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Environmental Effects of Ambient Charge in Courtnot Oligopoly

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Environmental Effects of Ambient Charge in Cournot Oligopoly.*

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Abstract

This paper investigates the effect caused by an increase in ambient charges on firm-specific and total pollutions in a Cournot oligopoly market. Formalizing profit-maximizing behavior in the n-firm framework with product differentiation, we show the static result that ambient change can reduce industrial pollution. We then demonstrate three dynamic results: the first that Cournot equilibrium can lose stability in the discrete time framework if the number of the firms is greater than four, the second that it is always stable in the continuous time framework and the third that stability can be switched to instability if a delay in production becomes large enough.

Keywords: Cournot competition, Ambient charge, Nonpoint source pollution, Production delay, Stability switch

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1 Introduction

This paper investigates the effects caused by ambient charges on the pollutions arose in an n-firm Cournot market. Its attention is put on nonpoint source (NPS) pollution such as water and air pollution from diffuse sources. Although any pollutant originates from a single point source, the long-range transport ability and multiple sources of the pollutant make it a nonpoint source of pollution. For the government that would like to control pollutions, it might be impossible to measure firm-specific emissions whereas possible to measure the total level of pollution. As suggested by Segerson (1988), the government adopts an environmental policy to establish a cut-off level of the whole pollution and to make the following rule: regardless of firm specific emission level, if the actual level of the total pollutions exceeds the cut-off level, then all firms levy the same penalty while if the actual level falls short of the cut-off level, then all firms are awarded the same subsidy. Ganguli and Raju (2012) show a "perverse" ambient charge effect on total pollution, an increase in the ambient charge could lead to greater pollution, in the Bertrand duopoly. On the other hand, Raju and Ganguli (2013) examine the same subject in a Cournot duopoly framework and attain the effective result such that an increase in the ambient charges reduce pollutant emissions. This study steps forward and demonstrate the effective environmental policy in an n-firm Cournot market.

This paper is organized as follows. Section 2 presents the basic mathematical model. Section 3 considers the effect of increasing ambient charges on individual as well as total pollutions. Section 4 examine the stability of Cournot equilibrium in the discrete and continuous time scales. Finally, Section 5 concludes the paper.

2 Cournot Model

There are n firms in the oligopoly market, producing a differentiated product. The production quantity and price of firm k are represented by q_k and p_k. It is assumed that the linear price function of good k is

$$p_k = \alpha - q_k - \gamma \sum_{i \neq k}^{n} q_i \text{ for } k = 1, 2, ..., n \tag{1}$$

where $n \geq 2$ and $0 < \gamma < 1$, implying that the goods are substitutes.[1] It is also assumed that all the firms have the same marginal production cost, c. Firm k emits pollutants $e_k q_k$ in connection with its production with positive emission coefficient $e_k > 0$. The government can measure the total emission quantity and has an exogenously determined environmental standard $\bar{\boldsymbol{E}}$. According to $m > 0$ times the difference between the total emission, $\sum_{k=1}^{n} e_k q_k$ and the standard, it will levy the penalty if the difference is positive and award the subsidy if

[1] The goods are complements if $-1 < \gamma < 0$.

negative. In consequence, the profit of firm k is expressed as

$$\pi_k = (p_k - c)q_k - m\left(\sum_{i=1}^{n} e_i q_i - \bar{\boldsymbol{E}}\right). \tag{2}$$

Under Cournot competition, firm k determines its output to maximize its profit subject to its price demand function, taking the other firms' quantities given. Assuming interior maximum and solving its first-order condition yield the best response of firm k,

$$q_k = \frac{\alpha - c_k}{2} - \frac{\gamma}{2}\sum_{i \neq k}^{n} q_i \text{ for } k = 1, 2, ..., n. \tag{3}$$

with the marginal cost of firm k is $c_k = c + me_k$. To avoid negative optimal production, we assume a large enough value of α so that $\alpha - c_k > 0$ holds for all k. It can be easily checked that the second-order condition is certainly satisfied. The Cournot equilibrium output for firm k is obtained by solving the following simultaneous equations:

$$q_k + \frac{\gamma}{2}\sum_{i \neq k}^{n} q_i = \frac{\alpha - c_k}{2} \text{ for } k = 1, 2, ..., n \tag{4}$$

or in vector form

$$\boldsymbol{B}\boldsymbol{q} = \boldsymbol{A}$$

where for $i, j = 1, 2, ..., n$,

$$\boldsymbol{q} = (q_i)_{(n,1)}\,,\ \boldsymbol{A} = \left(\frac{\alpha - c_i}{2}\right)_{(n,1)},\ \boldsymbol{B} = (B_{ij})_{(n,n)} \text{ with } B_{ii} = 1 \text{ and } B_{ij} = \frac{\gamma}{2} \text{ for } i \neq j.$$

