Monetary Stabilization Policy by Means of Taylor Rule in a Dynamic Keynesian Model with Capital Accumulation

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Abstract

In this paper, we investigate the macroeconomic effects of monetary stabilization policy by means of Taylor interest rate rule by using an analytical framework of the dynamic Keynesian model, which considers the capital accumulation effect of investment expenditure explicitly. We prove analytically that the dynamic stability/instability of the system depends on the sensitivities of the response of central bank's monetary policy and the parameters which characterize the way of expectation formation, and the cyclical fluctuations occur at the intermediate parameter values. We also present some numerical simulations which support the above mentioned analytical results.

Keywords：Monetary stabilization policy, Taylor rule, Dynamic Keynesian model, capital accumulation, inflation targeting, employment targeting, numerical simulations

1. Introduction

This paper is a sequel to a series of papers by the same author (Asada 2006a, 2006b, 2008, 2009) which dealt with the theoretical analyses of the monetary policy especially in reference to the Japanese economy in the period from the late 1990s to the 2000s, that was accompanied by the serious 'deflationary depression'.[1]

Asada(2006a, 2006b, 2008) applied the analytical method of the 'high-dimensional Keynesian macrodynamics' which was developed by Asada, Chen, Chiarella and Flaschel(2006), Asada, Chiarella, Flaschel and Franke(2003) etc. to the problems of the monetary policy in the period of the deflationary depression.[2] The spirit of these models is 'Keynesian' in that employment and production are determined by the 'effective demand' in the sense of Keynes(1936), and the possibility of the involuntary unemployment of labor as well as the under-utilization of capital stock is allowed for. Moreover, the capital accumulation effect of investment expenditure was explicitly considered in these models. In other words, these models can deal with the 'long run' effects as well as the 'short run' effects of the monetary policy.

By the way, in these models, the growth rate of nominal money supply was treated as a policy variable of the central bank, and the effectiveness of the inflation targeting as well as the importance of the inflation expectation formation was considered. It is well known, however, that the policy makers of BOJ (Bank of Japan) assert that their policy variable is not the (growth rate of) nominal money supply but the nominal rate of interest like the so called 'Taylor rule' that was proposed by Taylor(1993). Asada(2009) applied the 'Taylor rule' to the study of the monetary policy in the period of the deflationary depression, and showed that we can obtain essentially the same qualitative policy conclusion as that of the models with an alternative monetary policy rule.

It is worth noting, however, that Asada(2009) is a simplified small scale (in fact, two-dimensional) model that neglects the capital accumulation effect of the investment expenditure. In this paper, we study the effect of the monetary stabilization policy by means of the 'Taylor rule' in a dynamic Keynesian model that explicitly considers the capital accumulation effect of the investment expenditure.

This paper is organized as follows. In section 2, we present the analytical framework of the model. In section 3, we derive a system of fundamental dynamical equations,

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[1] As for the empirical analyses of the Japanese economy in this period, see Hamada(2004), Harada and Iwata (eds.)(2002) and Ito and Ban(2006) as well as Asada(2009). As for the theoretical analyses of the related topics, see Eggertsson and Woodford(2006), Gong(2005), Krugman(1998), McCallum(2000), and Reifschneider and Williams(2000).

[2] 'High-dimensional' macrodynamic model means the macrodynamic model with many variables.

which consists of three-dimensional system of nonlinear differential equations, and study the properties of its equilibrium solution. In section 4, we provide some analytical results on the local stability/instability of the equilibrium point and the existence of the cyclical fluctuations around the equilibrium point. In section 5, we provide some numerical simulations which support the analytical results in section 4. Section 6 is devoted to some concluding remarks, which refer to some popular approaches which we did not adopt in this paper *deliberately*. Somewhat complicated and lengthy mathematical proof of a proposition is relegated to the appendix.

2. The model

The model in this paper consists of the following system of equations.

$$y = c(y, \pi^e, \tau) + g(r - \pi^e) + h \tag{1}$$

$$m = \phi(r, \pi^e) y \tag{2}$$

$$\dot{w}/w = \varepsilon(e - \bar{e}) + n_2 + \pi^e \ ; \ \varepsilon > 0 \tag{3}$$

$$p = z(wN/Y) = zw/a \ ; \ z > 1 \tag{4}$$

$$\dot{e}/e = \dot{y}/y + g(r - \pi^e) - (n_1 + n_2) = \dot{y}/y + g(r - \pi^e) - n(e) \tag{5}$$

$$\dot{r} = \begin{cases} \alpha(\pi - \bar{\pi}) + \beta(e - \bar{e}) & if \quad r > 0 \\ \max[0, \alpha(\pi - \bar{\pi}) + \beta(e - \bar{e})] & if \quad r = 0 \end{cases} \ ; \ \alpha \geqq 0, \ \beta \geqq 0 \tag{6}$$

$$\dot{\pi}^e = \gamma[\theta(\bar{\pi} - \pi^e) + (1 - \theta)(\pi - \pi^e)] \ ; \ \gamma > 0, \ 0 \leqq \theta \leqq 1 \tag{7}$$

where the meanings of the symbols are as follows. $Y =$ real national income(real output). $K =$ real capital stock. $y = Y/K =$ output-capital ratio, which reflects the rate of capacity utilization of capital stock. $C =$ real private consumption expenditure. $c = C/K =$ consumption-capital ratio. $\dot{K} = I =$ real investment expenditure(real capital accumulation). $g = \dot{K}/K = I/K =$ rate of investment(rate of capital accumulation). $G =$ real government expenditure. $h = G/K =$ government expenditure-capital ratio(fixed). $\tau =$ marginal tax rate(fixed, $0 < \tau < 1$). $r =$ nominal rate of interest $\geqq 0$. $p =$ price level > 0. $\pi = \dot{p}/p =$ rate of price inflation. $\pi^e =$ expected rate of price inflation. $\bar{\pi} =$ target rate of price inflation. $r - \pi^e =$ expected real rate of interest. $M =$ nominal money supply. $m = M/(pK) =$ real money supply-capital ratio. $w =$ nominal wage rate > 0. $N =$ labor employment. $N^s =$ labor supply. $e = N/N^s =$ rate of employment = 1 – rate of unemployment $(0 \leqq e \leqq 1)$. $\bar{e} =$ 'natural' rate of employment $(0 < \bar{e} < 1)$. $a = Y/N =$ average labor productivity. $n_1 = \dot{N}^s/N^s =$ growth rate of labor supply. $n_2 = \dot{a}/a =$ growth rate of average labor productivity(rate of technical progress).

