IERCU

Institute of Economic Research, Chuo University
50th Anniversary Special Issues

Discussion Paper No.216

A Special Dynamic System with Two Time Delays

Akio Matsumoto
Chuo University

Ferenc Szidarovszky
University of Pécs

January 2014

INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

A Special Dynamic System with Two Time Delays*

Akio Matsumoto† and Ferenc Szidarovszky‡

Abstract

A special dynamic system is analyzed which describes Goodwin's business cycle model (Goodwin, 1951). In realistic economies there are time delays in both investment and consumption. The two time delays have a significant effect on the asymptotic behavior of the system. Without delay the system is locally asymptotically stable with reasonable parameter selection, however in the presence of delays stability might be lost. This paper gives a complete stability analysis of the delayed system by determining the stability switch curves and characterizing the directions of the stability switches based on the monotonic properties of the curves.

Keywords: Time delays, Stability, Bifurcation

AMS Classifications: 34K20, 37N40, 91B62

*The authors highly appreciate the financial supports from the MEXT-Supported Program for the Strategic Research Foundation at Private Universities 2013-207, the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (C) 24530202 and 25380238) and Chuo University (Grant for Special Research). The usual disclaimers apply.

†Professor, Department of Economics, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan; akiom@tamacc.chuo-u.ac.jp

‡Professor, Department of Applied Mathematics, University of Pécs, Ifúság, u. 6. H-7624, Pécs, Hungary; szidarka@gmail.com

1 Introduction

Physical and economic systems often deal with delayed data, so the dynamic equations describing the motion or development of such systems are usually delay differential equations. The asymptotical behavior of these systems became a central research topic recently. There are two different ways to model time delays (Cooke and Grossman, 1982). In applying the concept of *continuously distributed delays*, it is assumed that the length of the delay is uncertain following a particular distribution. Cushing (1977) provided a comprehensive summary of the relevant methodology with applications to population dynamics. If the length of the delay is known, then *fixed delays* are considered. Bellman and Cooke (1956) introduced the relevant methodology. The methods and stability conditions are model dependent, so researchers have examined particular model types and investigated their asymptotical behavior. The approach becomes much more complicated if multiple delays are present. The pioneering works of Hale (1979) and Hale and Huang (1993) can be considered as basic breakthrough in this area. The paper of Piotrowska (2007) examined some properties of the stability switch curves for important special models. More recently Matsumoto and Szidarovszky (2013) gave a complete description of the stability switches and asymptotical properties of a certain class of dynamic systems arising in the study of dynamic oligopolies. However the same approach cannot be used in the case of different dynamic models such as Goodwin's business cycle model (Goodwin, 1951). In this paper we will examine the local asymptotical behavior of the corresponding two-delay model. The paper is organized as follows. The classical Goodwin model is introduced in Section 2, and its single-delay extension is discussed in Section 3, and then the general case is investigated, where stability switches are determined, and conditions for the local asymptotical stability of the delay system are derived. The last section concludes the paper and further research directions are outlined.

2 The Model

Goodwin's classical model can be described by the following two-dimensional system:

$$\begin{aligned} \varepsilon \dot{y}(t) &= \dot{k}(t) - (1-\alpha)y(t) \\ \dot{k}(t) &= \varphi(\dot{y}(t)) \end{aligned} \tag{1}$$

where y is the national income, k is the capital stock, $\varphi(\dot{y})$ denotes the induced investment and α, ε are positive constants. By combining these equations a single-dimensional nonlinear equation is obtained:

$$\varepsilon \dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0. \tag{2}$$

The local asymptotical stability of this system can be examined by linearization around the steady state $\bar{y} = 0$:

$$\varepsilon \dot{y}(t) - \nu \dot{y}(t) + (1-\alpha)y(t) = 0 \tag{3}$$

where $\nu = \varphi^{'}(0)$. From economic consideration we discuss the case when $\nu < \varepsilon$. By assuming delays in both investment and consumption this equation becomes a delay differential equation with two delays:

$$\varepsilon\dot{y}(t) - \nu\dot{y}(t-\theta) + (1-\alpha)y(t-\sigma) = 0. \tag{4}$$

By introducing the notation

$$a = \frac{\nu}{\varepsilon} \text{ and } b = \frac{1-\alpha}{\varepsilon}$$

this equation simplifies as

$$\dot{y}(t) - a\dot{y}(t-\theta) + by(t-\sigma) = 0 \tag{5}$$

with characteristic equation

$$\lambda - a\lambda e^{-\theta\lambda} + be^{-\sigma\lambda} = 0. \tag{6}$$

The stability of system (5) can be examined by finding the locations of the eigenvalues.