Since $\boldsymbol{B}$ is invertible, the Cournot output vector is given by

$$\boldsymbol{q} = \boldsymbol{B}^{-1}\boldsymbol{A}$$

where the diagonal and off-diagonal elements of $\boldsymbol{B}^{-1}$ are, respectively,

$$\frac{2(2 + (n-2)\gamma)}{(2-\gamma)(2 + (n-1)\gamma)} \text{ and } -\frac{2\gamma}{(2-\gamma)(2 + (n-1)\gamma)}.$$

Hence the Cournot equilibrium output of firm k is

$$q_k^C = \frac{(\alpha - c_k)(2 + (n-1)\gamma) - \gamma\sum_{i=1}^{n}(\alpha - c_i)}{(2-\gamma)(2 + (n-1)\gamma)} \text{ for } k = 1, 2, ..., n. \tag{5}$$

We check the non-negativity condition for the Cournot output. Equation (5) can be written as

$$q_k^C = \frac{(\alpha - c_k)}{(2-\gamma)(2 + (n-1)\gamma)}\left\{(2-\gamma) + n\gamma(1 - \beta_k)\right\} \tag{6}$$

where β_k is defined as

$$\beta_k = \frac{\frac{1}{n}\sum_{i=1}^{n}(\alpha - c_i)}{\alpha - c_k}.$$

Substituting $c_k = c + me_k$ changes β_k to

$$\beta_k = \frac{\alpha - c - m\bar{e}}{\alpha - c - me_k} \text{ with } \bar{e} = \frac{1}{n}\sum_{i=1}^{n} e_i.$$

$\bar{e}$ is the average value of the emission coefficients. It can be shown that

$$\beta_k \gtreqless 1 \Longleftrightarrow e_k \gtreqless \bar{e}.$$

When $\beta_k < 1$, $\beta_k = 1$ and $\beta_k > 1$, firm k is called *lower-polluter*, *average-polluter* and *higher-polluter* as its emission level is lower than, equal to and larger than the average level. Equation (6) implies $q_k^C > 0$ if $\beta_k \leq 1$. In a case of $\beta_k > 1$, a different form of equation (6) is

$$q_k^C = \frac{(\alpha - c_k)(\beta_k - 1)\gamma}{(2-\gamma)(2+(n-1)\gamma)} \left\{ \frac{2-\gamma}{\gamma(\beta_k - 1)} - n \right\} \tag{7}$$

which leads to $q_k^C > 0$ if $\beta_k > 1$ and

$$n < \frac{2-\gamma}{\gamma(\beta_k - 1)}$$

which can be called the non-negativity condition for a higher-polluter firm. We then summarize these results,

Theorem 1 *Cournot output of firm k is positive if firm k is either lower-polluter or average-polluter or if it is higher-polluter and satisfies the non-negativity condition,*

$$n < \frac{2-\gamma}{\gamma(\beta_k - 1)}.$$

3 Ambient Charge

In this section we examine the effect of a change in the ambient charge on firm-specific production and then total pollution. Substituting $c_k = c + me_k$ into equation (5) and arranging the terms present

$$q_k^C = \frac{(\alpha - c)(2-\gamma) + m\left[\gamma \sum_{i=1}^{n} e_i - (2+(n-1)\gamma)e_k\right]}{(2-\gamma)(2+(n-1)\gamma)}. \tag{8}$$

Differentiating (8) with respect to m yields

$$\frac{\partial q_k^C}{\partial m} = \frac{\gamma \sum_{i=1}^{n} e_i - (2+(n-1)\gamma)e_k}{(2-\gamma)(2+(n-1)\gamma)}. \tag{9}$$

where the numerator can be written as

$$n\gamma\left(\bar{e}-e_k\right)+(\gamma-2)e_k.$$

The second term is negative and thus the numerator is also negative if $\bar{e}\le e_k$, that is, firm k is higher- or average-polluter. The sign seems to be ambiguous if $\bar{e}>e_k$. However, since the numerator is differently expressed as

$$n\gamma\left(\bar{e}-\tilde{e}_k\right)\ \text{ with } \tilde{e}_k=\frac{n\gamma+(2-\gamma)}{n\gamma}e_k>e_k,$$

changing m has a negative effect if $\bar{e}<\tilde{e}_k$ and a positive effect if $\bar{e}>\tilde{e}_k$. That is, the perverse effect on firm-specific production could be possible for the lower-polluter firms. Hence summarizing the results on firm-specific production levels gives the following.