$n = n_1 + n_2 = n(e) =$'natural' rate of growth$>0 \, (n_e = dn/de \geqq 0)$. [3]

Eq. (1) is the 'IS equation', which is nothing but the equilibrium condition of the goods market. In this equation, $c(y, \pi^e, \tau)$ is a standard Keynesian consumption function $(0 < c_y = \partial c/\partial y < 1, \; c_{\pi^e} = \partial c/\partial \pi^e \geqq 0, \; c_\tau = \partial c/\partial \tau < 0)$ and $g(r - \pi^e)$ is a standard Keynesian investment function $(g_{r-\pi^e} = dg/d(r - \pi^e) < 0)$. Solving this equation with respect to y, we obtain the following reduced form of the IS equation.

$$y = y(r - \pi^e, \pi^e, h, \tau) \; ; \; y_{r-\pi^e} = \partial y/\partial(r - \pi^e) < 0, \; y_{\pi^e} \geqq 0,$$

$$y_h = \partial y/\partial h > 0, \; y_\tau = \partial y/\partial \tau < 0 \qquad (8)$$

This equation shows that the rate of capacity utilization of capital stock is determined by the effective demand in our model.[4]

Eq. (2) is the 'LM equation', which is the equilibrium condition of the money market.[5] The right hand side of this equation is a standard Keynesian money demand function, where $\phi_r = \partial \phi/\partial r < 0$ and $\phi_{\pi^e} = \partial \phi/\partial \pi^e \leqq 0$. Differentiating Eq. (2) with respect to time, we obtain the following dynamic version of the LM equation.

$$\mu = \pi + g(r - \pi^e) + (\dot{y}/y) - \eta_r(\dot{r}/r) - \eta_{\pi^e}(\dot{\pi}^e/\pi^e) \; ; \; \mu = \dot{M}/M \qquad (9)$$

where $\eta_r = \left| (\frac{\partial \phi}{\partial r})/(\frac{\phi}{r}) \right| = -(\frac{\partial \phi}{\partial r})/(\frac{\phi}{r})$ and $\eta_{\pi^e} = \left| (\frac{\partial \phi}{\partial \pi^e})/(\frac{\phi}{\pi^e}) \right| = -(\frac{\partial \phi}{\partial \pi^e})/(\frac{\phi}{\pi^e})$ are elasticities of the real money demand per capital stock with respect to changes of the nominal rate of interest and the expected rate of price inflation respectively.

Eq. (3) is a standard expectations-augmented wage Phillips curve. Eq. (4) represents the markup pricing principle of the imperfect competitive firms. The parameter z is the average markup, which is assumed to be a constant that reflects the 'degree of

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[3] This means that in this model the 'natural rate of growth' can be an endogenous variable that is influenced by the rate of employment. As for the rationale of this formulation, see Asada(2008) pp. 87 – 88.

[4] In this model, the fixed technological coefficient is assumed. Nevertheless, the output-capital ratio (y) becomes a variable that reflects the rate of capacity utilization of the capital stock (δ) through the equation $y = \delta y_f$, where y_f is the output-capital ratio in case of full utilization of the capital stock, which is assumed to be a constant(cf. Asada 2008).

[5] In case of the so called 'liquidity trap' with $r = 0$, any value of m which is not smaller than the right hand side of Eq. (2) can become the equilibrium value of m.

monopoly' in the sense of Kalecki(1971). Differentiating Eq. (4) with respect to time, we have

$$\pi = (\dot{w}/w) - (\dot{a}/a) = (\dot{w}/w) - n_2. \tag{10}$$

Substituting this equation into Eq. (3), we have the following expectations-augmented price Phillips curve.

$$\pi = \varepsilon(e - \bar{e}) + \pi^e \ ; \ \varepsilon > 0 \tag{11}$$

Eq. (5) describes the dynamic of the rate of employment. We can derive this equation by the following way. By definition, we have

$$N = \frac{(Y/K)K}{Y/N} = yK/a. \tag{12}$$

Therefore, we have the following expression.

$$e = N/N^s = (yK)/(aN^s). \tag{13}$$

Differentiating this equation with respect to time, we obtain Eq. (5).

Eq. (6) is a version of the monetary policy by means of the 'Taylor interest rate policy rule' (so called the Taylor rule) that was introduced by Taylor(1993). In this formulation, it is assumed that the central bank raises or reduces the nominal rate of interest toward the realization of the target rate of inflation as well as the 'natural' rate of employment. This is a kind of 'inflation targeting' as well as 'employment targeting' monetary policy rule.[6] In this formulation, the nonnegative constraint on the nominal rate of interest is also considered.

Eq. (7) describes a hypothesis concerning the inflation expectation formation, which is a mixture of the 'forward looking' and the 'backward looking (or 'adaptive') inflation expectations. The parameter θ is the weight of the 'forward looking' inflation expectation formation, which can be considered to reflect the 'credibility' of the announcement by the central bank concerning the inflation targeting.[7]

A dynamic system which consists of six equations (5), (6), (7), (8), (9) and (11) is enough to determine the dynamics of six endogenous variables e, r, π^e, y, μ and π.

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[6] It is assumed that the central bank announces the target rate of inflation $(\bar{\pi})$ to the public, so that the public can use this information to form the inflation expectation (cf. Eq. (7)). As for the empirical and theoretical arguments on inflation targeting, see, for example, Asada(2006 a,b), Asada(2008), Asada(2009), Bernanke, Laubach, Mishkin and Posen(1999), Galí(2008), Ito and Ban(2006), Krugman(1988), and Woodford(2003).

[7] This type of 'mixed' expectation formation hypothesis was introduced by Asada, Chiarella, Flaschel and Franke(2003), Asada(2006 a,b), Asada(2008), and Asada(2009).

3. A system of fundamental dynamical equations and the properties of its equilibrium solution

In this section, we shall reduce a system of equations that was presented in the previous section to the more compact form.

Differentiating Eq. (8) with respect to time, we have the following expression.

$$\dot{y} = y_{r-\pi^e}(r,\pi^e)\dot{r} + \{-y_{r-\pi^e}(r,\pi^e) + y_{\pi^e}(r,\pi^e)\}\dot{\pi}^e \qquad (14)$$

Substituting equations (8), (11) and (14) into equations (5), (6) and (7), we have the following complete three dimensional system of nonlinear differential equations.

(i) $\quad \dot{r} = f_1(\pi^e,e) = \begin{cases} \alpha\{\varepsilon(e-\bar{e})+\pi^e-\bar{\pi}\}+\beta(e-\bar{e}) & if \quad r>0 \\ \max[0,\alpha\{\varepsilon(e-\bar{e})+\pi^e-\bar{\pi}\}+\beta(e-\bar{e})] & if \quad r=0 \end{cases}$

(ii) $\quad \dot{\pi}^e = f_2(\pi^e,e) = \gamma[\theta(\bar{\pi}-\pi^e)+(1-\theta)\varepsilon(e-\bar{e})]$

(iii) $\quad \dot{e} = f_3(r,\pi^e,e) = e[\{1/y(r-\pi^e,\pi^e)\}\{y_{r-\pi^e}(r,\pi^e)f_1(\pi^e,e)$

$$+\{-y_{r-\pi^e}(r,\pi^e)+y_{\pi^e}\}f_2(\pi^e,e)\}+g(r-\pi^e)-n(e)] \qquad (15)$$

We can consider that this is a system of 'fundamental dynamical equations' in our model. This system is supplemented by the dynamic LM equation (Eq. (9)), which determines the dynamic of the growth rate of the nominal money supply (μ) endogenously. However, this equation does not feed back to the dynamics of the system of equations (15), so that we can safely neglect Eq. (9) when we consider the fundamental dynamical system.

