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Akio Matsumoto
Chuo University

Ferenc Szidarovszky
University of Pécs

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INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

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Akio Matsumoto†
Chuo University

Ferenc Szidarovszky‡
University of Pécs

Abstract

Dynamics of Goodwin's accelerator business cycle model is reconsidered. The model is characterized by a nonlinear accelerator and an investment time delay. The role of the nonlinearity for the birth of persistent oscillations is fully discussed in the existing literature. On the other hand, not much of the role of the delay has yet been revealed. The purpose of this paper is to show that the delay really matters. In particular, two main results are obtained. In the original framework of Goodwin (1951), it is first demonstrated that limit cycles arise for smaller values of the delay and so do sawtooth oscillations for larger values and that the threshold value between these cases has initial point dependency. In the extended framework in which a consumption delay, in addition to the investment delay, is introduced, it is then demonstrated that there is an interval of delay in which the limit cycle coexists with the sawtooth oscillation. The possibility of the coexistence has an initial-point dependency.

Keywords: Investment delay, Nonlinear acceleration principle, Stability switch, Consumption delay, Limit cycle, Sawtooth oscillation

JEL number: C63, E12, E32

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†Professor, Department of Economics, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan. akiom@tamacc.chuo-u.ac.jp

‡Professor, Department of Applied Mathematics, University of Pécs, Ifúság, u. 6,. H-7624, Pécs, Hungary. szidarka@gmail.com

1 Introduction

This paper reconsiders dynamics of Goodwin's accelerator business cycle model (Goodwin, 1951). The model has two distinct features: one is a nonlinear accelerator and the other is an investment delay between the decision to investment and the corresponding outlay. In demonstrating emergence of persistently cyclic behavior in it, the roles of strong non-linearities are highlighted, however, the roles of the delays are made implicit. This is mainly because, as will be seen soon, a delay differential equation that describes behavior of the national income in Goodwin's model is approximated in a neighborhood of zero-delay to obtain a second-order nonlinear differential equation. The existence of the persistent oscillations is shown in this approximated model. As a result, considerations on delays lie outside the scope of the main discussions. In the existing literature since then, attention has been focused on non-linearities of the model. In this study, we pose the following question to shed light on the lost sight of Goodwin's model: *Is the delay responsible for the birth of the cyclic oscillations?* Although the Goodwin model, in addition to the classical multiplier-accelerator models (i.e., Samuelson (1930) and Hicks (1950)), can be found in macro-dynamics literature, there are only a limited number of works dealing with the delay and this question.[1]

Goodwin (1951) presents five different versions of the nonlinear accelerator-multiplier model with investment delay. The first version has the simplest form assuming a piecewise linear function with three levels of investment and aims to exhibit how non-linearities give rise to endogenous cycles without relying on structurally unstable parameters, exogenous shocks, etc. The second version replaces the piecewise linear investment function with a smooth nonlinear investment function. Although persistent cyclical oscillations are shown to exist, the second version includes unfavorable phenomena, that is, discontinuous investment jumps, which are not observed in the real economic world. "In order to come close to reality" (Goodwin, 1951, p.11), the third version introduces an investment delay. However, no analytical considerations are given to this version. The existence of a self-sustaining business cycle is confirmed in the fourth version, which is a linear approximation of the third version with respect to the investment delay. Finally, alternation of autonomous expenditure over time is taken into account in the fifth version, which becomes a forced oscillation system.

This paper reconstructs the third version and applies the recently developing mathematical results on the fixed delay differential equations to its analysis of cyclic dynamic behavior. It is a complement of Matsumoto and Suzuki (2008) and Matsumoto (2009) in which the dynamics of Goodwin's model is examined under the continuously distributed time delays and the existence of the multiple limit cycles are analytically and numerically shown. Our main concern in this paper is on the role of the fixed delay for macro dynamics and our main result is that the delay really matters. In particular, the delay not only affects convergent

[1] Gandolfo (1997) gives a short description that an gestation lag of investment in Kalecki (1935) can be a source of the existence of a business cycle.

dynamics when the steady state is locally stable but also plays a crucial role for persistent oscillations when the system is locally unstable.

The paper is organized as follows. In Section 2, the basic elements of Goodwin's model are recapitulated. In Section 3, the notable features of the delay are explicitly considered. This section is divided into two parts. In its first part, effects of investment delays are examined and in the second part, consumption delays are introduced to extend Goodwin's model. Section 4 contains some concluding remarks.

2 Basic Model

To find out how nonlinearity works to generate endogenous cycles, we review the second version of Goodwin's model, which we call the *basic model*,

$$\begin{cases} \varepsilon\dot{y}(t) = \dot{k}(t) - (1-\alpha)y(t), \\ \\ \dot{k}(t) = \varphi(\dot{y}(t)). \end{cases} \tag{1}$$

Here k is the capital stock, y the national income, α the marginal propensity to consume, which is positive and less than unity, and the reciprocal of ε is a positive adjustment coefficient. The dot over variables stands for time differentiation. The first equation of (1) defines an adjustment process of the national income. Accordingly, national income rises or falls if investment is larger or smaller than savings. The second equation, in which $\varphi(\dot{y}(t))$ denotes the induced investment, describes an accumulation process of capital stock based on the acceleration principle. According to this principle, investment depends on the rate of changes in the national income. A distinctive feature of Goodwin' model is to introduce a nonlinearity into the investment function in such a way that the investment is proportional to the change in the national income in the neighborhood of the equilibrium income but becomes inflexible (i.e., less elastic) for extremely larger or smaller values of the income. This *nonlinear* acceleration principle is crucial in obtaining endogenous cycles in Goodwin's model. We will retain this nonlinearity assumption. On the other hand, we depart from Goodwin's non-essential assumption of positive autonomous expenditure and will work with zero autonomous expenditure for the sake of simplicity. A direct consequence of this alternation is that a stationary point of the basic model is $y(t) = \dot{y}(t) = 0$ for all t.

Inserting the second equation of (1) into the first one and moving the terms on the right hand side to the left give the single dynamic equation for the national income y,

$$\varepsilon\dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0. \tag{2}$$

This is a nonlinear differential equation. Although (2) is one-dimensional, its nonlinearity due to the acceleration principle prevents deriving an explicit form of the solution. It is, however, possible to detect local dynamics by examining its linearized version in a neighborhood of the stationary point and global dynamics by numerical simulations. This is the approach we take in this study.