Theorem 2 *An increase in the ambient charge decreases production of higher- or average-polluter firm whereas it decreases, increases or does not change production of lower-polluter firm k according to whether $\bar{e}<\tilde{e}_k$, $\bar{e}>\tilde{e}_k$ or $\bar{e}=\tilde{e}_k$.*

Concerning the total quantity of emission at the Cournot equilibrium,

$$\boldsymbol{E}^C=\sum_{k=1}^{n}e_k q_k^C,$$

we have the following result:

Theorem 3 *Given positive $e_k>0$ for all k, then increasing the policy parameter m decreases the total emission.*

Proof. Differentiating $\boldsymbol{E}^C$ with respect to m and then substituting equation (9) give

$$\begin{aligned}\frac{\partial \boldsymbol{E}^C}{\partial m}&=\textstyle\sum_{k=1}^{n}e_k\dfrac{\partial q_k^C}{\partial m}\\&=\dfrac{\gamma\left(\sum_{k=1}^{n}e_k\right)^2-(2+(n-1)\gamma)\sum_{k=1}^{n}e_k^2}{(2-\gamma)(2+(n-1)\gamma)}\end{aligned}$$

The denominator is definitely positive. Let us denote the numerator by $S(\gamma)$,

$$S(\gamma)=\gamma\left(\sum_{k=1}^{n}e_k\right)^2-(2+(n-1)\gamma)\sum_{k=1}^{n}e_k^2.$$

We then obtain the trivial result

$$S(0)=-2\sum_{k=1}^{n}e_k^2<0$$

and

$$S(1) \quad = \left(\sum_{k=1}^{n} e_k\right)^2 - (1+n)\sum_{k=1}^{n} e_k^2 < 0$$

where the direction of the inequality is shown as follows. The Cauchy inequality implies

$$\left(\sum_{k=1}^{n} e_k \cdot 1\right)^2 \leq \sum_{k=1}^{n} e_k^2 \cdot \sum_{k=1}^{n} 1^2,$$

so

$$\left(\sum_{k=1}^{n} e_k\right)^2 \leq n\sum_{k=1}^{n} e_k^2.$$

Using the last inequality above we can arrive at $S(1) < 0$ because

$$S(1) = \left(\sum_{k=1}^{n} e_k\right)^2 - (1+n)\sum_{k=1}^{n} e_k^2 < \left(\sum_{k=1}^{n} e_k\right)^2 - n\sum_{k=1}^{n} e_k^2 \leq 0.$$

Since $S(\gamma)$ is linear in γ and both $S(0)$ and $S(1)$ are negative, the value of $S(\gamma)$ is also negative for any $\gamma \in (0,1)$. Therefore we have the negative derivative,

$$\frac{\partial \boldsymbol{E}^C}{\partial m} < 0.$$

■

4 Stability

The effective ambient charge summarized in Theorem 3 is a comparative static result and thus is economically meaningful only in the stable economy in which any disturbances of the equilibrium caused by changes in exogenous factors sooner or later can be eliminated. Dividing this section into three subsections, we draw attention to stability of the n-firm Cournot equilibrium from various view points. First, dynamics with discrete-time scales is considered and then dynamics with continuous-time scales is examined. In the third subsection delay dynamics is investigated that is thought to be a hybrid of the two previous cases.

4.1 Discrete-time Dynamics

Assuming naive expectation, the simplest form of expectation, in which each firm believes that the other firms remain unchanged with their outputs from the previous period. With discrete time scales, the best response (3) gives rise to the linear dynamic system,

$$q_k(t+1) = \frac{\alpha - c_k}{2} - \frac{\gamma}{2}\sum_{i \neq k}^{n} q_i(t) \text{ for } k = 1, 2, ..., n. \tag{10}$$

The coefficient matrix of system (10) is

$$\boldsymbol{J}_D = \begin{pmatrix} 0 & -\frac{\gamma}{2} & \cdots & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & 0 & \cdots & -\frac{\gamma}{2} \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{\gamma}{2} & -\frac{\gamma}{2} & \cdots & 0 \end{pmatrix}$$

and the corresponding characteristic equation reads

$$|\boldsymbol{J}_D - \lambda \boldsymbol{I}| = (-1)^n \left(\lambda - \frac{\gamma}{2}\right)^{n-1} \left(\lambda + \frac{(n-1)\gamma}{2}\right) = 0$$

where $\boldsymbol{I}$ is the identity matrix. Accordingly there are $n-1$ identical eigenvalues and one different eigenvalue. Without loss of generality, the first $n-1$ are assumed to be identical,

$$\lambda_1^D = \lambda_2^D = ... = \lambda_{n-1}^D = \frac{\gamma}{2} \text{ and } \lambda_n^D = -\frac{(n-1)\gamma}{2}.$$