3 The Single-Delay Case

Assume first that $\theta = 0$, so equation (6) becomes

$$\lambda(1-a) + be^{-\sigma\lambda} = 0. \tag{7}$$

At $\sigma = 0$ the eigenvalue is $-b/(1-a)$, so the system is stable if $a < 1$, which is the case, since $\nu < \varepsilon$. At any stability switch $\lambda = i\omega$, where we can assume that $\omega > 0$, since the conjugate of any eigenvalue is also an eigenvalue. By substitution into equation (7), we have

$$i\omega(1-a) + b(\cos\sigma\omega - i\sin\sigma\omega) = 0 \tag{8}$$

and separating the real and imaginary parts gives two equations for unknowns ω and σ as

$$\begin{aligned} b\cos\sigma\omega &= 0 \\ \omega(1-a) - b\sin\sigma\omega &= 0 \end{aligned} \tag{9}$$

from which we conclude that $\cos\sigma\omega = 0$ and $\sin\sigma\omega = 1$. So

$$\omega = \frac{b}{1-a}$$

$$\sigma = \frac{1-a}{b}\left(\frac{\pi}{2} + 2k\pi\right) \text{ for } k = 0, 1, 2, ...,$$

that is, we have infinitely many potential stability switches. In order to see if there are actual stability switches we select σ as the bifurcation parameter and consider the eigenvalues as functions of σ, $\lambda = \lambda(\sigma)$. By implicitly differentiating equation (7) with respect to σ, we have

$$\frac{d\lambda}{d\sigma}(1-a) + be^{-\sigma\lambda}\left(-\lambda - \sigma\frac{d\lambda}{d\sigma}\right) = 0 \tag{10}$$

implying that

$$\begin{aligned}\frac{d\lambda}{d\sigma} &= \frac{\lambda b e^{-\sigma\lambda}}{1-a-b\sigma e^{-\sigma\lambda}} \\ &= -\frac{\lambda^2(1-a)}{1-a+\sigma\lambda(1-a)} \\ &= -\frac{\lambda^2}{1+\sigma\lambda}\end{aligned} \tag{11}$$

where we used equation (7). If $\lambda = i\omega$, then

$$\frac{d\lambda}{d\sigma} = \frac{\omega^2}{1+i\sigma\omega} \tag{12}$$

with real part

$$\mathrm{Re}\left[\frac{d\lambda}{d\sigma}\right] = \frac{\omega^2}{1+(\sigma\omega)^2} > 0. \tag{13}$$

Therefore by gradually increasing the value of σ from zero, at each potential stability switch an eigenvalue changes its real part from negative to positive. So the system becomes unstable at the smallest such value,

$$\sigma_0 = \frac{1-a}{b}\frac{\pi}{2}, \tag{14}$$

and the stability cannot be regained later. Hence we have the following result:

Proposition 1 *System (5) with $\theta = 0$ and $a < 1$ is locally asymptotically stable if $\sigma < \sigma_0$ and unstable for $\sigma > \sigma_0$. At $\sigma = \sigma_0$, Hopf bifurcation occurs giving the possibility of the birth of limit cycles.*

4 The General Case

The characteristic equation of system (5) is considered now. We know that its eigenvalues have negative real parts if $\theta = 0$ and $\sigma < \sigma_0$. At any stability switch $\lambda = i\omega$ and by substituting it into equation (6) we get

$$i\omega - ia\omega(\cos\theta\omega - i\sin\theta\omega) + b(\cos\sigma\omega - i\sin\sigma\omega) = 0. \tag{15}$$

By separating the real and imaginary parts we have two equations for three unknowns:

$$\begin{aligned} -a\omega\sin\theta\omega + b\cos\sigma\omega &= 0, \\ a\omega\cos\theta\omega + b\sin\sigma\omega &= \omega. \end{aligned} \tag{16}$$