The linear version of equation (2) is

$$\varepsilon\dot{y}(t) - \nu\dot{y}(t) + (1-\alpha)y(t) = 0, \tag{3}$$

where $\nu = \varphi'(0)$ is the slope of the investment function at the stationary point. This is a first-order ordinary differential equation. If $\nu = \varepsilon$, then $y(t) = 0$ for all $t \geq 0$, so we may assume that $\nu \neq \varepsilon$. Applying separation of variables gives the complete solution,

$$y(t) = y_0 e^{\lambda t} \text{ with } \lambda = \frac{1-\alpha}{\nu - \varepsilon}, \tag{4}$$

where y_0 is an initial condition. λ is a real root and no oscillation occurs. The stationary point is locally asymptotically stable or unstable according to whether the eigenvalue λ is negative or positive. Since $1-\alpha$ is the positive marginal propensity to save, the sign of the eigenvalue depends on whether the numerator is positive or negative. Roughly speaking, a smaller adjustment coefficient (i.e., a larger ε) is responsible for local stability while a larger response of the investment accelerator is a local destabilizer. We summarize this result on the basic model:

Theorem 1 *The zero solution of the basic model (2) is locally asymptotically stable if $\nu < \varepsilon$ and locally unstable if $\nu > \varepsilon$.*

It is easy to see that if $\nu < \varepsilon$, then any time trajectory of equation (2) eventually converges to the zero solution as $t \longrightarrow \infty$, although it exhibits different transient oscillations according to the different selections of the initial point. We turn our attention to global dynamics in the case of local instability (i.e., $\nu > \varepsilon$). In order to conduct numerical analysis, we first specify the investment function as well as the values of the coefficients of the basic model and then perform simulations to see what dynamics of y can be generated. Although Goodwin (1951) assumed the piecewise linear investment function, we, for the sake of analytical convenience, adopt a smooth nonlinear investment function of the form of an arctangent,

$$\varphi(\dot{y}(t)) = \delta\left\{\tan^{-1}(\dot{y}(t) - \eta) - \tan^{-1}(-\eta)\right\}, \ \ \delta > 0 \text{ and } \eta > 0. \tag{5}$$

This function has endogenous "ceiling" and "floor" and is asymmetric when the parameter, η, is non-zero. In what follows, since the same set of the parameter values will be repeatedly used, we make the following assumption for convenience. Needless to say, these particular values of the parameters are selected only for analytical simplicity and do not affect qualitatively the results to be obtained.

Assumption 1 $\alpha = 0.8, \ \varepsilon = 0.5, \ \delta = 1.5$ and $\eta = 1$

Notice that the investment function (5) under Assumption 1 passes through the origin with the slope $\varphi'(0) = \nu = 0.75$ and its ceiling is three time higher than its floor as it was the case in Goodwin's model. An alternative expression of dynamic equation (2) is

$$y(t) = \frac{\varphi(\dot{y}(t)) - \varepsilon \dot{y}(t)}{1-\alpha}, \tag{6}$$

which describes a mirror-imaged N-shaped curve in the $(\dot{y}, y)$ plane when equation (5) is substituted into (6). Differentiating (6) with respect to $\dot{y}(t)$ and equating the resultant derivative to zero, we can obtain the maximizer $(\dot{y}_M)$ and the minimizer $(\dot{y}_m)$,

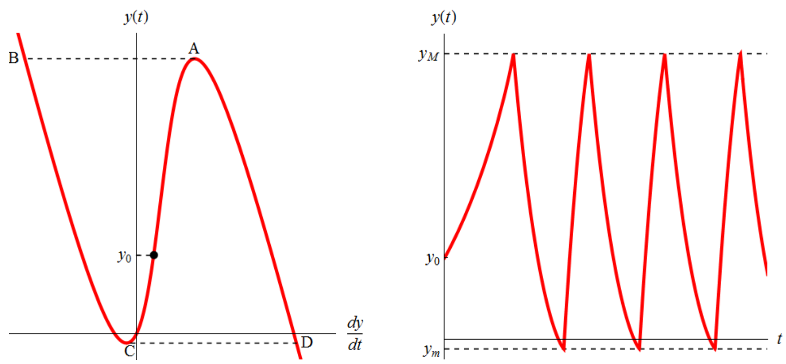
$$\dot{y}_M = \eta + \frac{\sqrt{(\delta - \varepsilon)\varepsilon}}{\varepsilon} \text{ and } \dot{y}_m = \eta - \frac{\sqrt{(\delta - \varepsilon)\varepsilon}}{\varepsilon}.$$

These then determine the local maximum (y_M) and minimum (y_m) of y via equation (6).[2] The phase diagram is presented in Figures 1(A) in which a birth of an endogenous cycle is illustrated under Assumption 1 when the initial point is taken at $y_0 = 2$. Along the locus, the initial point is displaced slightly upward to point $A = (\dot{y}_M, y_M)$ so that the output is increasing to the highest level of the national income, y_M. Investment immediately switches discontinuously from positive to negative. Graphically, the orbit jumps from point A to point B. With negative $\dot{y}(t)$ at point B, the national income gradually declines from point B to point $C = (\dot{y}_m, y_m)$ so that the output is decreasing to the lowest level, y_m. Once point C is reached, investment switches again discontinuously from negative to positive. That is, the orbit jumps again from point C to point D from which the national income glides toward point A, and then the process repeats itself. Thus the differential equation (2) with the nonlinear investment function (5) can give rise to a closed orbit constituting a self-sustaining slow-rapid (or relaxation) oscillation. Notice that regardless of the initial point, a trajectory can converge to the same slow-rapid oscillation. Since $\dot{y}(t)$ discontinuously jumps at the points A and C, it makes the sharp kinks at the highest and lowest levels of the corresponding time trajectory $y(t)$ as shown in Figure 1(B). We call it a *sawtooth* oscillation.[3] This is a simple exhibition of emerging an endogenous cycle of output. We summarize the global results obtained in the basic model:

Theorem 2 *The zero solution of the basic model (2) is globally asymptotically stable if $\nu < \varepsilon$ whereas a slow-rapid cycle emerges, and the corresponding time trajectory displays sawtooth oscillations if $v > \varepsilon$.*

[2] Explicit forms of y_M and y_m are not given only because they are long and clumsy.

[3] The implicit function theorem states that $\dot{y}(t)$ is a continuous function of $y(t)$ if $\varphi'(\dot{y}(t)) \neq 0$. Solving equation $\varphi'(\dot{y}(t)) = 0$ leads to two solutions for $\dot{y}(t)$, $\dot{y}^A(t)$ at point A and $\dot{y}^C(t)$ at point C, at each of which a jump of $\dot{y}(t)$ occurs. At these points, the $y(t)$ values are the kinks, in other words, the continuity of $\dot{y}(t)$ as function of $y(t)$ (which is continuous in t) is violated. So $\dot{y}(t)$ is discontinuous in t there.



(A) Slow-rapid cycle (B) Kinked trajectory

Figure 1. Birth of an endogenous oscillation

3 Fixed Delay Model

We investigate how the delay affects time paths of national income. This section is subdivided into two subsections. The investment delay of the Goodwin model is presented in the first subsection. We then introduce, in addition to the investment delay, a delay in consumption, following the spirit of the multiplier-accelerator models in which the delay in consumption is one of the main ingredients.