Next, let us consider the 'normal' equilibrium solution of the system (15) such that $\dot{r} = \dot{\pi}^e = \dot{e} = 0$ and $e = \bar{e}$. If we neglect the nonnegative constraint on the nominal rate of interest, we have the following 'normal' equilibrium solution $(r*,\pi^e*,e*)$.

$$e* = \bar{e} \qquad (16)$$

$$\pi^e* = \pi* = \bar{\pi} \qquad (17)$$

$$g(r*-\bar{\pi}) = n(\bar{e}) \qquad (18)$$

In this case we also have $\dot{y} = 0$ from Eq. (14). Substituting the relationships $\dot{y} = \dot{r} = \dot{\pi}^e = 0$ and equations (17) and (18) into Eq. (9), we obtain the following equilibrium growth rate of nominal money supply.

$$\mu* = \bar{\pi} + n(\bar{e}) \qquad (19)$$

Equations (16) and (18) mean that the rate of employment and the rate of capital accumulation are settled down to their 'natural' values at the normal equilibrium point. Eq. (17) means that the 'actual' and the 'expected' rates of inflation are equal to the

'target' rate of inflation at the normal equilibrium point. Eq. (19) implies that the equilibrium growth rate of nominal money supply is determined by the 'target' rate of inflation and the equilibrium value of the 'natural' rate of growth, and not the other way round in this model of endogenous money supply.

We can interpret the determination of the equilibrium nominal rate of interest (r^*) as follows. The equilibrium real rate of interest (ρ^*) is determined by

$$g(\rho^*) = n(\bar{e}). \tag{20}$$

Then, r^* is determined by

$$r^* = \rho^* + \bar{\pi}. \tag{21}$$

Needless to say, we have $r^* > 0$ if and only if the inequality

$$\bar{\pi} > -\rho^* \tag{22}$$

is satisfied. This inequality can *always* be satisfied if the target rate of inflation $(\bar{\pi})$ is set to be sufficiently large, *even if* $\rho^* < 0$.[8] On the other hand, the deflationary-biased central bank such as BOJ in the late 1990s and the 2000s may fail to satisfy the inequality (22). In case of $\bar{\pi} < -\rho^*$, the economically meaningful 'normal' equilibrium does not exist.

4. Stability/instability analysis and the existence of cyclical fluctuations

Next, we shall investigate the local stability/instability of the 'normal' equilibrium point of the system (15) by *assuming* that the inequality (22) is in fact satisfied. We can write the Jacobian matrix of this system that is evaluated *at the equilibrium point* as follows.

$$J = \begin{bmatrix} 0 & \alpha & \alpha\varepsilon + \beta \\ 0 & -\gamma\theta & \gamma\varepsilon(1-\theta) \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \tag{23}$$

where $f_{31} = \bar{e}\, g^*_{r-\pi^e} < 0$, $f_{32} = \bar{e}[\{y^*_{r-\pi^e}\alpha + (y^*_{r-\pi^e} - y^*_{\pi^e})\gamma\theta\}/y^* - g^*_{r-\pi^e}]$, and
$\underset{(-)}{}$ $\underset{(-)}{}$ $\underset{(-)}{}$ $\underset{(+or0)}{}$ $\underset{(-)}{}$

$$f_{33} = \bar{e}[y^*_{r-\pi^e}(\alpha\varepsilon + \beta)/y^* + \{-y^*_{r-\pi^e} + y^*_{\pi^e}\}\gamma\varepsilon(1-\theta)/y^* - n_e].$$
$\underset{(-)}{}$ $\underset{(-)}{}$ $\underset{(+or0)}{}$ $\underset{(+or0)}{}$

Then, the characteristic equation of this system *at the equilibrium point* becomes as follows(cf. mathematical appendix of Asada, Chiarella, Flaschel and Franke 2003).

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[8] The equilibrium real rate of interest ρ^* can be negative if the equilibrium 'natural' rate of growth $n(\bar{e})$ is too large.

$$\Delta(\lambda) = |\lambda I - J| = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{24}$$

where

$$a_1 = -traceJ = \underset{(?)}{\gamma\theta - f_{33}}, \tag{25}$$

$a_2 = $ sum of all principal second-order minors of J

$$= \begin{vmatrix} 0 & \alpha \\ 0 & -\gamma\theta \end{vmatrix} + \begin{vmatrix} 0 & \alpha\varepsilon + \beta \\ f_{31} & f_{33} \end{vmatrix} + \gamma\begin{vmatrix} -\theta & \varepsilon(1-\theta) \\ f_{32} & f_{33} \end{vmatrix}$$

$$= \underset{(-)}{-(\alpha\varepsilon + \beta)f_{31}} - \gamma\{\underset{(?)}{\theta f_{33}} + \underset{(?)}{\varepsilon(1-\theta)f_{32}}\}, \tag{26}$$

$$a_3 = -\det J = -f_{31}\gamma\begin{vmatrix} \alpha & \alpha\varepsilon + \beta \\ -\theta & \varepsilon(1-\theta) \end{vmatrix}$$

$$= \underset{(-)}{-f_{31}}\gamma\{\alpha\varepsilon(1-\theta) + (\alpha\varepsilon + \beta)\theta\} > 0. \tag{27}$$

It is well known that the equilibrium point of the system (15) is locally asymptotically stable if and only if the following set of inequalities is satisfied(cf. Gandolfo 1996 p.221).[9]

$$a_1 > 0, \quad a_3 > 0, \quad a_1 a_2 - a_3 > 0 \tag{28}$$

In our model, the inequality $a_3 > 0$ is always satisfied, so that we can reduce the local stability conditions to the following set of two inequalities.

$$a_1 > 0, \quad a_1 a_2 - a_3 > 0 \tag{29}$$

Now, we can easily prove the following proposition that describes a set of *sufficient* conditions for *local instability*.

Proposition 1.

Suppose that the monetary policy parameter values $\alpha \geqq 0$ and $\beta \geqq 0$ are fixed arbitrarily. Furthermore, suppose that the value of the credibility parameter $\theta \in [0,1]$ is sufficiently close to zero (including the case of $\theta = 0$) and the adjustment speed of expectation adaptation $\gamma > 0$ is sufficiently large. Then, the equilibrium point of the system (15) is locally unstable.