By introducing the notation

$$x = \cos\theta\omega \text{ and } y = \sin\sigma\omega$$

and using the first equation of (16) we get

$$a\omega\sqrt{1-x^2} = b\sqrt{1-y^2} \tag{17}$$

so

$$a^2\omega^2 - b^2 = a^2\omega^2x^2 - b^2y^2. \tag{18}$$

From the second equation of (16) we have

$$y = \frac{\omega - a\omega x}{b}, \tag{19}$$

and by substituting it into (18),

$$a^2\omega^2 - b^2 = a^2\omega^2x^2 - (\omega - a\omega x)^2 \tag{20}$$

implying that

$$\cos\theta\omega = x = \frac{(1+a^2)\omega^2 - b^2}{2a\omega^2} \tag{21}$$

and from (19),

$$\sin\sigma\omega = y = \frac{(1-a^2)\omega^2 + b^2}{2b\omega}. \tag{22}$$

Feasible solutions exist only if both x and y are in interval $[-1, 1]$ which can be reduced to

$$\frac{b}{1+a} \leq \omega \leq \frac{b}{1-a}. \tag{23}$$

From (16) it is clear that $\sin\theta\omega$ and $\cos\sigma\omega$ have the same sign, therefore we have two parametric curves describing the set of potential stability switches:

$$C_1(k,n) = \begin{cases} \sigma = \dfrac{1}{\omega}\left[\sin^{-1}\left(\dfrac{(1-a^2)\omega^2 + b^2}{2b\omega}\right) + 2k\pi\right] \\ \theta = \dfrac{1}{\omega}\left[\cos^{-1}\left(\dfrac{(1+a^2)\omega^2 - b^2}{2a\omega^2}\right) + 2n\pi\right] \end{cases} \tag{24}$$

and

$$C_2(k,n) = \begin{cases} \sigma = \dfrac{1}{\omega}\left[\pi - \sin^{-1}\left(\dfrac{(1-a^2)\omega^2 + b^2}{2b\omega}\right) + 2k\pi\right] \\ \theta = \dfrac{1}{\omega}\left[2\pi - \cos^{-1}\left(\dfrac{(1+a^2)\omega^2 - b^2}{2a\omega^2}\right) + 2n\pi\right] \end{cases} \tag{25}$$

with $k, n = 0, 1, 2, ...$ and

$$\omega \in \left[\frac{b}{1+a}, \frac{b}{1-a}\right].$$

Notice first that at $\omega = b/(1+a)$,

$$\frac{(1+a^2)\omega^2 - b^2}{2a\omega^2} = -1, \ \frac{(1-a^2)\omega^2 + b^2}{2b\omega} = 1$$

and at $\omega = b/(1-a)$,

$$\frac{(1+a^2)\omega^2 - b^2}{2a\omega^2} = \frac{(1-a^2)\omega^2 + b^2}{2b\omega} = 1.$$

Therefore the initial and end points of $C_1(k,n)$ are

$$I_1(k,n) = \frac{1+a}{b}\left(\frac{\pi}{2} + 2k\pi, \pi + 2n\pi\right), \ E_1(k,n) = \frac{1-a}{b}\left(\frac{\pi}{2} + 2k\pi, 2n\pi\right)$$

and these for $C_2(k,n)$ are

$$I_2(k,n) = \frac{1+a}{b}\left(\frac{\pi}{2} + 2k\pi, \pi + 2n\pi\right), \ E_2(k,n) = \frac{1-a}{b}\left(\frac{\pi}{2} + 2k\pi, 2\pi + 2n\pi\right).$$

Clearly $C_1(k,n)$ and $C_2(k,n)$ have the same initial point and $C_1(k,n+1)$ and $C_2(k,n)$ have identical endpoints. Figure 1 shows these connecting curves for $k = 0$ and $n = 0, 1, 2, ..$with the parameter specification of $\alpha = 17/20$, $\varepsilon = 3/4$, $\delta = 6/5$ and $\nu = 9/80$. These curves are shifted to the right by increasing the value of k.