3.1 Investment Delay

Observing the fact that, in real economy, plans and their realizations need time to take effects, Goodwin (1951) introduces the investment delay, θ, between decisions to invest and the corresponding outlays in order, first, to come closer to reality and second, to eliminate unrealistic discontinuous jumps. Inserting θ into the investment function of the basic model yields the third version of his model,

$$\varepsilon\dot{y}(t) - \varphi(\dot{y}(t-\theta)) + (1-\alpha)y(t) = 0 \tag{7}$$

with an initial function,

$$y(t) = f(t) \text{ for } -\theta \le t \le 0.$$

This is a *delayed differential equation of neutral type*, which we call the *delay model*. The initial function gives behavior of y prior to time zero. For simplicity, we assume a constant initial function $f(t) = y_0$ and call y_0 an initial point. Goodwin (1951) does not analyze dynamics generated by this model. Furthermore, to the best of our knowledge, no analytical solutions of equation (7) are available yet. Since a cyclic oscillation has been shown to exist in the basic model, our main concern here is to see how the presence of the investment delay affects characteristics of such a slow-rapid cycle. We analytically investigate the local stability of the cycle generated in the linearized model and numerically detect the effects caused by the delay on cyclical dynamics.

3.1.1 Local Stability

The delay model is autonomous and its special solution is constant (i.e., $y(t) = 0$) so that its linearized version takes the form of a *linear neutral autonomous delay* differential equation,

$$\varepsilon\dot{y}(t) - v\dot{y}(t-\theta) + (1-\alpha)y(t) = 0. \tag{8}$$

It is well known that if the characteristic polynomial of a linear neutral equation has roots only with negative real parts, then the stationary point is locally asymptotically stable. The normal procedure for solving this equation is to try an exponential form of the solution. Substituting $y(t) = y_0 e^{\lambda t}$ into (8) and rearranging terms, we obtain the corresponding characteristic equation:

$$\varepsilon\lambda - \nu\lambda e^{-\lambda\theta} + (1-\alpha) = 0.$$

To check stability, we determine conditions under which all roots of this characteristic equation lie in the left or right half of the complex plane. Dividing both sides of the characteristic equation by ε and introducing the new variables

$$a = \frac{\nu}{\varepsilon} > 0 \text{ and } b = \frac{1-\alpha}{\varepsilon} > 0, \tag{9}$$

we rewrite the characteristic equation as

$$\lambda - a\lambda e^{-\lambda\theta} + b = 0. \tag{10}$$

Freedman and Kuang (1991) derive explicit conditions for stability/instability of the n-th order linear scalar neutral delay differential equation with a single delay. Since (8) is a special case of the n-th order equation, applying their result (i.e., Theorem 2.1) leads to the following: the real parts of the solutions of equation (10) are positive for all $\theta > 0$ if $a > 1$. The first result on the fixed delay model is summarized as follows:

Lemma 1 *If $\nu > \varepsilon$, then the zero solution of the fixed delay model (7) is locally unstable for all $\theta > 0$.*

On the other hand, if $v \leq \varepsilon$ or $a \leq 1$, characteristic equation (10) has at most finitely many eigenvalues with positive real parts. It is shown in equation (4) that the eigenvalue is real and negative when $\theta = 0$. The roots of the characteristic equation are functions of the delay. Although it is expected that all roots have negative real parts for small values of θ, the real parts of some roots may change their signs to positive from negative as the lengths of the delay increases. The stability of the zero solution may change. Such phenomena are often referred to as *stability switches*. We will next prove that stability switchings, however, cannot take place in the delayed model.

Lemma 2 *If $\nu \leq \varepsilon$, then the zero solution of the fixed delay model (7) is locally stable for all $\theta > 0$.*

Proof. (i) We first deal with the case of $v < \varepsilon$. It can be checked that $\lambda = 0$ is not a solution of (10) because substituting $\lambda = 0$ yields $b = 0$ that contradicts $b > 0$. If the stability switches at $\theta = \bar{\theta}$, then (10) must have a pair of pure conjugate imaginary roots with $\theta = \bar{\theta}$. Thus to find the critical value of $\bar{\theta}$, we assume that $\lambda = i\omega$, with $\omega > 0$, is a root of (10) for $\theta = \bar{\theta} > 0$. Substituting $\lambda = i\omega$ into (10), we have

$$b - a\omega \sin \omega\theta = 0,$$

and

$$\omega - a\omega \cos \omega\theta = 0.$$

Moving b and ω to the right hand sides and adding the squares of the resultant equations, we obtain

$$b^2 + (1 - a^2)\omega^2 = 0.$$

Since $b > 0$ and $1 - a^2 > 0$ as $a < 1$ is assumed, there is no ω that satisfies the last equation. In other words, there are no roots of (10) crossing the imaginary axis when θ increases. No stability switch occurs and thus the zero solution is locally asymptotically stable for any $\theta > 0$.

(ii) In case of $\varepsilon = \nu$ in which $a = 1$, the characteristic equation becomes

$$\lambda(1 - e^{-\lambda\theta}) + b = 0. \tag{11}$$

It is clear that $\lambda = 0$ is not a solution of (11) since $b > 0$. Thus we can assume that a root of (11) has non-negative real part, $\lambda = u + iv$ with $u \geq 0$ for some $\theta > 0$. From (11), we have

$$(u + b)^2 + v^2 = e^{-2u\theta}(u^2 + v^2) \leq (u^2 + v^2),$$

where the last inequality is due to $e^{-2u\theta} \leq 1$ for $u \geq 0$ and $\theta > 0$. Hence

$$2ub + b^2 \leq 0,$$

where the direction of inequality contradicts the assumption that $u \geq 0$ and $b > 0$. Hence it is impossible for the characteristic equation to have roots with nonnegative real parts. Accordingly, all roots of (11) must have negative real parts for all $\theta > 0$. ■

Lemmas 1 and 2 imply the following theorem concerning local stability of the delay model (7).

Theorem 3 *For any $\theta > 0$, the zero solution of the delay model (7) is locally asymptotically stable if $\nu \leq \varepsilon$ and unstable if $\nu > \varepsilon$.*

3.1.2 Global Stability

Theorems 1 and 3 show that the delay model and the basic model have the same stability condition except the critical case of $\nu = \varepsilon$. Introducing fixed time delay does not affect the local stability condition of the basic model. However this does not necessarily mean that the delay has no effects on global behavior. To see the delay effect, we conduct numerical simulations.

We start with the locally stable case under Assumption 1 but with changing the value of ε to 0.85 to hold the stability condition $\nu < \varepsilon$. Figure 2(A) illustrates three simulation results of the global behavior for $0 < t < 50$ when the same initial point $y_0 = 0.1$ is taken and the lengths of the delay are different. The red trajectory has no delay, $\theta = 0$, the blue has $\theta = 3$ and the green has $\theta = 6$. At least the following three issues can be observed, implying that the delay matters in the stable case:

(i) The delay generates oscillations as the blue and green trajectories exhibit dampening oscillations while the red trajectory monotonically converges to the zero solution.

(ii) The delay causes slow convergence to the zero solution since the blue and green trajectories take more time to arrive at the zero solution than the red trajectory, which rapidly converges.