Since $0 < \gamma < 1$, the first $n-1$ eigenvalues are positive and less than unity. It hence depends on the value of λ_n^D whether the Cournot output is stable or not. It is clear that $\left|\lambda_n^D\right| < 1$ for $n = 2$ and 3, implying that the Cournot equilibrium is definitely asymptotically stable in the duopoly and triopoly markets. Solving $\lambda_n^D > -1$ for $n > 3$ presents the stability condition,

$$n < \frac{2+\gamma}{\gamma}$$

where $1 + 2/\gamma > 3$ for $0 < \gamma < 1$. Notice the following,

$$\frac{2+\gamma}{\gamma} \leq \frac{2-\gamma}{\gamma(\beta_k - 1)} \text{ if } 1 < \beta_k \leq \frac{4}{2+\gamma}.$$

Hence there are $n \geq 2$ and $\beta_k > 1$ for which Cournot output for each firm is positive and stable. To simplify the analysis, we assume that the non-negativity condition does not violate the stability condition even for $\beta_k > 1$. We now summarize the stability results as follows.

Theorem 4 *The Cournot output with discrete time scales is stable if the number of firms does not exceed three, otherwise it is stable, marginally stable or unstable according to the number of the firms is less than, equal to or greater than* $(2+\gamma)/\gamma$.

This result reminds us the Theocharis theorem concerning the homogeneous product in which the stability of the Cournot equilibrium depends only on the number of the firms involved in the economy. It is stable for $n = 2$, marginally stable for $n = 3$ and unstable for $n \geq 4$. Comparing Theocharis theorem with Theorem 4 reveals that substitutability (i.e., production differentiation) works to partially stabilize the $n \geq 4$ economy.

4.2 Continuous-time Dynamics

If continuous time scales are assumed and $q_k(t+1) - q_k(t)$ in equation (10) is replaced with $\dot{q}_k(t) = dq_k(t)/dt$, then the dynamic system (10) turns to be

$$\dot{q}_k(t) = \frac{\alpha - c_k}{2} - q_k(t) - \frac{\gamma}{2}\sum_{i\neq k}^{n} q_i(t) \text{ for } k = 1, 2, ..., n. \tag{11}$$

Notice that the steady state of system (11) is the same as the one of system (10). A Jacobi matrix is

$$\boldsymbol{J}_C = \begin{pmatrix} -1 & -\frac{\gamma}{2} & \cdots & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & -1 & \cdots & -\frac{\gamma}{2} \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{\gamma}{2} & -\frac{\gamma}{2} & \cdots & -1 \end{pmatrix} = \boldsymbol{J}_D - \boldsymbol{I}$$

Since $\boldsymbol{I}$ is the identity matrix, $\lambda_i^C = \lambda_i^D - 1$ for $i = 1, 2, ..., n$, that is,

$$\lambda_1^C = \lambda_2^C = ... = \lambda_{n-1}^C = -1 + \frac{\gamma}{2} < 0 \text{ and } \lambda_n^C = -1 - \frac{(n-1)\gamma}{2} < 0$$

which are summarized as follows:

Theorem 5 *The Cournot output with continuous time scales is always asymptotically stable regardless of the number of the firms and the degree of substitutability.*

It has been known that the n-firm Cournot equilibrium with the homogenous product is stable in the continuous-time framework. Theorem 5 implies that production differentiation can not be a destabilizing factor. Theorems 4 and 5 show sharply different dynamic results that is sensitive to selection of time scales. A natural question to arise concerns stability in a delay system that is a hybrid of these two systems.