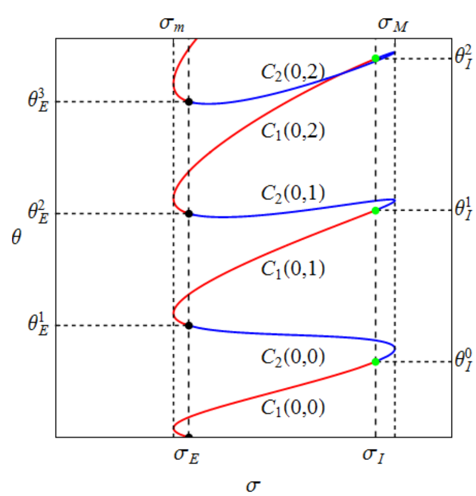

Figure 1. Shapes of curves $C_1(0,n)$ and $C_2(0,n)$ for $n = 0, 1, 2$

Notice that with fixed value of k, all initial points with different values of n have the same abscissas, and the same holds for the endpoints as well. The common abscissa values are

$$\sigma_I = \frac{1+a}{b}\left(\frac{\pi}{2} + 2k\pi\right) \text{ and } \sigma_E = \frac{1-a}{b}\left(\frac{\pi}{2} + 2k\pi\right)$$

respectively. Notice that

$$E_1(0,0) = \left(\frac{1-a}{b} \frac{\pi}{2}, 0 \right)$$

so from the previous section we know that the system is stable for

$$\theta = 0 \text{ and } \sigma < \frac{1-a}{b} \frac{\pi}{2}$$

which is the linear segment connecting the origin with $E_1(0,0)$. At the points of the horizontal axis being to the right of $E_1(0,0)$ the system is unstable. Select and fix a value of $\theta > 0$ and gradually increase the value of σ from zero. The resulting horizontal line will have infinitely many intersections with the curves $C_1(k,n)$ and $C_2(k,n)$. The directions of stability switches at the intersections can be determined by considering σ as the bifurcation parameter, and considering the eigenvalues as functions of σ, $\lambda = \lambda(\sigma)$. By implicitly differentiating equation (6) with respect to σ we get a simple equation for $d\lambda/d\sigma$:

$$\frac{d\lambda}{d\sigma} - a\frac{d\lambda}{d\sigma}e^{-\theta\lambda} - a\lambda e^{-\theta\lambda}\left(-\theta\frac{d\lambda}{d\sigma}\right) + be^{-\sigma\lambda}\left(-\lambda - \sigma\frac{d\lambda}{d\sigma}\right) = 0 \tag{26}$$

implying that

$$\frac{d\lambda}{d\sigma} = \frac{b\lambda e^{-\sigma\lambda}}{1 - ae^{-\theta\lambda} + a\lambda\theta e^{-\theta\lambda} - b\sigma e^{-\sigma\lambda}}. \tag{27}$$

From (6) we see that

$$ae^{-\theta\lambda} = \frac{1}{\lambda}\left(\lambda + be^{-\sigma\lambda}\right), \tag{28}$$

so

$$\frac{d\lambda}{d\sigma} = \frac{b\lambda^2}{\lambda^2\theta e^{\sigma\lambda} + (-b + b\lambda\theta - b\lambda\sigma)}. \tag{29}$$

At $\lambda = i\omega$ we have

$$\frac{d\lambda}{d\sigma} = \frac{-b\omega^2}{-\omega^2\theta(\cos\sigma\omega + i\sin\sigma\omega) + (-b + i\omega\theta b - i\omega b\sigma)} \tag{30}$$

with real part having the same sign as

$$b\omega^2(\omega^2\theta\cos\sigma\omega + b).$$

So at any point of the curve $C_1(k,n)$ or $C_2(k,n)$, stability is lost if $\omega^2\theta\cos\sigma\omega + b > 0$ and stability may be regained if $\omega^2\theta\cos\sigma\omega + b < 0$. Notice first that on $C_1(k,n)$,

$$\sigma\omega \in \left[2k\pi, \frac{\pi}{2} + 2k\pi\right],$$

so $\cos\sigma\omega > 0$ implying that on all intercepts with $C_1(k,n)$ at least one eigenvalue changes its sign from negative to positive. Consider next a curve $C_2(k,n)$. On this curve

$$\begin{aligned}\frac{\partial\theta}{\partial\omega} &= -\frac{1}{\omega^2}\left[2\pi - \cos^{-1}\left(\tfrac{(1+a^2)\omega^2-b^2}{2a\omega^2}\right) + 2n\pi\right] + \frac{1}{\omega}\frac{1}{\sqrt{1-\left(\frac{(1+a^2)\omega^2-b^2}{2a\omega^2}\right)^2}}\frac{2b^2}{2a\omega^3} \\ &= -\frac{1}{\omega^2}\omega\theta + \frac{1}{\omega}\frac{1}{-\sin\theta\omega}\frac{b^2}{a\omega^3}\end{aligned} \tag{31}$$