(Proof.)

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[9] In this case, the inequality $a_2 > 0$ is automatically satisfied.

First, let us consider the case of $\theta = 0$. In this case, it is easy to see that we have $a_1 < 0$ for all sufficiently large values of $\gamma > 0$, which violates one of the local stability conditions (29). By continuity, this conclusion also applies to the case of $\theta > 0$ as long as θ is close to zero. □

On the other hand, the following two propositions present some economically meaningful *sufficient* conditions for *local stability.*

Proposition 2.
(1) Suppose that the parameter values of $\beta \geqq 0$, $\theta \in [0,1]$, and $\gamma > 0$ are fixed arbitrarily. Then, the equilibrium point of the system (15) is locally asymptotically stable for all sufficiently large values of $\alpha > 0$.
(2) Suppose that the parameter values of $\alpha \geqq 0$, $\theta \in [0,1]$, and $\gamma > 0$ are fixed arbitrarily. Then, the equilibrium point of the system (15) is locally asymptotically stable for all sufficiently large values of $\beta > 0$.

(Proof.)

(1) We can easily see that a_1 is a linear increasing function of α, and $a_1 a_2 - a_3$ is a

quadratic function of α in which the coefficient of the term α^2 is positive. This means that both of two inequalities in (29) are satisfied for all sufficiently large values of $\alpha > 0$.
(2) The method of the proof of **Proposition 2** (1) is essentially the same as that of **Proposition 2** (1). □

Proposition 3.
Suppose that the parameter values $\alpha > 0$, $\beta \geqq 0$, and $\gamma > 0$ are fixed arbitrarily. Then, the equilibrium point of the system (15) is locally asymptotically stable for all $\theta \in (0,1]$ which are sufficiently close to 1 (including the case of $\theta = 1$).

(Proof.)

Suppose that $\theta = 1$. In this, case, we have $f_{33} < 0$ and the following relationships are satisfied.

$$a_1 = \gamma - \underset{(-)}{f_{33}} > 0 \tag{30}$$

$$a_2 = -(\alpha\varepsilon + \beta)\underset{(-)}{f_{31}} - \gamma\underset{(-)}{f_{33}} > 0 \tag{31}$$

$$a_3 = -\underset{(-)}{f_{31}}\,\gamma(\alpha\varepsilon + \beta) > 0 \tag{32}$$

$$a_1 a_2 - a_3 = \underset{(-)}{f_{33}}\underset{(-)}{f_{31}}(\alpha\varepsilon + \beta) + (\underset{(-)}{f_{33}} - \gamma)\gamma\underset{(-)}{f_{33}} > 0 \tag{33}$$

Inequalities (30) and (33) mean that all of the local stability conditions (29) are satisfied in case of $\theta = 1$. By continuity, these inequalities are satisfied even if $0 < \theta < 1$, as long as θ is sufficiently close to 1. □

Next, we shall consider the bifurcation analysis by choosing the monetary policy parameter $\alpha \geqq 0$ as a bifurcation parameter.[10]

Proposition 4.

Suppose that the parameter values $\beta \geqq 0$ and $\theta \in [0,1)$ are fixed at the levels such that $a_1 < 0$ when $\alpha = 0$. Then, we have the parameter value $\alpha_0 > 0$ such that the following properties (1) and (2) are satisfied.

(1) The equilibrium point of the system (15) is locally unstable for all $\alpha \in [0, \alpha_0)$, and it is locally asymptotically stable for all $\alpha \in (\alpha_0, +\infty)$.

(2) There exists a family of non-constant closed orbits around the equilibrium point of the system (15) for some range of the parameter value α which is sufficiently close to α_0.

(Proof.) See **Appendix**.

Proposition 1 means that the high adjustment speed of price expectations that is combined with the public's highly backward-looking (adaptive) expectation formation is a *destabilizing* factor of the macroeconomic system. We can represent this destabilizing positive feedback mechanism schematically as follows.

$$e \downarrow \Rightarrow \pi \downarrow \Rightarrow \pi^e \downarrow \Rightarrow (r - \pi^e) \uparrow \Rightarrow g \downarrow \Rightarrow y \downarrow \Rightarrow e \downarrow \tag{M_1}$$

Proposition 2 says that the quick response of the monetary authority (central bank) to

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[10] We can obtain essentially the same conclusion as **Proposition 4** even if we choose the parameter $\beta \geqq 0$ or $\theta \in [0,1]$ as a bifurcation parameter.

the macroeconomic conditions is a *stabilizing* factor. This stabilizing negative feedback mechanism can be expressed as follows.

$$e \downarrow \Rightarrow r \downarrow \Rightarrow (r - \pi^e) \downarrow \Rightarrow g \uparrow \Rightarrow y \uparrow \Rightarrow e \uparrow \qquad (M_2)$$

Proposition 3 implies that the public's highly forward-looking expectation formation (high credibility of the monetary authority's announcement on inflation targeting) is a *stabilizing* factor. This stabilizing negative feedback mechanism may be expressed as follows.[11]

$$e \downarrow \Rightarrow \pi \downarrow \Rightarrow \pi^e \downarrow \Rightarrow \bar{\pi} > \pi^e \Rightarrow \pi^e \uparrow \Rightarrow (r - \pi^e) \downarrow \Rightarrow g \uparrow \Rightarrow y \uparrow \Rightarrow e \uparrow \qquad (M_3)$$

Proposition 4 means that the cyclical fluctuations occur at the intermediate values of the monetary policy parameters.

5. A numerical illustration

In this section, we shall present some numerical simulations which support the analytical results in the previous section. Let us assume the following parameter values and the functional form of the investment function.[12]

$$\bar{e} = 0.97, \quad \bar{\pi} = 0.03 = 3\% \text{ per year}, \quad n = 0.05 = 5\% \text{ per year} \qquad (34)$$

$$g(r - \pi^e) = -0.2(r - \pi^e - 0.03) + 0.05 = 0.2(\pi^e - r) + 0.056 \qquad (35)$$

These data are enough to determine the equilibrium values $(e^*, \pi^e*, r^*, \mu^*)$ which are given by equations $(16) - (19)$. In fact, we obtain the following results.

$$e^* = \bar{e} = 0.97, \quad \pi^e* = \pi^* = \bar{\pi} = 0.03 = 3\% \text{ per year}, \quad r^* = 0.06 = 6\% \text{ per year},$$

$$\mu^* = \bar{\pi} + n = 0.03 + 0.05 = 0.08 = 8\% \text{ per year}. \qquad (36)$$

In this case, the equilibrium real rate of interest ρ^* becomes

$$\rho^* = r^* - \bar{\pi} = 0.06 - 0.03 = 0.03 = 3\% \text{ per year}. \qquad (37)$$

Next, let us derive the explicit functional form of the function $y(r - \pi^e, \pi^e, h, \tau)$ in Eq. (8) under the simplified specification of the consumption function. For simplicity, we

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[11] In this schematic representation, it is assumed that at the initial stage of expectation formation, the 'backward-looking' factor is strong, but at the latter stage, the 'forward-looking' factor becomes strong because of the increase of the credibility of the central bank's announcement of the target rate of inflation.