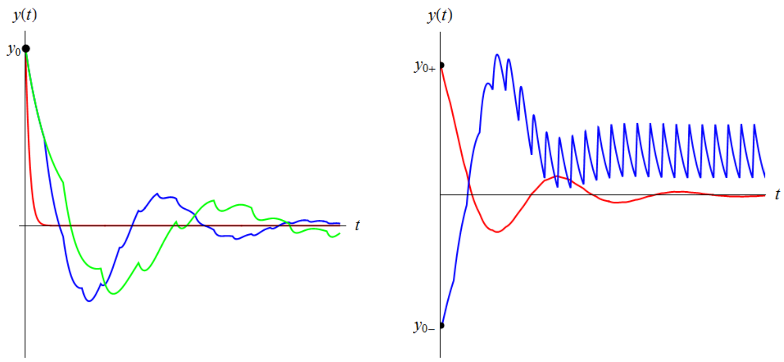
(iii) The delay makes time trajectories kinked, implying discontinuous jumps of $\dot{y}(t)$.

Increasing the value of y_0 does not change the qualitative features of the transient dynamics just described above as far as y_0 is positve. However, dynamics drastically is changed if the inital value is taken to be negative. In the next two simulations, the initial point is changed but the length of the delay is kept to be constant at $\theta = 2$. In Figure 2(B), the red trajectory starts at $y_0 = 2$ and shows oscillatory convergence to the zero solution as in Figure 2(A) and the blue trajectory starting at $y_0 = -2$ finally converges to sawtooth oscillations. Since the parameter values are not changed, these simulations exhibit a coexistence of persistent oscillations and convergent trajectory due to different choices of the initial points. Further, careful observations reveal that these trajectories are also kinked many times. In short, we have the following.

(iv) The delay generates initial point dependency that leads to the coexistence of qualitatively different dynamics.

In both simulations, introducing delay into the basic model does not get rid of kinked oscillations. These numerical results are summarized:

Proposition 1 *If $\nu < \varepsilon$, then a positive delay can be a source of oscillations but unable to eliminate kinked oscillations including sawtooth oscillations while different initial functions can lead to qualitative different dynamics although the zero solution is locally stable.*



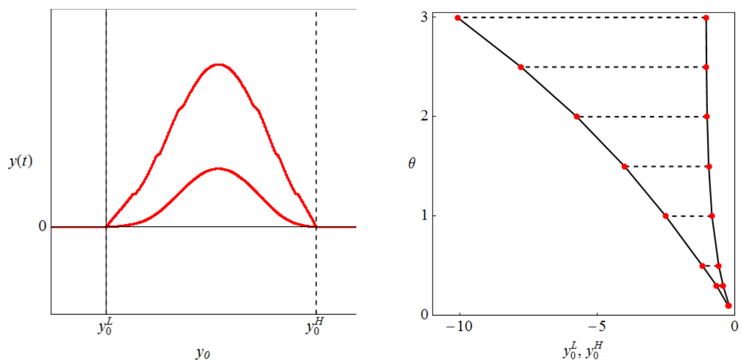
(A) Different delays (B) Different initial points

Figure 2. Delay effects on stable trajectories

To confirm the initial point dependency, we further simulate the delay model. Given θ, Figure 3(A) describes how the dynamics changes as the initial point changes. Indeed, with $\theta = 2$, the initial point y_0 is increased from -7 to -0.1 with increment of 0.01. There are two critical values, $y_0^L \simeq -5.7578$ and $y_0^H \simeq -1.0169$,[4] and the time trajectory converges to the zero solution if $y_0 < y_0^L$ or $y_0 > y_0^H$. On the other hand, it converges to the sawtooth oscillations if $y_0^L < y_0 < y_0^H$. The maximum and minimum values of the oscillations are presented by the upper and lower mound-shape curves. It is seen that amplitude of the oscillations gradually increases, arrives at the maximum and then decreases to zero as y_0 increases to y_0^H from y_0^L. Figure 3(B) illustrates the sensitivity of these critical values to a change of the length of delay. We take eight different values of θ, $0.1,\ 0.3,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5$ and 3. The corresponding values of y_0^L and y_0^H are denoted by the red dots. Connecting the values of y_0^L or y_0^H is given by the flatter sloping curve or the steeper sloping curve. Within the interval $[y_0^L,\ y_0^H]$, sawtooth oscillations emerge. The length of the interval is expressed by the dotted line between y_0^L and y_0^H. It can be seen that the interval becomes

[4] These values are numerically obtained.

longer as the value of θ increases.[5]



(A) Dependency on y_0 (B) Interval $[y_0^L,\ y_0^H]$

Figure 3. Delay effects on stable trajectories

We now draw attention to the global dynamics under the local unstable conditions. Goodwin (1951) expands the delay model (7) with respect to θ to obtain his fourth version, given as a second-order nonlinear ordinary differential equation,

$$\varepsilon\theta\ddot{y}(t) + [\varepsilon + (1-\alpha)\theta]\dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0. \tag{12}$$

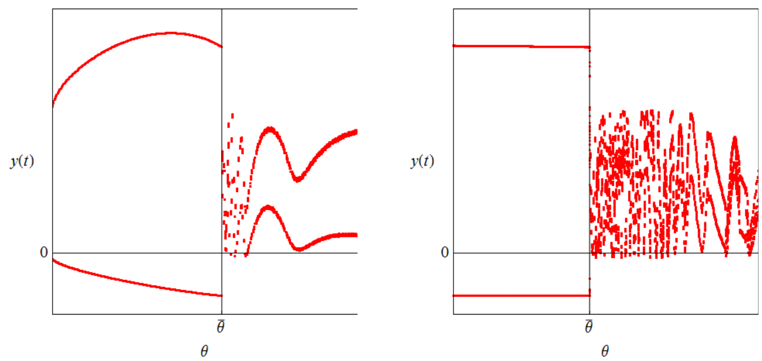
This is an equation of the Rayleigh type. It is already shown that this equation has a unique periodic solution without any jumps in the locally instable case when $\nu > \varepsilon + (1-\alpha)\theta$.[6] We call such a smooth oscillation a Goodwinian cycle or oscillation hereafter.[7] Since this approximated version is valid only for smaller values of θ, it is natural and interesting to address a question on whether a Goodwinian cycle might emerge for larger values of θ. To answer this question, we numerically simulate the delay model when the zero solution is locally unstable (i.e., $\nu > \varepsilon$). Figure 4 represents the bifurcation diagrams of $y(t)$ with respect to the delay parameter θ. For given value of θ, the solutions of (7) for $0 < t < 1000$ are calculated and the local maximum and minimum values of $y(t)$ for $950 < t < 1000$ are plotted against θ. The bifurcation parameter θ increases from 0 to 3 with increment of 0.01 and for each value of θ, the same calculating procedure is repeated to obtain the bifurcation diagram. Figure 4(A) implies that Goodwinian oscillations of $y(t)$ are possible not only for smaller values of θ but also for larger values. More precisely, under Assumption 1 and

[5] For $\theta = 0.1$, $y_0^H \simeq -0.2173$ and $y_0^L \simeq -0.2567$ so that their difference, $y_0^H - y_0^L \simeq 0.04$, is almost invisible. Although two points seem to stick together in Figure 3(B), they are distinct.