From the first equation of (16), we have

$$\sin\theta\omega = \frac{b}{a\omega}\cos\sigma\theta \tag{32}$$

so

$$\begin{aligned}\frac{\partial\theta}{\partial\omega} &= -\frac{1}{\omega}\left(\theta + \frac{1}{\dfrac{b}{a\omega}\cos\sigma\theta}\frac{b^2}{a\omega^3}\right) \\ &= -\frac{1}{\omega^3\cos\sigma\omega}\left(\theta\omega^2\cos\sigma\omega + b\right).\end{aligned} \tag{33}$$

Since $\cos\sigma\omega < 0$ on $C_2(k,n)$, we conclude that stability is lost when $\partial\theta/\partial\omega > 0$ and might be regained if $\partial\theta/\partial\omega < 0$. The first case occurs when the curve $C_2(k,n)$ is increasing in θ from right to left and the second case occurs when the curve is decreasing in θ from right to left.

Next we show that at each intersection only one eigenvalue can change the sign of its real part. In contrary, assume that λ is a multiple eigenvalue. Then it solves the characteristic equation and its derivative:

$$\lambda - a\lambda e^{-\theta\lambda} + be^{-\sigma\lambda} = 0 \tag{34}$$

and

$$1 - ae^{-\theta\lambda} + a\lambda\theta e^{-\theta\lambda} - b\sigma e^{-\sigma\lambda} = 0. \tag{35}$$

If $\lambda = i\omega$, then

$$i\omega - ia\omega(\cos\theta\omega - i\sin\theta\omega) + b(\cos\sigma\omega - i\sin\sigma\omega) = 0 \tag{36}$$

and

$$1 - a(\cos\theta\omega - i\sin\theta\omega) + ia\theta\omega(\cos\theta\omega - i\sin\theta\omega) - b\sigma(\cos\sigma\omega - i\sin\sigma\omega) = 0. \tag{37}$$

By separating the real and imaginary parts, four equations are obtained for the four unknowns, $\sin\theta\omega$, $\cos\theta\omega$, $\sin\sigma\omega$ and $\cos\sigma\omega$:

$$-a\omega\sin\theta\omega + b\cos\sigma\omega = 0, \tag{38}$$

$$\omega - a\omega\cos\theta\omega - b\sin\sigma\omega = 0, \tag{39}$$

$$1 - a\cos\theta\omega + a\theta\omega\sin\theta\omega - b\sigma\cos\sigma\omega = 0, \tag{40}$$

$$a\sin\theta\omega + a\theta\omega\cos\theta\omega + b\sigma\sin\sigma\omega = 0. \tag{41}$$

Simple calculation shows that the solution is the following:

$$\sin\theta\omega = -\frac{\theta\omega}{a\left(1+\omega^2(\sigma-\theta)^2\right)},\ \cos\theta\omega = \frac{1+\omega^2\sigma(\sigma-\theta)}{a\left(1+\omega^2(\sigma-\theta)^2\right)}, \tag{42}$$

$$\sin\sigma\omega = \frac{\omega^3\theta(\theta-\sigma)}{b\left(1+\omega^2(\sigma-\theta)^2\right)},\ \cos\sigma\omega = -\frac{\theta\omega^2}{b\left(1+\omega^2(\sigma-\theta)^2\right)}, \tag{43}$$

and now from (33) at these values,

$$\frac{\partial\theta}{\partial\omega} = \frac{\theta^2\omega^4 - b^2\left(1+\omega^2(\sigma-\theta)^2\right)}{\omega^3(\cos\sigma\omega)b\left(1+\omega^2(\sigma-\theta)^2\right)} = 0, \tag{44}$$

since from (43),

$$\begin{aligned} 1 &= \sin^2\sigma\omega + \cos^2\sigma\omega \\ &= \frac{\theta^2\omega^4[1+\omega^2(\sigma-\theta)^2]}{b^2[1+\omega^2(\sigma-\theta)^2]^2} \\ &= \frac{\theta^2\omega^4}{b^2\left(1+\omega^2(\sigma-\theta)^2\right)}. \end{aligned} \tag{45}$$

Consequently multiple eigenvalues are possible only at the extreme values of θ with respect to ω on $C_2(k,n)$. This is not an intersection since the horizontal line is tangent to the curve at the extreme points.

