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Abstract

Delay has been considered as one of destabilizing factors in macroeconomic dynamics since the seminal work of Kalecki (1935). In this paper introducing two fixed delays into the traditional neoclassical growth model, we first rigorously determine the conditions for which the stability is lost and then numerically confirm the analytical results. We add one interesting feature of the delay dynamics. Stability loss and gain repeatedly occur as a delay parameter increases. This implies that the delay is not only a destabilizer but also a stabilizer.

Keywords: Two fixed delays, Stability crossing curve, Hopf bifurcation, Neoclassical growth model

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1 Introduction

For more than a half-century, the neoclassical growth model of Solow (1956) and Swan (1956) has been a prototype model for analyzing long-run economic growth. It has the general equilibrium structure although very simple and brings out how an economy can enjoy positive growth rates in a very clear way. On the other hand, business cycles have been often observed in a real economy, however, the neoclassical model fails to explain such dynamics since its steady state is locally asymptotically stable implying that its dynamics is monotonic. There are several turning points at which its basic structure is modified so as to give rise to cyclical fluctuations. Among others, Day (1982) incorporates the two opposite effects of increasing capital stock into the neoclassical model, the positive effect that is an essential source of economic growth and the negative effects caused by environmental distortion of high economic growth such as pollutions It is demonstrated in a discrete-time framework that persistent irregular fluctuations can be generated when nonlinearities due to the two effects get stronger. Since the seminal work of Kalecki (1935), it has been conjectured that a production delay could be a source of economic fluctuations. Zak (1999) rebuilds the model in which the current capital stock is adjusted by the savings at some preceding time and shows an emergence of a cycle via a Hopf bifurcation. Matsumoto and Szidarovszky (2011) reconsider Day's discrete time model in a continuous-time framework with production delay and numerically confirm the birth of chaotic dynamics through ala period-doubling cascade. Recently Bianca et al. (2013) extend analysis to the case in which the neoclassical model has two distinct delays, one refers to the time when capital is used for production and the other to the necessary time for the capital to be depreciated. By applying the normal theory and center manifold argument, they demonstrate the existence, the direction and stability of a Hopf bifurcation. However, their approach can be improved more as it does not allow to obtain analytical results on couples of two delays that generate a stable or an unstable stationary state. The aim of this study is to reconfirm and extend their results in more systematic way by applying the mathematical method developed by Gu et al. (2005) to deal with two delay models, through the use of the stability crossing curves, which are defined as the curves that separate the stable and unstable regions in the two delay plane.

The rest of the paper is organized as follows. Section 2 derives the complete form of the stability switching curve under two delays and determines the direction of crossing the imaginary axis. Section 3 confirms the analytical results numerically in our own example and reexamine two other examples provided by Bianca et at. (2013) in our way. Finally, concluding remarks are given in Section 4.

2 The model

We consider two fixed delays in the neoclassical growth model,

$$\dot{k}(t) = sf(k(t-\tau_1)) - \delta k(t-\tau_2), \tag{1}$$

where $s \in (0,1)$, $\delta \in (0,1]$ and $f(k)$ is well-behaved neoclassical production function implying that it is continuous, increasing, strictly concave, $f(0) = 0$ and satisfies Indada's conditions. An equilibrium point of Eq. (1) is a solution of $sf(k) - \delta k = 0$, where existence and number of equilibria depend on properties of function f. To examine stability of an equilibrium point of (1), say k_*, Eq. (1) is linearized at $k = k_*$,

$$\dot{k}(t) = s\alpha[k(t-\tau_1) - k_*] - \delta[k(t-\tau_2) - k_*], \tag{2}$$

where $\alpha = f'(k_*)$ and $s\alpha < \delta$ due to Inada's conditions. Looking for exponential solutions of (2), that is solutions of the form $k(t) = e^{\lambda t}u$ with u a constant, then, substituting it into (2), we obtain the corresponding characteristic equation

$$\lambda - s\alpha e^{-\lambda\tau_1} + \delta e^{-\lambda\tau_2} = 0. \tag{3}$$

In case of absence of delays, Eq. (3) becomes $\lambda = s\alpha - \delta < 0$. Thus, the equilibrium k_* of (1) is locally asymptotically stable. Let $\tau_1 > 0$ and $\tau_2 > 0$.[1] As τ_1 and τ_2 change, Eq. (1) can switch from stability to instability, or vice versa, only when at least one characteristic root moves to the imaginary axis. Thus, the stability analysis of (1) requires calculating the characteristic roots $\lambda = i\omega$ of the characteristic equation (3). To study the change of stability when τ_1 and τ_2 both vary, we will follow the methodology of Gu et al. (2005) with the use of the stability crossing curves, which are defined as the curves that separate the stable and unstable regions in the (τ_1, τ_2) plane. To apply this method, we rewrite Eq. (3) as

$$p(\lambda, \tau_1, \tau_2) = p_0(\lambda) + p_1(\lambda)e^{-\lambda\tau_1} + p_2(\lambda)e^{-\lambda\tau_2} = 0, \tag{4}$$

where

$$p_0(\lambda) = \lambda, \quad p_1(\lambda) = -s\alpha, \quad p_2(\lambda) = \delta.$$

Next we check the following assumptions on $p(\lambda, \tau_1, \tau_2)$ to exclude some obvious trivial cases:

I) $deg[p_0(\lambda)] \geq \max\{deg[p_1(\lambda)], deg[p_2(\lambda)]\}]$ (existence of a principal term).