[6] See Sasakura (1996).

[7] This is called the first mode of oscillation while the sawtooth oscillations the higher modes of oscillations in Strotz et al. (1953).

$y_0 = 15$, there is a threshold value $\bar{\theta} \simeq 1.6723$.[8] A Goodwinian cycle having one maximum and one minimum emerges for $\theta < \bar{\theta}$ and its diameter measured by the distance between these extremum values changes as θ increases. The smooth oscillations suddenly disappear when θ arrives at $\bar{\theta}$ and a sawtooth oscillation appears for $\theta > \bar{\theta}$ as shown in Figure 4(B) which is an enlargement of Figure 4(A) in the neighborhood of $\bar{\theta}$.[9]



(A) $0 \leq \theta \leq 3$ (B) $1.65 \leq \theta \leq 1.70$

Figure 4. Bifurcation diagrams with respect to θ

Taking $\theta < \bar{\theta}$, we will first compare the Goodwinian cycle arisen in the delay model with one in the approximated model and then turn to examine dynamics for $\theta > \bar{\theta}$. Two phase planes are presented in Figure 5 in which the national income is on the vertical axis, its derivative on the horizontal axis and the origin denotes the singular point with $y(t) = \dot{y}(t) = 0$ for all $t \geq 0$. The blue phase trajectory forms the closed loop generated by the delay model and the green one by the approximated model while the red curve by equation (6), the basic model. As a smaller delay, $\theta = 0.1$ is assumed in Figure 5(A), the blue and green cycles are almost the same. Further it can be seen that these cycles have no jumps and are located on the negative- and positive-sloping parts of the red curve. These facts indicate the followings:

(i) introducing the delay works to eliminate unfavorable jumps observed in the basic model;

(ii) the nonlinear differential equation well approximates the delay differential equation when θ is small;

(iii) the cycles are stable and have no initial point dependency.

[8] This value is numerically obtained by rule of thumb and is sensitive to the selected value of the initial point. See Figure 7(A).

[9] Antonova *et al.* (2013) rigorously investigates the properties of the sawtooth oscillations.

These are the issues which Goodwin (1951) emphasizes. In Figure 5(B) a larger delay is taken, $\theta = 1$, and it is observed that the delay model generates a Goodwinian cycle. It is also observed that although the approximated model still gives rise to an oscillatory movement, the resultant green cycle shows large differences from the blue cycle. This dissimilarity implies that the nonlinear differential equation (12) does not approximate the delay differential equation (7) anymore.

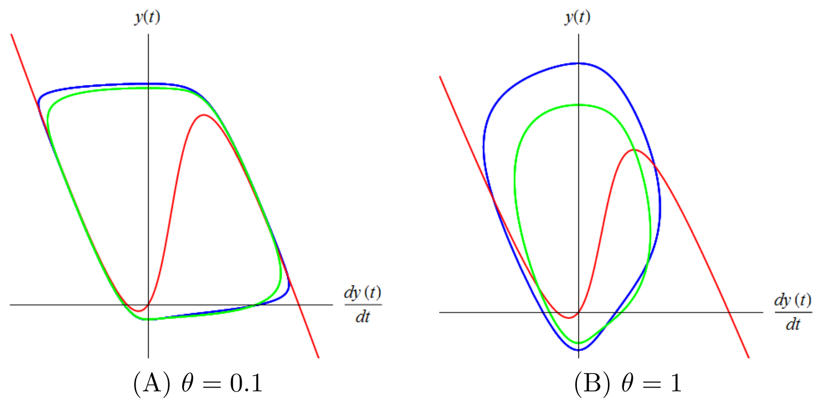


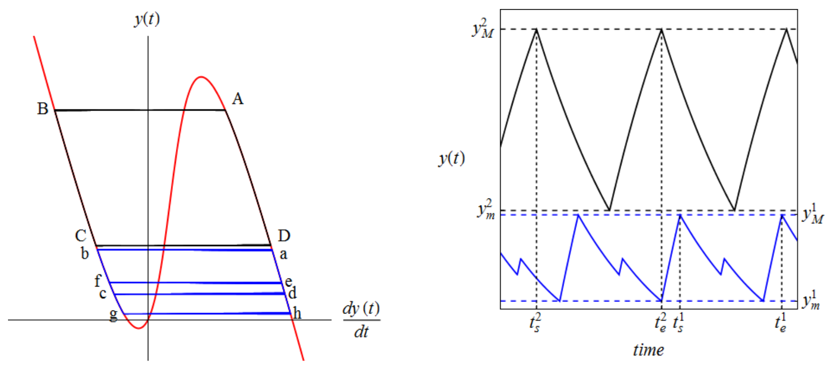
Figure 5. Delay Goodwinian and approximated oscillations

For much larger delay $\theta > \bar{\theta}$, the delay model does not generate a Goodwinian cycle but exhibits sawtooth oscillations as the basic model (2). Given $y_0 = 15$, the delay model (7) with different delays, $\theta_1 = 1.71$ and $\theta_2 = 2.1$ generates two different slow-rapid cycles, the blue cycle denoted by points $abcdefgh$ and the black cycle by points $ABCD$ as shown in Figure 6(A). Both take parallelogram-wise shaped forms and jump at points a, c, d and g or at A and C. These are the solutions of the delay model (2) with the same parameter values and indicate that there are more additional cycles corresponding to the different initial points.[10] Accordingly, its corresponding time trajectories have kinks as shown in Figure 6(B). The slow parts of the slow-rapid cycle are located exactly on the mirror-imaged N-shaped red curve described by the basic delay (2).[11] The blue cycle has period $t_e^1 - t_s^1 = \theta_1$ and amplitude $y_M^1 - y_m^1 \simeq 1.87$ and the black cycle has period $t_e^2 - t_s^2 = \theta_2$ and amplitude $y_M^2 - y_m^2 \simeq 3.94$. Notice that the minimum values y_m^1 or y_m^2 of the trajectory are positive and thus greater than the zero solution. The trajectories oscillate around some positive levels of output. On the other hand, the kinked trajectory in Figure 1(B) oscillates around the equilibrium level of national income. In all, contrary to Goodwin's

[10] Strotz *et al.* (1953) describe "... we found...at least twenty-five other limit cycles." (p.398)

[11] This has been already pointed out by Antonova *et al.* (2010). Furthermore, it is analytically shown by Antonova *et al.* (2013) that the sawtooth oscillations obtained with a positive delay satisfy the basic model (2) without delay.

intention, the delay model with a large value of θ fails to eliminate discontinuous jumps.