4.3 Delay Dynamics

In this section we introduce delay into the continuous-time system (11) and examine how the delays affect dynamics.[2]

4.3.1 Off-Diagonal Delays

We first consider the case in which the firms have delays for obtaining information about the competitor's decisions that we will call *information delays*. The dynamic system (11) is modified as follows:

$$\dot{q}_k(t) = \frac{\alpha - c_k}{2} - q_k(t) - \frac{\gamma}{2}\sum_{i\neq k}^{n} q_i(t-\tau) \text{ for } k = 1, 2, ..., n \tag{12}$$

[2] This section depends on Matsumoto and Szidarovszky (2015).

where $\tau > 0$ is the length of the information delay and is assumed to be identical for all firms for the sake of analytical simplicity. Linearizing equation (12) at the equilibrium point and assuming exponential solutions, $q_k(t) = e^{\lambda t}u_k$ for $i = 1, 2, ..., n$ and substituting these into it, we then obtain the following from of the coefficient matrix,

$$\boldsymbol{J}_1 = \begin{pmatrix} -1-\lambda & -\frac{\gamma}{2}e^{-\lambda\tau} & \cdots & -\frac{\gamma}{2}e^{-\lambda\tau} \\ -\frac{\gamma}{2}e^{-\lambda\tau} & -1-\lambda & \cdots & -\frac{\gamma}{2}e^{-\lambda\tau} \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{\gamma}{2}e^{-\lambda\tau} & -\frac{\gamma}{2}e^{-\lambda\tau} & \cdots & -1-\lambda \end{pmatrix}. \tag{13}$$

Equation $\det(\boldsymbol{J}_1) = 0$ has the form

$$\left(\lambda + 1 - \frac{1}{2}\gamma e^{-\lambda\tau}\right)^{n-1}\left(\lambda + 1 + \frac{n-1}{2}\gamma e^{-\lambda\tau}\right) = 0$$

that generates two independent equations, the first $n-1$ solutions satisfy the first equation

$$\lambda + 1 - \frac{1}{2}\gamma e^{-\lambda\tau} = 0 \tag{14}$$

and the last n^{th} solution solves the second equation

$$\lambda + 1 + \frac{n-1}{2}\gamma e^{-\lambda\tau} = 0. \tag{15}$$

We investigate the possibility of *stability switch* leading to stability loss (i.e., stability switches to instability) or stability regain (i.e., instability switches to stability). It could occur when the real parts of the eigenvalue are zero. Since it is apparent that $\lambda = 0$ is not a solution of either (14) or (15), we suppose that $\lambda = i\omega$ with $\omega > 0$ could be a solution.[3] We start with equation (14) and then substitute this purely imaginary solution into it to investigate whether an appropriate ω can be obtained,

$$i\omega + 1 - \frac{\gamma}{2}e^{-i\omega\tau} = 0$$

that is divided into the real and imaginary parts,

$$1 - \frac{\gamma}{2}\cos\omega\tau = 0,$$

and

$$\omega + \frac{\gamma}{2}\sin\omega\tau = 0.$$

Moving the constant terms to the right hand side and adding the squares of these equations yield

$$\omega^2 = \left(\frac{\gamma}{2}\right)^2 - 1 < 0$$

[3] We obtain the same result to be obtain if $\omega < 0$ is assumed as complex roots are conjugate.

where the inequality is due to the assumption of substitutability $0 < \gamma < 1$ and thus leads to the result that there is no $\omega > 0$. This implies no occurrence of stability switch.

We now turn attention to the solution of equation (15) and repeat the same procedure: substituting the purely imaginary solution $\lambda = i\omega$ with $\omega > 0$ into equation (15) and then separating the real part from the imaginary parts give

$$1 + \frac{n-1}{2}\gamma \cos \omega\tau = 0 \tag{16}$$

and

$$\omega - \frac{n-1}{2}\gamma \sin \omega\tau = 0. \tag{17}$$

Again, moving the constant terms to the right hand sides and then adding the squares of these expressions present

$$\omega^2 = \frac{[(n-1)\gamma - 2]\,[(n-1)\gamma + 2]}{4}$$

from which we can derive two results,

(1) if $n \leq \dfrac{2+\gamma}{\gamma}$, then there is no $\omega > 0$, implying no stability switch.

(2) if $n > \dfrac{2+\gamma}{\gamma}$, then there is the positive solution

$$\omega^* = \frac{\sqrt{[(n-1)\gamma - 2]\,[(n-1)\gamma + 2]}}{2}.$$

Notice that $\dfrac{2}{n-1} \geq 1 > \gamma$ for $n = 2$ and $n = 3$. Therefore the Cournot equilibrium is stable in the duopoly and triopoly markets. Substituting ω^* into equation (16) and solving it for τ determine the threshold value of τ for which some of the characteristic roots are purely imaginary[4],

$$\tau_m^* = \frac{1}{\omega^*}\left[\cos^{-1}\left(-\frac{2}{(n-1)\,\gamma}\right) + 2m\pi\right] \text{ for } m = 0, 1, 2, \ldots$$

Since it is already shown that the system is asymptotically stable for $\tau = 0$, stability is switched to instability when increasing the value of τ form zero arrives at the smallest threshold value,

$$\tau_0^* = \frac{2\cos^{-1}\left(-\dfrac{2}{(n-1)\,\gamma}\right)}{\sqrt{[(n-1)\gamma - 2]\,[(n-1)\gamma + 2]}}.$$

Given the number of n, the critical threshold value τ_0^* decreases as a value of γ increases, implying a destabilizing effect in the sense that the stability region in the (γ, τ) region shrinks.