[12] In this example, it is assumed that the 'natural' rate of growth n does not depend on the rate of employment for simplicity.

assume the linear consumption function $c = \sigma(1-\tau)y + \bar{c}$ neglecting the effect of π^e on c, where $\sigma \in (0,1)$ is the marginal propensity to consume from the disposable income. In this case, we obtain the following relationship solving Eq. (1) with respect to y.

$$y(r - \pi^e, \pi^e, h, \tau) = \frac{1}{1 - \sigma(1-\tau)}\{g(r - \pi^e) + \bar{c} + h\} \tag{38}$$

Furthermore, let us assume the following parameter values.

$$\sigma = 0.7, \quad \tau = 0.3, \quad \bar{c} = 0.03, \, h = 0.03 \tag{39}$$

Substituting equations (35) and (39) into Eq. (38), we obtain the following approximate solution.

$$y(r - \pi^e, \pi^e, h, \tau) \cong 1.96\{-0.2(r - \pi^e - 0.03) + 0.11\}$$
$$= 0.392(\pi^e - r) + 0.22736 \tag{40}$$

In this case, we have

$$y_{r - \pi^e} = -0.392, \quad y_{\pi^e} = 0. \tag{41}$$

Substituting $r = r* = 0.06$ and $\pi^e = \pi^e* = 0.03$ into Eq. (40), we have the following approximate solution of the equilibrium value $y*$.

$$y* \cong 1.96 \times 0.11 = 0.2156 \tag{42}$$

Suppose, furthermore, that

$$\beta = 0, \quad \varepsilon = 0.2, \quad \gamma = 0.3. \tag{43}$$

Then, the system of equations (15) becomes as follows.[13]

（i）$\dot{r} = f_1(\pi^e, e; \alpha) = \begin{cases} \varphi_1(\pi^e, e; \alpha) & if \quad r > 0 \\ \max[0, \varphi_1(\pi^e, e; \alpha)] & if \quad r = 0 \end{cases}$

（ii）$\dot{\pi}^e = f_2(\pi_e, e; \theta) = 0.3[\theta(0.03 - \pi^e) + 0.2(1 - \theta)(e - 0.97)]$

（iii）$\dot{e} = f_3(r, \pi^e, e; \alpha, \theta) = \begin{cases} \varphi_2(r, \pi^e, e; \alpha, \theta) & if \quad 0 < e < 1 \\ \min[0, \varphi_2(r, \pi^e, e; \alpha, \theta)] & if \quad e = 1 \end{cases} \tag{44}$

where

$$\varphi_1(\pi^e, e; \alpha) = \alpha\{0.2(e - 0.97) + \pi^e - 0.03\}, \tag{45}$$

$$\varphi_2(r, \pi^e, e; \alpha, \theta) = e[\frac{-f_1(\pi^e, e; \alpha) + f_2(\pi^e, e; \theta)}{\pi^e - r + 0.58} + 0.2(\pi^e - r) + 0.006]. \tag{46}$$

Figures 1 − 5 are the results of our numerical simulation of the 'out of equilibrium'

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[13] In this formulation, an additional natural nonlinearity, which means that the rate of employment cannot exceed 1, is introduced.

dynamics corresponding to the following initial conditions.

$$r(0) = r*+0.01 = 0.07, \quad \pi^e(0) = \pi^e*-0.01 = 0.02, \quad e(0) = e*-0.02 = 0.95 \quad (47)$$

We consider the following three alternative scenarios, where t denotes the time period, and the unit time period is interpreted as a year.[14]

Case A： $\alpha = 0$ and $\theta = 0$ for $0 \leqq t < 5$, $\alpha = 0.1$ and $\theta = 0$ for $t \geqq 5$.

Case B： $\alpha = 0$ and $\theta = 0$ for $0 \leqq t < 5$, $\alpha = 0.13$ and $\theta = 0$ for $t \geqq 5$.

Case C： $\alpha = 0$ and $\theta = 0$ for $0 \leqq t < 5$, $\alpha = 0.13$ and $\theta = 0$ for $5 \leqq t < 10$, $\alpha = 0.13$ and $\theta = 0.1$ for $t \geqq 10$.

Case D： $\alpha = 0$ and $\theta = 0$ for $0 \leqq t < 5$, $\alpha = 0.13$ and $\theta = 0$ for $5 \leqq t < 10$, $\alpha = 0.13$ and $\theta = 0.3$ for $t \geqq 10$.

Comparison of the cases A and B suggests that the increase of the speed of the response of central bank's monetary policy (α) has a stabilizing effect, but slight increase of the value of the parameter α may not sufficient to restore the 'normal' equilibrium because of the existence of the nonnegative constraint on the nominal rate of interest, if the central bank does not announce the target rate of inflation at all or the announcement of the inflation targeting by the central bank is highly incredible for the public.[15]

Comparison of the cases B, C, and D suggests that even the slight increase of the 'credibility' of the inflation targeting by the central bank has surprisingly strong power of macroeconomic stabilization. In fact, in our numerical example the economy moves, although relatively slowly, toward the 'normal' equilibrium point, if only 10% of the population think that the central bank's announcement of the 3% target rate of inflation per year is credible.[16] As our numerical example shows, this conclusion is true even when the economy is stuck to the so called 'liquidity trap' with the lower bound of the nominal rate of interest like the Japanese economy in the late 1990s and the 2000s. In our numerical example, the economy moves to the state of full employment fairly rapidly if 30% of the population believe the central bank's target rate of inflation.

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[14] For the numerical simulation, we adopted Euler's algorithm and the time interval $\Delta t = 0.1$ (years).

[15] In fact, we can easily see that the normal equilibrium point becomes stable in both of the cases A and B if the *negative* nominal rate of interest is allowed for, but Figures 1 – 3 show that it becomes unstable if the nonnegative constraint on the nominal rate of interest is explicitly considered.

[16] In this explanation, we interpret the parameter value θ as the ratio of the people who believe the central bank's announcement of the target rate of inflation to the total amount of population.