(A) Slow-rapid cycles (B) Kinked trajectories

Figure 6. Sawtooth oscillations with $\theta = 2$

The critical value $\bar{\theta}$ is numerically confirmed to have initial point dependency. Figures 7(A) and 7(B) depict the locus of negative y_0 denoted as y_0^m and $\bar{\theta}$ and the locus of positive y_0 denoted as y_0^M and $\bar{\theta}$ on which the Goodwinian cycle of the delay model just disappears. The critical values $\bar{\theta}$ is obtained for $y_0 = \pm 0.1$, ± 0.5 and integers from ± 1 to ± 15. These 17 combinations are depicted as the red dots and the boundary curves are constructed by connecting these dots. Hence a Goodwinian cycle emerges for (y_0, θ) in the yellow region and a sawtooth oscillation is born in the white region. Notice further the followings:

(i) dynamics is almost symmetric with respect to $y_0 = 0$;

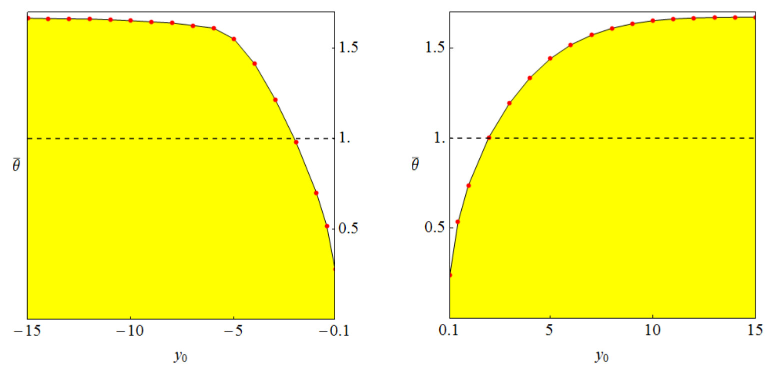
(ii) Goodwinian oscillations emerge for $y_0 < y_0^m$ or $y > y_0^M$;

(iii) the interval $[y_0^m, 0) \cup (0, y_0^M]$ becomes longer as θ increases.

Figures 8(A) and 8(B) describe dependency of the resultant oscillations on y_0 that change from ± 0.01 to ± 3 along the dotted horizontal line at $\theta = 1$ in Figures 7(A) and 7(B). The horizontal line crosses the loci of (y_0, θ) approximately at $y_0 = \pm 2$ (more precisely, $y_0^m \simeq -2.05$ and $y_0^M \simeq 1.98$). It is seen that the Goodwinian cycle has a constant diameter whereas the maximum and minimum values of the sawtooth oscillation are sensitive to the value of y_0. It is also seen that the threshold value $\bar{\theta}$ seems to have an upper bound above which only the sawtooth oscillations appear regardless of the initial point.[12]

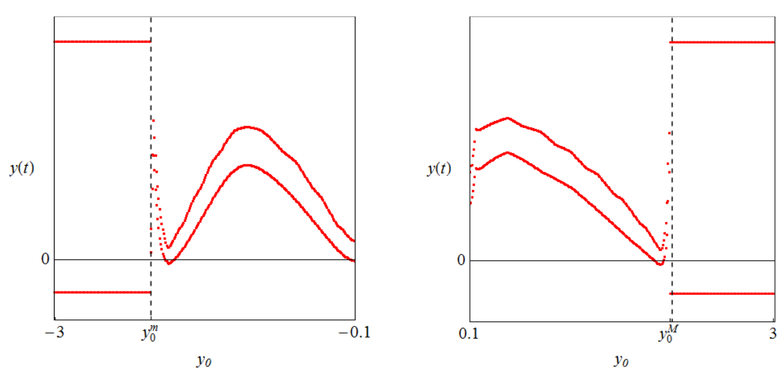
[12] The changes of $\bar{\theta}$ for a unit increase of y_0 from 12 to 15 are 25/10000, 12/10000 and 6/10000. On the other hand, the changes $\bar{\theta}$ for a unit decrease of y_0 from -12 to -15 are 28/10000, 23/10000 and 21/10000. The marginal changes are decreasing.

Proposition 2 *If $\nu > \varepsilon$, then the delay model gives rise to a Goodwinian cycle if $\theta < \bar{\theta}$ and a sawtooth oscillation if $\theta > \bar{\theta}$ where the critical value $\bar{\theta}$ positively depends on the initial point y_0.*



(A) Locus of y_0^m and $\bar{\theta}$ (B) Locus of y_0^M and $\bar{\theta}$

Fgure 7. Boundary curves between Goodwinian and sawtooth ocillations



(A) Dependency of $y_0 < 0$ (B) Dependency on $y_0 > 0$

Figure 8. Initial point dependency of two oscillations wth $\theta = 1$

3.2 Consumption Delay

Goodwin (1951) introduces the investment delay in order to "come closer to reality" in which discontinuous jumps are not observed. Although he did not examine the delay model, our numerical results indicate that his attempt is half-success and half-failure because smooth oscillations without jumps are obtained for $\theta < \bar{\theta}$ and sawtooth oscillations are still obtained for $\theta > \bar{\theta}$. In order to improve Goodwin's delay model (7), we introduce, in addition to the investment delay, a consumption delay with two reasons. First, as seen in the classical multiplier-acclerator models like Samuelson (1939) and Hicks (1950), it is natural to assume that current consumption is determined by the income in the past periods. Second, according to Sordi and Vercelli (2006), Goodwin (1946) takes into account a fixed delay between consumption expenditure and income generation to construct an earlier version of Goodwin business cycle model. However, for only analytical simplicity, in this study, we deal with the special case where the consumption delay denoted as σ is identical with the investment delay, $\sigma = \theta$.[13] Equation (7) is now modified to

$$\varepsilon \dot{y}(t) - \varphi(\dot{y}(t-\theta)) + (1-\alpha)y(t-\theta) = 0. \tag{13}$$

3.2.1 Local Stability

The linear version of equation (13) is

$$\varepsilon \dot{y}(t) - \nu \dot{y}(t-\theta) + (1-\alpha)y(t-\theta) = 0.$$

or

$$\dot{y}(t) - a\dot{y}(t-\theta) + by(t-\theta) = 0$$

where a and b are already defined in (9). The corresponding characteristic equation is

$$\lambda - a\lambda e^{-\lambda\theta} + be^{-\lambda\theta} = 0. \tag{14}$$

Due to Freedman and Kuang (1991, Theorem 1.2) again, we have the following:

Lemma 3 *If $\nu > \varepsilon$, then the zero solution of equation (13) is locally unstable for all $\theta > 0$.*

In the following discussions we assume $\nu < \varepsilon$ or $a < 1$ for a while. It is easily checked that $\lambda = 0$ is not a solution of equation (14) as $b > 0$. Hence if stability of the zero solution of equation (13) switches at $\theta = \tilde{\theta}$, then equation (14) must have a pair of pure conjugate imaginary roots. In order to check stability switches, we now suppose that $\lambda = i\omega$ with $\omega > 0$ is a root of equation (14) for $\theta = \tilde{\theta} \geq 0$.