[4] Substituting ω^* into equation (17) and then solving it for τ give the same value in a different form.

Theorem 6 *If there are information delays in the competitors' production, then the Cournot production is stable for any* $\tau \geq 0$ *if* $n \leq \dfrac{2+\gamma}{\gamma}$ *whereas it is stable for* $\tau < \tau_0^*$ *and unstable for* $\tau > \tau_0^*$ *if* $n > \dfrac{2+\gamma}{\gamma}$.

4.3.2 Diagonal Delays

We now examine the case in which the firms have delays in their own production level, delays of which are called *implementation delays*. The dynamic equation (11) is modified as follows:

$$\dot{q}_k(t) = \frac{\alpha - c_k}{2} - q_k(t-\tau) - \frac{\gamma}{2}\sum_{i\neq k}^{n} q_i(t) \text{ for } k = 1, 2, ..., n \tag{18}$$

where $\tau > 0$ now denotes the length of the implementation delay and is assumed to be identical for all firms for the sake of analytical simplicity. As usual, given an exponential solution, $q_k(t) = e^{-\lambda t}u_k$, the coefficient matrix becomes

$$\boldsymbol{J}_2 = \begin{pmatrix} -e^{-\lambda\tau} - \lambda & -\frac{\gamma}{2} & \cdots & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & -e^{-\lambda\tau} - \lambda & \cdots & -\frac{\gamma}{2} \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{\gamma}{2} & -\frac{\gamma}{2} & \cdots & -e^{-\lambda\tau} - \lambda \end{pmatrix}. \tag{19}$$

and the corresponding equation $\det(\boldsymbol{J}_2) = 0$ has the form

$$\left(\lambda + e^{-\lambda\tau} - \frac{\gamma}{2}\right)^{n-1}\left(\lambda + e^{-\lambda\tau} + \frac{n-1}{2}\gamma\right) = 0$$

that is divided into the two independent equations,

$$\lambda + e^{-\lambda\tau} - \frac{\gamma}{2} = 0 \tag{20}$$

and

$$\lambda + e^{-\lambda\tau} + \frac{n-1}{2}\gamma = 0. \tag{21}$$

As in the same way as in the previous case, we suppose that there is a purely imaginary solution $\lambda = i\omega$, $\omega > 0$ and substitute it into equation (20) to obtain the real and imaginary parts,

$$\cos\tau\omega = \frac{\gamma}{2}$$

and

$$\sin\tau\omega = \omega.$$

Adding the squares of these equations and then solving it for ω present

$$\omega_a^* = \frac{\sqrt{(2-\gamma)(2+\gamma)}}{2} > 0$$

and solving the real part for τ yields

$$\tau^*_{a,m} = \frac{1}{\omega^*_a}\left[\cos^{-1}\left(\frac{\gamma}{2}\right) + 2m\pi\right] \text{ for } m = 0, 1, 2, ...$$

The threshold value at which the stability switch takes place is

$$\tau^*_{a,0} = \frac{2\cos^{-1}\left(\frac{\gamma}{2}\right)}{\sqrt{(2-\gamma)(2+\gamma)}}. \tag{22}$$

Notice that this value is independent from the number of the firms.

We now turn attention to equation (21). Solving it with $\lambda = i\omega$ and $\omega > 0$ presents

$$\omega^2 = \frac{[2-(n-1)\gamma]\,[2+(n-1)\gamma]}{4}$$

from which we have two results:

(1) if $n \geq \dfrac{2+\gamma}{\gamma}$, then there is no $\omega > 0$, implying no stability switch.