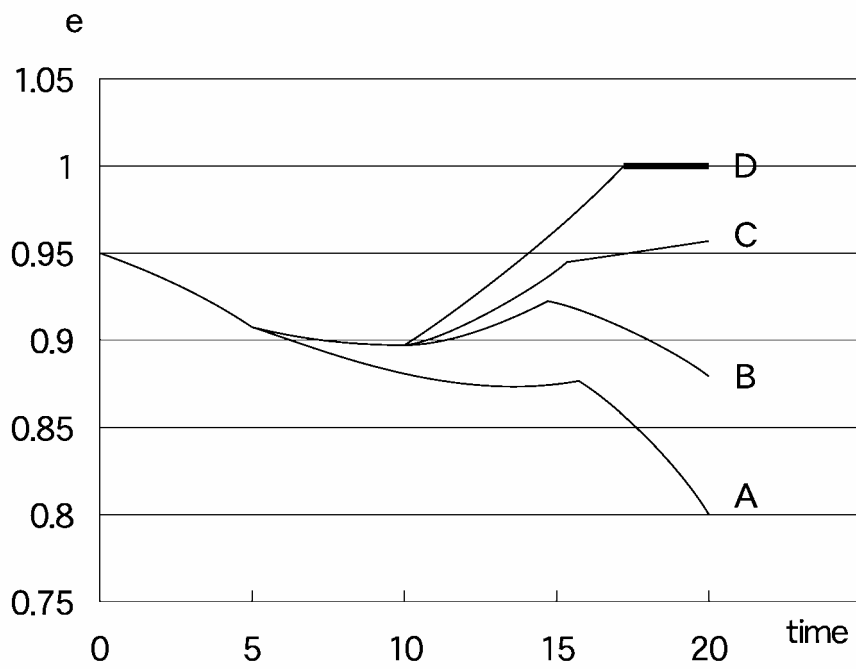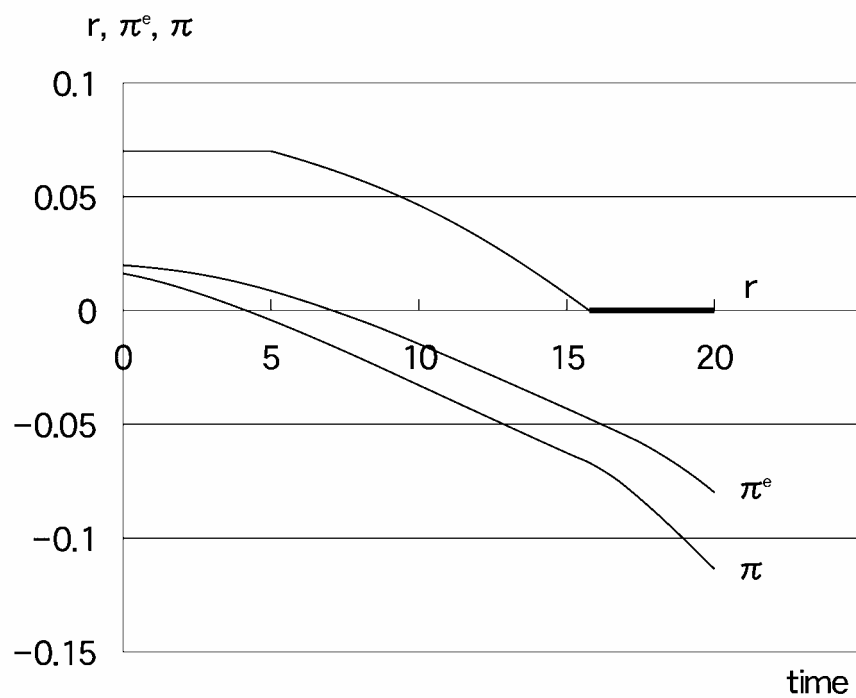Fig.1. Alternative time paths of e
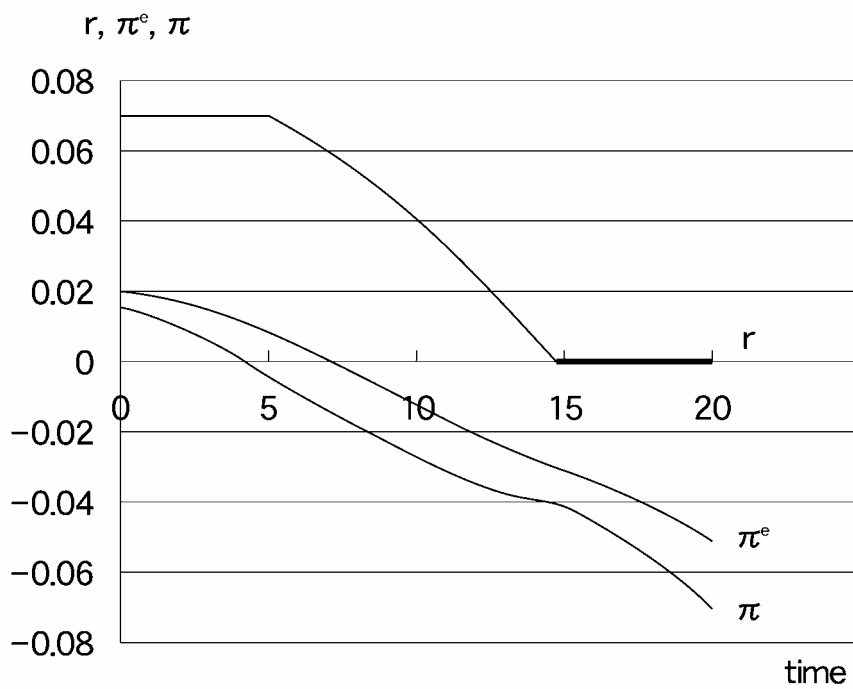(Cases A, B, C, D)

Fig.2. Time paths of r, π^e and π
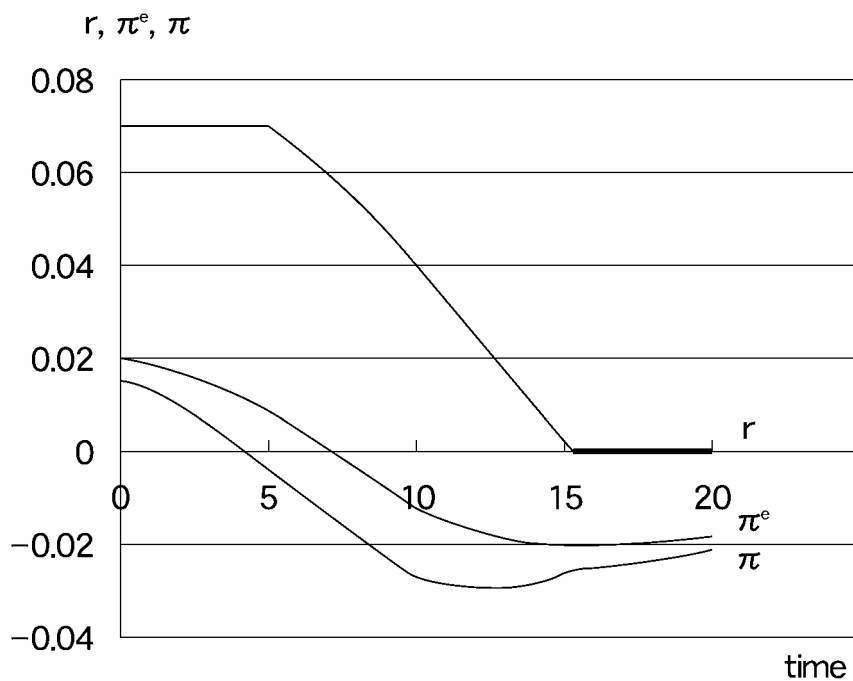(Case A)