Substituting this root into the characteristic equation (14) and separating the real and imaginary parts give

$$\begin{aligned} -a\omega \sin \omega\theta + b\cos\omega\theta &= 0 \\ a\omega\cos\omega\theta + b\sin\omega\theta &= \omega. \end{aligned} \tag{15}$$

[13] A general case of $\sigma \neq \theta, \sigma > 0, \theta > 0$ and a special case of $\theta = 0$, $\sigma > 0$ are fully considered in Matsumoto and Szidarovszky (2014).

Squaring both equations and adding them together, we obtain

$$a^2\omega^2 + b^2 = \omega^2.$$

Hence

$$\omega^2 = \frac{b^2}{1-a^2} > 0$$

which implies that purely imaginary roots of equation (14) exist,

$$\omega = \frac{b}{\sqrt{1-a^2}} > 0.$$

From (15), we have

$$\sin\omega\theta = \frac{b\omega}{a^2\omega^2 + b^2} > 0 \tag{16}$$

and

$$\cos\omega\theta = \frac{a\omega^2}{a^2\omega^2 + b^2} > 0. \tag{17}$$

Hence the inequalities imply that there is a unique $\omega\theta$, $0 < \omega\theta < \pi/2$ for which both equations (16) and (17) hold. Solving these equations for θ yields

$$\begin{aligned} \theta &= \frac{1}{\omega}\left[\sin^{-1}\left(\frac{b\omega}{a^2\omega^2 + b^2}\right) + 2n\pi\right] \\ &= \frac{\sqrt{\varepsilon^2 - \nu^2}}{1-\alpha}\left[\sin^{-1}\left(\frac{\sqrt{\varepsilon^2 - \nu^2}}{\varepsilon}\right) + 2n\pi\right] \end{aligned} \tag{18}$$

or

$$\begin{aligned} \theta &= \frac{1}{\omega}\left[\cos^{-1}\left(\frac{a\omega^2}{a^2\omega^2 + b^2}\right) + 2n\pi\right] \\ &= \frac{\sqrt{\varepsilon^2 - \nu^2}}{1-\alpha}\left[\cos^{-1}\left(\frac{\nu}{\varepsilon}\right) + 2n\pi\right] \end{aligned} \tag{19}$$

Notice that equations (18) and (19) have different forms giving the same value of θ for which the stability switch takes place.

We determine the sign of the real part of the derivative of λ at the stability switches. Since λ is θ-dependent, differentiating (14) with respect to θ yields

$$\left\{1 - (b\theta + a(1-\lambda\theta))e^{-\lambda\theta}\right\}\frac{d\lambda}{d\theta} - \lambda(b - a\lambda)e^{-\lambda\theta} = 0.$$

Solving this, only for convenience, for $(d\lambda/d\sigma)^{-1}$, we have

$$\left(\frac{d\lambda}{d\theta}\right)^{-1} = \frac{e^{\lambda\theta} - (b\theta + a(1-\lambda\theta))}{\lambda(b - a\lambda)}.$$

Notice that from (14),

$$e^{\lambda\theta} = \frac{a\lambda - b}{\lambda},$$

therefore, for $\lambda = i\omega$,

$$\text{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right] = \frac{b^2}{\omega^2\left((a\omega)^2 + b^2\right)} > 0.$$

This inequality implies that the only crossing with the imaginary axis is from left to right as θ increases. That is, stability is lost at the smallest stability switch and it cannot be regained later. As seen in the left part of Figure 9, the locus described by equation (18) or (19) divides the first quadrant of the (ν, θ) plane into two regions. The zero solution is locally stable under a positive delay in the region below this locus and locally unstable in the region above. In other words, a stability switch occurs on this locus which we call the *partition curve.*

Lemma 4 *If $\nu < \varepsilon$, then stability of the zero solution of (13) is switched to instability at $\theta = \theta_s$ where*

$$\theta_s = \frac{\sqrt{\varepsilon^2 - \nu^2}}{1-\alpha}\cos^{-1}\left(\frac{\nu}{\varepsilon}\right)$$

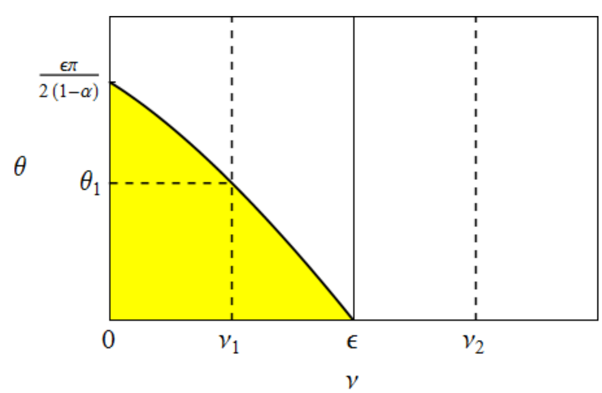


Figure 8. Partition curve and stability region

Lemmas 3 and 4 together imply the local stability condition of the zero solution of the delay model (13) as follows:

Theorem 4 *The zero solution of the delay model (13) is locally unstable if $\nu > \varepsilon$ while it is locally asymptotically stable for (ν, θ) in the yellow region below the partition curve and loses stability on the curve.*

3.2.2 Global Stability

We examine the delay effects on global stability and start with the case where the zero solution is locally stable. We address the following questions: *does* *it destabilize the locally stable zero solution and if so, then what kind of global dynamics the delay model (13) can have.*

Taking $\nu_1 = 0.25$ with other parameter values specified in Assumption 1 that yields $a < 1$, we increase the value of θ along the vertical dotted line at $\nu = \nu_1$ in Figure 8. The threshold value $\theta_1 = 5\sqrt{3}\pi/12(\simeq 2.267)$ is the point at which the vertical dotted line crosses the partition curve. To see how the delay affects global dynamics, we obtain bifurcation diagrams with respect to θ with three different initial points y_0, in the same way as shown in Figure 3. To highlight the dissimilarities among these diagrams, we restrict the domain of θ to a very small interval, $[\theta_m,\ \theta_M]$ with $\theta_m = \theta_1 - 0.145$ and $\theta_M = \theta_1 + 0.105$. The numerical results with three different initial points are presented in Figure 10(A) where the red curve is obtained with $y_0 = 0.1$, the blue with $y_0 = 1.5$ and the green with $y_0 = 5$. The green curve is plotted first, the blue one is then put on it and finally the red is over these two diagrams. The green curve seems to jump at $\theta = \theta_0 \simeq 2.18$ and so does the blue curve for $\theta = \tilde{\theta} \simeq 2.23$.[14] The followings are observed:

(i) For $\theta < \theta_0$, all the three curves are identical with the horizontal axis implying that all converge irrespective of the initial values;

(ii) For $\theta_0 \leq \theta < \tilde{\theta}$, the red and blue curves are still on the horizontal axis but the green curve bifurcates to two upper and lower branches, implying that a limit cycle emerges;

(iii) For $\tilde{\theta} < \theta \leq \theta_1$, the red curve is still on the horizontal axis while the blue curve bifurcates to two branches which are exactly on the green branches;