(2) if $n < \dfrac{2+\gamma}{\gamma}$, then there is the positive solution,

$$\omega^*_b = \frac{\sqrt{[2-(n-1)\gamma]\,[2+(n-1)\gamma]}}{2} > 0.$$

and

$$\tau^*_{b,m} = \frac{1}{\omega^*_b}\left[\cos^{-1}\left(-\frac{(n-1)\,\gamma}{2}\right) + 2m\tau\right] \text{ for } m = 0, 1, 2, ...$$

The smallest threshold value is

$$\tau^*_{b,0} = \frac{2\cos^{-1}\left(-\frac{(n-1)\,\gamma}{2}\right)}{\sqrt{[2-(n-1)\gamma]\,[2+(n-1)\gamma]}}. \tag{23}$$

Since we have two stability switching curves (22) and (23), we determine which is effective in particular. For $n = 2$,

$$\tau^*_{b,0} - \tau^*_{a,0} = \frac{2}{\sqrt{(2-\gamma)(2+\gamma)}}\left[\cos^{-1}\left(-\frac{\gamma}{2}\right) - \cos^{-1}\left(\frac{\gamma}{2}\right)\right] > 0 \text{ for } \gamma > 0$$

implying that $\tau^*_{b,0}$ should be located above $\tau^*_{a,0}$ for $\gamma > 0$ and $n = 2$ where

$$\tau^*_{a,0} = \tau^*_{b,0} = \frac{\pi}{2} \text{ for } \gamma = 0 \text{ regardless of the number of } n.$$

The numerator of (23) increases in n, and the denominator decreases in n, so $\tau^*_{b,0}$ increases as n becomes larger. Therefore

$$\tau^*_{b,0} > \tau^*_{a,0} \text{ for any } 0 < \gamma < 1 \text{ and } n \geq 2.$$

This result is partially visualized in Figure 1 where the blue curves describe $\tau^*_{b,0}$ with $n = 3, 4, 5$ and the red curve does $\tau^*_{a,0}$.

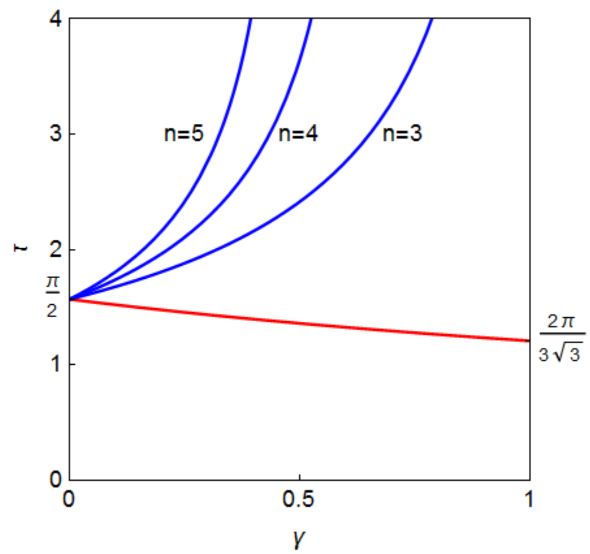

Figure 1. The $\tau^*_{a,0}$ red curve and the $\tau^*_{b,0}$ blue curves with $n = 3, 4, 5$

Conditions given in (23) imply the following

Theorem 7 *If there are implementation delays in the firms' own productions, then the Cournot production is stable for* $\tau < \tau^*_{a,0}$ *and is unstable at* $\tau \geq \tau^*_{a,0}$.

4.3.3 Diagonal and Off-Diagonal Delays

Next we deal with the case in which both the implementation and information delays coexist. However, for the sake of analytical simplicity, both delays are assumed to be identical. So the both-delay system is

$$\dot{q}_k(t) = \frac{\alpha - c_k}{2} - q_k(t-\tau) - \frac{\gamma}{2}\sum_{i\neq k}^{n} q_i(t-\tau) \text{ for } k = 1, 2, ..., n \tag{24}$$

where $\tau > 0$ denotes the common length of the both delays. The determinant equation of (24) is obtained by combining $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$,

$$\boldsymbol{J}_3 = \begin{pmatrix} -e^{-\lambda\tau} - \lambda & -\frac{\gamma}{2}e^{-\lambda\tau} & \cdots & -\frac{\gamma}{2}e^{-\lambda\tau} \\ -\frac{\gamma}{2}e^{-\lambda\tau} & -e^{-\lambda\tau} - \lambda & \cdots & -\frac{\gamma}{2}e^{-\lambda\tau} \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{\gamma}{2}e^{-\lambda\tau} & -\frac{\gamma}{2}e^{-\lambda\tau} & \cdots & -e^{-\lambda\tau} - \lambda \end{pmatrix}. \tag{25}$$

and the corresponding equation $\det(\boldsymbol{J}_3) = 0$ has the form

$$\left(\lambda + e^{-\lambda\tau} - \frac{\gamma}{2}e^{-\lambda\tau}\right)^{n-1}\left(\lambda + e^{-\lambda\tau} + \frac{(n-1)\gamma}{2}e^{-\lambda\tau}\right) = 0$$

that generates two independent equations

$$\lambda + e^{-\lambda\tau} - \frac{\gamma}{2}e^{-\lambda\tau} = 0 \tag{26}$$

and

$$\lambda + e^{-\lambda\tau} + \frac{(n-1)\gamma}{2}e^{-\lambda\tau} = 0. \tag{27}$$