Fig.3. Time paths of r, π^e and π
(Case B)
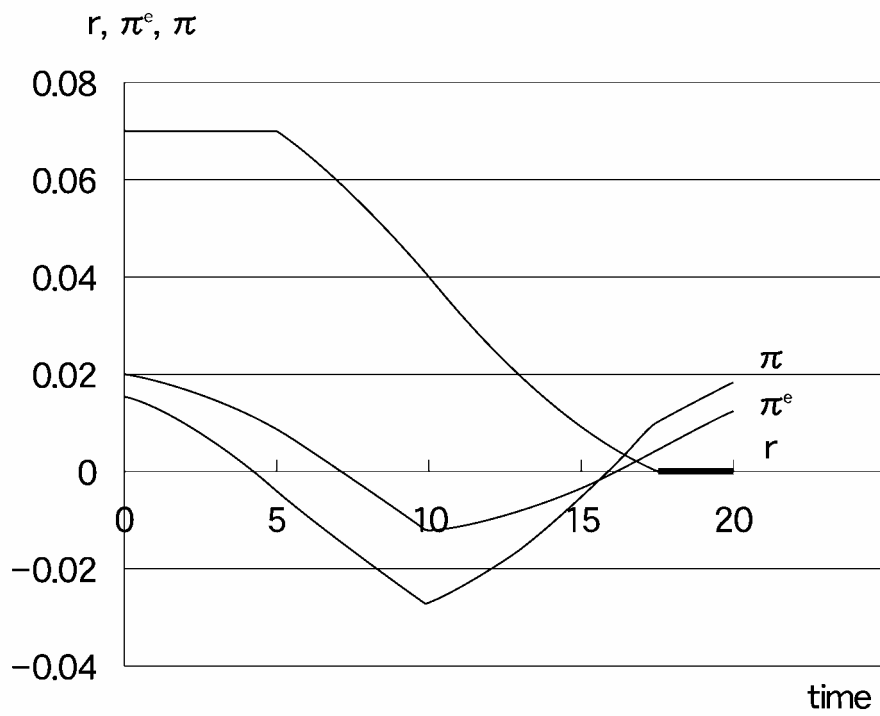

Fig.4. Time paths of r, π^e and π
(Case C)

Fig.5. Time paths of r, π^e and π
(Case D)

It seems that the numerical example in this section suggests the importance of the 'regime switching' of monetary policy, especially the importance of the change of the 'credibility' of the central bank's announcement, for the drastic change of the macroeconomic performance.[17]

6. Concluding remarks

In this final section, we shall briefly comment on two popular mainstream approaches which we did not adopt *deliberately* in this paper, namely, the 'New Keynesian' dynamic model and the 'optimal' monetary policy model.[18] [19]

The so called 'New Keynesian' dynamic model, that is explained extensively in Woodford(2003), Galí(2008) and others, also adopts a kind of IS curve (New Keynesian IS curve) and a kind of Phillips curve (New Keynesian Phillips curve) together with some version of the Taylor interest rate monetary policy rule by the central bank. In this sense, the building blocks of our model are apparently similar to those of the 'New Keynesian' dynamic model. However, our approach is more traditional (so to speak, 'Old Keynesian') than 'New Keynesian' approach, so that we can avoid the notorious anomalies of the dynamic behavior in 'New Keynesian' dynamic model, which was pointed out by Mankiw(2001), Flaschel and Schlicht(2006), Asada, Chen, Chiarella and Flaschel(2006) etc. It is well known that 'New Keynesian' dynamic model is a forward-looking 'rational expectation' model with the so called microfounded optimization behavior of the single 'representative agent' other than the central bank and the government. However, as Mankiw(2001) pointed out correctly, this 'New Keynesian' formulation causes the counter-factual 'sign reversal' problem, which means

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[17] Miyao(2006) chap. 3 provided an argument against the inflation targeting policy which was proposed by Krugman(1998). Miyao(2006) admits that the increase of the expected rate of inflation is effective to increase the production and employment through the increase of current aggregate demand. He points out, however, that the increase of the expected future income also induces the increase of current aggregate demand in Krugman(1998)'s model, and asserts that the so called 'structural reform' can increase the expected future income and it is better than the inflation targeting. However, we can point out following two important facts. First, Miyao(2006)'s argument by no means imply the ineffectiveness of the inflation targeting. Second, the so called 'structural reform' by Koizumi's administration in the early 2000s in Japan, the most notable one is the privatization of the postal service, had nothing to do with the increase of the expected future income of the Japanese public. See also chap. 2 by A. Noguchi in Harada and Iwata(eds.)(2002).

[18] Obviously, these two approaches are sometimes closely related to each other.

[19] The more detailed discussions on the topics in this section are contained in Asada(2009).

that the *rate of change* of the inflation rate becomes the *decreasing* function of the current output level and the actual rate of employment. In other words, in this model the rate of inflation *accelerates* whenever the actual output level is *below* the 'natural' output level.[20] Another problem is that the 'New Keynesian' rational expectation approach treats the expected and the actual rates of inflation as well as the nominal rate of interest as the 'jump variables' or 'not pre-determined variables' which can make the unstable equilibrium point (in the traditional sense) 'stable' by allowing for the discontinuous 'jump' to the convergent path. However, as Mankiw(2001) pointed out, this postulate also contradicts the empirical fact.[21]

On the other hand, in our model the more traditional 'Old Keynesian' postulate is adopted, which means that the expected and actual rates of inflation as well as the nominal rate of interest are treated as the 'pre-determined variables', although in our formulation some 'forward looking' element as well as the 'backward looking' element of the inflation expectation formation is allowed for. In our rather traditional formulation, the anomalies of the behavior of the main macroeconomic variables, which are peculiar to the 'New Keynesian' formulation, do not occur. Next, let us quote our general evaluation of the 'New Keynesian' dynamic model from Asada, Chiarella, Flaschel and Franke(2009) chap. 5.[22]

" While the *microfoundation* of economic behavior is per se an important desideratum to be reflected also by behaviorally oriented macrodynamics, the use of 'representative' consumers and firms for the explanation of macroeconomic phenomena is too simplistic and also too narrow to allow for a proper treatment of what is *really* interesting on the economic behavior of economic agents – the interaction of heterogeneous agents –, and it is also not detailed enough to discuss the various feedback channels present in the real world. *Market Clearing*, the next ingredient of such approaches, may however be a questionable device for studying the macroeconomy in particular on its real side." "Yet, neither microfoundations per se nor market clearing assumptions are the true dividing