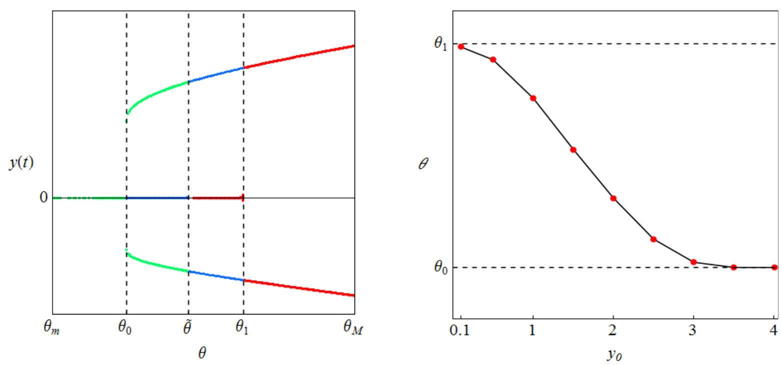
(iv) For $\theta > \theta_1$, the red curve bifurcates to two branches implying that all the three curves are identical again irrespective of the initial values.

The critical value $\tilde{\theta}$ depends on the initial value of y_0. This dependency is described by the negative-sloping curve in Figure 10(B). The nine dotted points in red are numerically obtained and the curve is constructed by connecting these points. It is seen that $\tilde{\theta}(y_0)$ gets closer to θ_0 as y_0 increases and to θ_1 as y_0 decreases to zero.[15]

Proposition 3 *If $\nu \leq \varepsilon$ and the initial function takes an initial point close to* 0 *at $t \leq 0$, then the zero solution is locally stable for $\theta < \tilde{\theta}$, its stability is lost at $\theta = \tilde{\theta}$ and then it bifurcates to a limit cycle with increasing amplitude as θ increases further.*

[14] It seems that the zero point jumps to a limit cycle just after it loses stability at $\theta = \theta_0$ or $\theta = \tilde{\theta}$. However, it does not jump but bifurcates to the cycle via the supercritical bifurcation in the very small (almost invisible) parameter interval of θ.

[15] It it also obtained that $\tilde{\theta}(y_0) \simeq 2.2664$ for $y_0 = 0.05$ and $\tilde{\theta}(y_0) \simeq 2.2668$ for $y_0 = 0.01$.



(A) Bifurcation diagrams (B) Dependency of $\tilde{\theta}$

Figure 10. Goodwinian cycle for $\nu < \varepsilon$

In the second simulation we increase the value of ν to $\nu_2 = 0.75$ and deal with the unstable case. Two different initial functions are selected and accordingly, two bifurcation diagrams are illustrated in Figure 11(A) in which the zero solution is locally unstable and no stability switch occurs. The diagram colored in blue has the initial point $y_0 = 0.1$ and the red one has $y_0 = 1.1$. The former is located on the latter. Notice that the two diagrams are different in the interval $[\theta_L,\ \theta_H]$ and identical otherwise. The extreme values of the interval have initial point dependency and the dependency is numerically confirmed in Figure 11(B) where the red curve is a locus of $\theta_L(y_0)$, the blue curve is a locus of $\theta_H(y_0)$ and the vertical dotted line stands at $\bar{y}_0 \simeq 0.1714$.[16] In Figure 11(A), since $y_0 = 0.1$ is assumed, the extreme values are calculated as $\theta_L \simeq 1.554$ and $\theta_H \simeq 2.254$. From the numerical results that are conducted under $\alpha = 0.8,\ \varepsilon = 0.5,\ \nu = 0.75$, we have the following three issues:

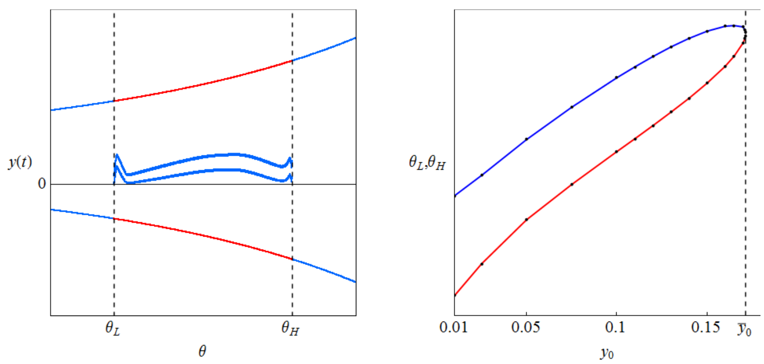
(i) A Goodwinian oscillation (i.e., red curve) coexists with a sawtooth oscillation (i.e., blue curve) for θ in the interval $[\theta_L,\ \theta_H]$ if $y_0 \leq \bar{y}_0$;

(ii) For $\theta < \theta_L$ or $\theta > \theta_H$, the two diagrams are identical, implying that only the Goodwinian cycle emerges.

(iii) Only Goodwinian cycle emerges if $y_0 > \bar{y}_0$.

Proposition 4 *If $\nu > \varepsilon$, then the delay model (13) generates a Goodwinian cycle unless the initial point is in a neighborhood of the locally unstable zero solution.*

[16] This value could depend on the specification of the model's parameters.



(A) Bifurcation diagrams (B) Interval $[\theta_L(y_0), \theta_H(y_0)]$

Figure 11. Goodwinian cycle for $\nu > \varepsilon$

4 Concluding Remarks

We have reconsidered the dynamics of Goodwin's multiplier-accelerator model with fixed delays. As a benchmark, the basic model that has no delays is investigated. It is shown that its steady state is locally and globally stable when the product of the marginal propensity to invest and the adjustment coefficient of the national income dynamic process is less than unity. It is also shown that a stable rapid-slow cycle with discontinuous jumps emerges when it is locally unstable. Following the spirit of Goodwin, we introduce the investment delay and obtain the following results. It is first confirmed that the local stability condition of the basic model still holds in the delay model. This means that the delay has no effects on the local behavior. However the delay definitely affects the global behavior. Second, the delay prolongs the convergence of the stable time trajectories and makes trajectories kinked. There is a threshold value of the delay, and the time trajectories are smooth (i.e., no sudden changes) for the delay less than this value and kinked for larger delay. Furthermore, the threshold value has an initial point dependency. Following the spirit of Hicks, we add the consumption delay and find that the time trajectory has an initial point dependency and the stable limit cycle can coexist with the convergent stable point when the steady state is locally stable. In the case of locally unstable steady state, the smooth trajectory can coexist with the sawtooth oscillations for relatively small values of the delay and only the smooth trajectory emerges for larger values.

From the numerical point of view, we can demonstrate that the delay matters in the motion of macroeconomic variables like the national income. However, from the analytical point of view, we still have some open questions. In particular, it is unclear why the sudden change of oscillations from the smooth motion to zigzag motion occurs when the investment delay is involved. It is also unclear why is there an interval in which the smooth and sawtooth oscillations coexist when investment and consumption delays are present. These are what we will address in a future study.

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