1) yields a two dimensional system in the form

$$q(t+1) = q(t) + \alpha q(t) \{a - 2(b+c) [\omega q(t) + (1-\omega)q^e(t)]\}$$

$$q^e(t+1) = \omega q(t) + (1-\omega)q^e(t).$$

The positive equilibrium q_M is the only positive steady state of this system. The coefficient matrix of the linearized equation is

$$\boldsymbol{J} = \begin{pmatrix} 1 - a\alpha\omega & -a\alpha(1-\omega) \\ \omega & 1-\omega \end{pmatrix}$$

and its characteristic equation is quadratic,

$$\lambda^2 - [2 - (1 + a\alpha)\omega]\,\lambda + (1 - \omega) = 0.$$

The condition for locally asymptotically stability of q_M is confirmed to be

$$a\alpha < \frac{2(2-\omega)}{\omega}$$

which is a simple consequence of relation (6). It is also confirmed that the monopoly equilibrium is destabilized only through the PD bifurcation.

Proposition 11 *When the expected output has geometric delay, then the stability region of the monopoly equilibrium is*

$$S = \{(\omega, a\alpha) \mid 0 \le \omega \le 1,\ a\alpha < a\alpha^{III}_{PD}\}$$

where $a\alpha^{III}_{PD}$ *is a period-doubling boundary defined by*

$$a\alpha^{III}_{PD} = \frac{2(2-\omega)}{\omega}.$$

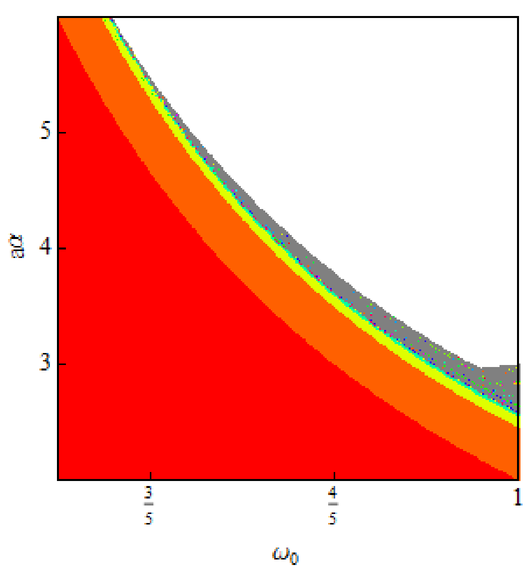


Figure 9. Bifurcation diagram with geometric delay

6 Concluding Remarks

In this paper dynamic monopolies were examined with discrete time scales and time delays in demand expectations. For mathematical simplicity linear price and quadratic costs were assumed. The cases of one, two, three and geometric delays were considered. The stability regions were determined in all cases and when the steady state loses stability, period-doubling or Neimark-Sacker bifurcation occurs depending on the number of delays and model parameters. The stability region is either a connected set in the $(\omega_0, a\alpha)$ space or a union of two mutually exclusive connected sets. Numerical examples illustrated the theoretical findings, and the computed bifurcation diagrams supported very well the theory.

References

[1] Agiza, H. N., A. S. Hegazi and A. A. Elsadany, "The Dynamics of Bowley's Model with Bounded Rationality," *Chaos, Solitons and Fractals*, 12, 1705-1717, 2001.

[2] Bellman, R. and K. L. Cooke, *Differential-difference Equations,* Academic Press, New York, 1956.

[3] Bischi, G-I, C. Chiarella, M. Kopel and F. Szidarovszky, *Nonlinear Oligopolies: Stability and Bifurcations*, Berlin/Heidelberg/New York, Springer-Verlag, 2010.

[4] Cushing, J., *Integro-differential Equations and Delay Models in Population Dynamics*, Springer-Verlag, Berlin/Heidelberg/New York, 1977.

[5] Matsumoto, A. and F. Szidarovszky, "Nonlinear Delay Monopoly with Bounded Rationality," Discussion Paper #166, Institute of Economic Research, Chuo University, 2011a.

[6] Matsumoto, A. and F. Szidarovszky, "Boundedly Rational Monopoly with Continuously Distributed Time Delays," mimeo, 2011b.

[7] Okuguchi, K. and K. Irie, "The Schur and Samuelson Conditions for a Cubic Equation," *Macnhester School*, 58, 414-418, 1990.

[8] Okuguchi, K. and F. Szidarovszky, *The Theory of Oligopoly with Multi-Product Firms*, Berlin/Heidelberg/New York, Springer-Verlag, 1990.

[9] Puu, T., *Attractions, Bifurcations and Chaos: Nonlinear Phenomena in Economics, 2nd Edition*, Berlin/Heidelberg/New York, Springer-Verlag, 2003.

[10] Sun, Z., "Effect of Delay on Stability of Monopoly System with Delayed Bounded Rationality," forthcoming in *Chinese Journal of Physics.*