From (24) we know that in $C_1(k,n)$ the value of θ decreases as ω increases, so as the curve moves from right to left from $I_1(k,n)$ to $E_1(k,n)$ the value of θ decreases, so at the intersection with $C_1(k,n)$ one eigenvalue changes its real part from negative to positive. The same is true with intersections on increasing segments of $C_2(k,n)$ as well. However on the decreasing segment of $C_2(k,n)$ one eigenvalue changes its real part from positive to negative, so stability is regained here when only one eigenvalue had positive real part before, that is, only one intersection with stability loss can be found before.

Figure 2 shows again the continuous curves, $C_1(0,n)$ and $C_2(0,n)$ ($n = 0,1,2,3,...$) under the same specification of the parameters as before. The horizontal line shows the stability losses and gains. When we increase the value of σ along the horizontal line, stability is lost at point A, regained at point B and lost again at point C. However system is unstable after point C. The stability region is the yellow region. If (σ,θ) is any point, then we have to consider the linear segment connecting points $(0,\theta)$ and (σ,θ) and count the number of intersections with stability loss (L) and number of intersections with stability gain (G). The point (θ,σ) is a stability point if $G \geq L$.

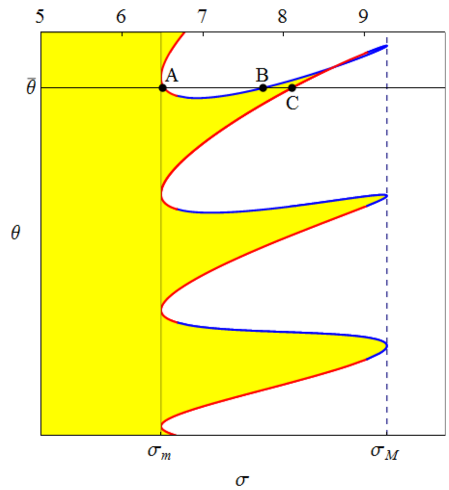

Figure 2. Stability switches

5 Conclusions

In this paper a special dynamic system with two delays was examined. The stability switch curves were determined and the directions of the stability switches were characterized by the monotonicity of the different segments of the curves. Small values of σ are harmless, since system is stable with any values $\theta > 0$. With large values of σ, the stability region is an irregular domain depending on both values of θ and σ.

This study discovered only local asymptotic stability. The global asymptotic behavior of the system in case of local instability is an interesting research issue which can be examined by computer simulation. This is our next project.

References

[1] Bellman, R. and Cooke, K. L. (1956), *Differential-difference Equations.* Academic Press, New York.

[2] Cooke, K. L. and Grossman, Z. (1982). Discrete Delay, Distributed Delay and Stability Switches. *J. Math. Anal. Appl.* 86: 592-627.

[3] Cushing, J. M. (1977) *Integro-Differential Equations and Delay Models in Population Dynamics.* Springer-Verlag, Berlin/ Heidelberg/ New York.

[4] Goodwin, R. (1951) The Nonlinear Accelerator and the Persistence of Business Cycles. *Econometrica* 19: 1-17.

[5] Hale, J. (1979) Nonlinear Oscillations in Equations with Delays. In *Nonlinear Oscillations in Biology* (Happenstadt, L.C. editor). Lecture Notes in Mathematics 17: 157-185.

[6] Hale, J. and Huang, W. (1993) Global Geometry of the Stable Regions for Two Delay Differential Equations. *J. Math. Anal. Appl.* 178: 344-362.

[7] Matsumoto, A. and Szidarovszky, F. (2013), An Elementary Study of a Class of Dynamic Systems with Two Delays. *CUBO, A Mathematical Journal* 14: 103-113.

[8] Piotrowska, M. (2007) A Remark on the ODE with Two Discrete Delays. *J. Math. Anal. Appl.* 329: 664-674.