In the same way as before, we suppose that $\lambda = i\omega$ with $\omega > 0$ and then substituting it into equation (26) to obtain

$$\omega_A^* = \frac{2-\gamma}{2} > 0$$

and

$$\tau_{A,m}^* = \frac{1}{\omega_A^*}\left(\frac{\pi}{2} + 2m\pi\right) \text{ for } m = 0, 1, 2, ...$$

with

$$\tau_{A,0}^* = \frac{\pi}{2-\gamma}. \tag{28}$$

Similarly substituting $\lambda = i\omega$ with $\omega > 0$ into equation (27), we can derive

$$\omega_B^* = \frac{2+(n-1)\gamma}{2}$$

and

$$\tau_{B,m}^* = \frac{1}{\omega_B^*}\left(\frac{\pi}{2} + 2m\pi\right) \text{ for } m = 0, 1, 2, ...$$

with

$$\tau_{B,0}^* = \frac{\pi}{2+(n-1)\gamma} < \tau_{A,0}^*. \tag{29}$$

Relation (29) implies the following:

Theorem 8 *If the information and implementation delays coexist, then both-delay system (24) is stable for* $\tau < \tau_{B,0}$ *and loses stability for* $\tau \geq \tau_{B,0}$.

Figure 2 illustrates three stability switch curves with $n = 9$, the τ_0^* curve that is asymptotic to the vertical dotted line at $\gamma_0 = 0.25$, the $\tau_{a,0}^*$ black curve and the $\tau_{B,0}^*$ blue curve, the last two curves of which intersect at $\gamma_1 \simeq 0.412$. For $\gamma < \gamma_0$, the off-diagonal delay model is always stable for any $\tau \geq 0$ whereas increasing the value of τ first destabilizes the both-delay model and then the diagonal delay model. For $\gamma_0 < \gamma \leq \gamma_1$, the off-diagonal model loses stability last. Finally for $\gamma_1 < \gamma < 1$, stability is lost in order of the both-delay model, the off-diagonal model and the diagonal model when τ increases from zero.

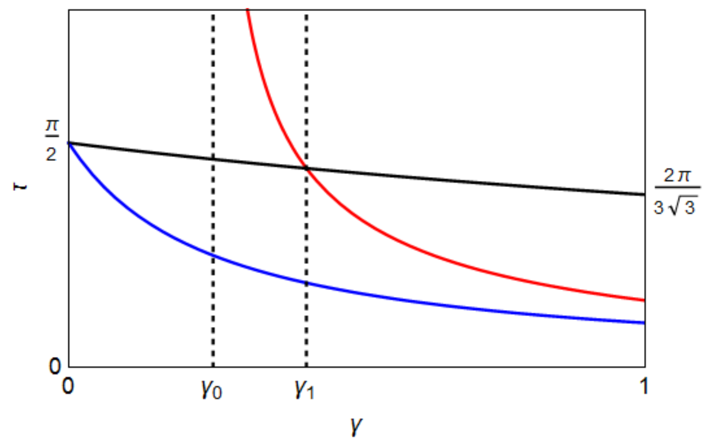

Figure 2. the stability switching curves of τ_0^* (red), $\tau_{a,0}^*$ (black) and $\tau_{B,0}^*$ (blue), given $n = 9$.

5 Concluding Remarks

The effect of the ambient charges on total pollution in a Cournot oligopoly was examined. First the Cournot equilibrium was determined and condition was given to the positivity of the individual output levels of the firms. An increase of the ambient charge can have a diverse effect on the production levels: if a firm is higher- or average-polluter, then its output decreases, otherwise it can increase, decrease or remain the same depending on a simple condition derived in the paper. It is also proved that the total emission decreases by increasing the policy parameter.

Dynamic models were then constructed in both discrete and continuous time scales based on naive expectations of the firms. In the case of discrete time scales, stability depends on the number of the firms, which result reduces to the classical theorem of Theocharis in the case of $\gamma = 1$. The dynamics with continuous time scales always generates the stability of the equilibrium. As a hybrid of these two dynamic models, we constructed a production-delay dynamic model in which the stability can be lost if the length of the delay becomes large enough.

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