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[20] Mankiw(2001) wrote as follows. "Although the new Keynesian Phillips curve has many virtues, it also has one strikingly vice : It is completely at odd with the facts." (Mankiw 2001, p. C52)

[21] Mankiw(2001) wrote as follows. "In these models of staggered price adjustment, the price level adjusts slowly, but the inflation rate can jump quickly. Unfortunately for the model, that is not what we see in the data." (Mankiw 2001, p. C54)

[22] This chapter is entitled as "New Keynesian equilibrium vs. Keynesian disequilibrium dynamics : Two competing approaches". Incidentally, the inflation expectation formation hypothesis that is expressed by Eq. (7) in our model can be considered to be one of the simplest formulations of the 'heterogeneous' expectations hypothesis that is described in this quotation.

line between the approaches we are advocating and the ones considered in this section. It is the ad hoc, that is not behaviorally microfounded assumption of *Rational Expectations* that by the chosen analytical methods makes the world in general loglinear (by construction) and the generated dynamics convergent (by assumption) to its unique steady state which is the root of the discontent that this chapter tries to make explicit. Indeed, agents are heterogeneous, form heterogeneous expectations along other lines than suggested by the rational expectations theory, and have differentiated short- and long-term views about the economy."

Another popular approach that was not adopted in our model is the 'optimal' monetary policy approach. Although this approach is sometimes combined with the 'New Keyneian' dynamic model, this approach can be combined with any other macrodynamic model. In fact, we can also formulate such an approach in the context of our model. For example, let us suppose that the central bank tries to control the time path of $r \geqq 0$ that minimizes the following 'loss function' subject to the constraints of the equations (1) – (5) and (7) in this paper.

$$L = \int_0^\infty \{\xi(\pi - \overline{\pi})^2 + (1-\xi)(e - \overline{e})^2\}e^{-\nu t}dt \quad ; \quad 0 < \xi < 1, \quad \nu > 0 \tag{48}$$

where ξ and ν as well as $\overline{\pi}$ and $\overline{e}$ are constant parameters.

We can solve such a dynamic optimization problem by means of Pontryagin's maximum principle(cf. Gandolfo 1996 chap. 22), and in a typical case the optimal path converges to the steady state with 'natural' rate of employment, positive nominal rate of interest and positive rate of inflation.[23] In this case, the 'deflationary depression' that is accompanied by the 'liquidity trap' with the lower bound of the nominal rate of interest cannot occur at the 'optimal' path.

However, such an 'optimal' monetary policy approach requires that the central bank has omniscient power with the accurate knowledge of the macroeconomic structure (rational expectations), even if it is not assumed that the public has such an omniscient power.[24] In fact, in Japan in the late 1990s and the 2000s the deflationary depression with the liquidity trap *did* occur, which means that not only the Japanese public but also BOJ had no such omniscient power at that period.

It is not practical to stick to the 'optimal' monetary policy to evaluate the performance of actual monetary policy because of the above mentioned reason. Instead, we would like

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[23] See also Asada(2009).

[24] It is worth noting that in the 'New Keynesian' dynamic model it is assumed that the public as well as the central bank has such a power.

to propose to use more practical criterion of the policy evaluation, namely, the 'permissibility' criterion. Asada(2009) proposed the following simple and practical criterion of the 'permissibility' of the monetary policy.

"The monetary policy that can avoid the deflationary depression with the 'liquidity trap' is permissible, while the monetary policy that cannot avoid it is impermissible."

We can easily see that the 'optimal' policy in the above formulation is one of the 'permissible' policies, but there are infinite numbers of 'permissible' but non-optimal policies. The approach in this paper presented an analytical apparatus to evaluate the 'permissibility' rather than 'optimality' of the monetary policy. It is clear that the monetary policy of BOJ during the period from the late 1990s to the 2000s was not only 'non-optimal' but also 'impermissible' in view of our criterion.

Appendix : Proof of Proposition 4.

(1) It is easy to see that a_1 is a linear increasing function of α, and by assumption we have $a_1 < 0$ when $\alpha = 0$. Hence, there exists the unique parameter value $\alpha_1 > 0$ that satisfies the following property.

(P_1) We have $a_1 < 0$ for all $\alpha \in [0, \alpha_1)$, we have $a_1 = 0$ at $\alpha = \alpha_1$, and we have $a_1 > 0$ for all $\alpha \in (\alpha_1, +\infty)$.

On the other hand, $a_1 a_2 - a_3$ is a quadratic function of α in which the coefficient of α^2 is positive, and we have $a_1 a_2 - a_3 = -a_3 < 0$ when $\alpha = \alpha_1$. Therefore, there exists the unique parameter value $\alpha_0 > \alpha_1$ that satisfies the following property.

(P_2) We have $a_1 a_2 - a_3 < 0$ for all $\alpha \in [\alpha_1, \alpha_0)$, we have $a_1 a_2 - a_3 = 0$ at

$\alpha = \alpha_0$, and we have $a_1 a_2 - a_3 > 0$ for all $\alpha \in (\alpha_0, +\infty)$.

Property (P_1) means the equilibrium point of the system (15) is locally unstable for all $\alpha \in [0, \alpha_1)$, because one of the local stability conditions (29) is violated in this range of α. Property (P_2) means that the equilibrium point of this system becomes locally unstable also in the range $\alpha \in [\alpha_1, \alpha_0)$ because of the same reason. In sum, the equilibrium point of this system is locally unstable for all $\alpha \in [0, \alpha_0)$. On the other hand, properties (P_1) and (P_2) mean that the equilibrium point of this system is locally asymptotically stable for all $\alpha \in (\alpha_0, +\infty)$, because all of the local stability conditions (29) are satisfied in this rage of the parameter value α.

(2) At the point $\alpha = \alpha_0$, we have the following property.

(P_3) $a_1 > 0$, $a_3 > 0$, $a_1 a_2 - a_3 = 0$

We can show that the characteristic equation (24) has a pair of pure imaginary roots and one negative real root when the property (P_3) is satisfied (cf. Theorem A 13 in the mathematical appendix of Asada, Chiarella, Flaschel and Franke 2003). Furthermore, we can see that the real part of the complex roots is a decreasing function of α at the point $\alpha = \alpha_0$, since we have $\dfrac{\partial(a_1 a_2 - a_3)}{\partial \alpha} > 0$ at $\alpha = \alpha_0$. This means that the point $\alpha = \alpha_0$ is in fact the 'Hopf bifurcation' point, so that it is ensured that there exists a family of non-constant closed orbits for some range of the parameter value α which is sufficiently close to α_0 (cf. Theorem A 10 of the mathematical appendix of Asada, Chiarella, Flaschel and Franke 2003). $\square$

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