II) $p_0(0) + p_1(0) + p_2(0) \neq 0$ ("0" is not a solution of (4) for any pair (τ_1, τ_2)).

III) The polynomials $p_0(\lambda), p_1(\lambda)$ and $p_2(\lambda)$ do not have common roots (in order to simplify the expressions).

IV) $\lim_{\lambda\to\infty} \left(\left|\frac{p_1(\lambda)}{p_0(\lambda)}\right| + \left|\frac{p_2(\lambda)}{p_0(\lambda)}\right| \right) < 1$ (restriction on difference operator).

Now, conditions I), II) and IV) hold since $deg[p_0(\lambda)] = 1$ and $deg[p_1(\lambda)]=deg[p_2(\lambda)]] = 0$, $p_0(0)+p_1(0)+p_2(0) = -s\alpha+\delta \neq 0$, and the limit as $\lambda \to \infty$ is equal to zero, respectively. Finally, condition III) is clearly satisfied.

A pair $(\tau_1, \tau_2) \in R_+^2$ is said to be a crossing point if $p(\lambda, \tau_1, \tau_2) = 0$ has at least one solution for λ on the imaginary axis. The set of all crossing points is known as the stability crossing set, and is denoted by T. An $\omega > 0$ is known as a crossing frequency if there exists at least one pair (τ_1, τ_2) such that $p(i\omega, \tau_1, \tau_2) = 0$. The set Ω of all crossing frequencies is called the crossing frequency set, i.e.,

$$\Omega = \{\omega > 0 : p(i\omega, \tau_1, \tau_2) = 0 \text{ for some } (\tau_1, \tau_2) \in R_+^2\}$$

Considering that $p_0(\lambda)$ has no nonzero roots on the imaginary axis, the stability analysis of (4) can be reduced to the analysis of the equation

$$a(\lambda, \tau_1, \tau_2) = 1 + a_1(\lambda)e^{-\lambda\tau_1} + a_2(\lambda)e^{-\lambda\tau_2} = 0, \tag{5}$$

where

$$a_1(\lambda) = \frac{p_1(\lambda)}{p_0(\lambda)} = -\frac{s\alpha}{\lambda}, \quad a_2(\lambda) = \frac{p_2(\lambda)}{p_0(\lambda)} = \frac{\delta}{\lambda}.$$

[1] See Zak (1999) for the model with $\tau_1 = \tau_2 = \tau$.

The form of Eq. (5) allows to replace the investigation on crossing the imaginary axis into the geometric problem of a triangle. Specifically, for each given $\lambda = i\omega$, $\omega > 0$, the term $a(\lambda, \tau_1, \tau_2)$ is represented in the complex plane as the sum of three vectors 1, $a_1(\lambda)e^{-\lambda\tau_1}$ and $a_2(\lambda)e^{-\lambda\tau_2}$, with magnitudes 1, $|a_1(i\omega)|$ and $|a_2(i\omega)|$, respectively, which are independent of τ_1 and τ_2. If these vectors create a triangle (i.e., $a(\lambda, \tau_1, \tau_2) = 0$), then the characteristic equation has a solution $\lambda = i\omega$ for some delays τ_1 and τ_2. Since the length of each edge of a triangle cannot exceed the sum of the length of the remaining two edges, we derive that the range of ω to parameterize T are the solution of the following three inequalities:

$$|a_1(i\omega)| + |a_2(i\omega)| \geq 1, \quad -1 \leq |a_1(i\omega)| - |a_2(i\omega)| \leq 1. \tag{6}$$

Since

$$|a_1(i\omega)| = \left|-\frac{s\alpha}{i\omega}\right| = \frac{s\alpha}{\omega} \text{ and } |a_2(i\omega)| = \left|\frac{\delta}{i\omega}\right| = \frac{\delta}{\omega},$$

the triangle conditions (6) give

$$s\alpha + \delta \geq \omega, \quad -\omega \leq s\alpha - \delta \leq \omega.$$

Recalling $s\alpha - \delta < 0$, we obtain

$$-s\alpha + \delta \leq \omega \leq s\alpha + \delta. \tag{7}$$

As a result, we get

$$\Omega = [-s\alpha + \delta, s\alpha + \delta].$$

For any $\omega \in \Omega$, the characteristic equation (2) has a pair of purely imaginary roots and it is now possible to identify solutions (τ_1, τ_2) of $p(\lambda, \tau_1, \tau_2) = 0$ as the following two sets of curves in the first quadrant of the (τ_1, τ_2)-region:

$$C^+(m,n) : \begin{cases} \tau^+_{1,m} = \tau^+_{1,m}(\omega) = \dfrac{\arg[a_1(i\omega)] + (2m-1)\pi + \theta_1(\omega)}{\omega} \\[2ex] \tau^+_{2,n} = \tau^+_{2,n}(\omega) = \dfrac{\arg[a_2(i\omega)] + (2n-1)\pi - \theta_2(\omega)}{\omega} \end{cases} \tag{8}$$

and

$$C^-(m,n) : \begin{cases} \tau^-_{1,m} = \tau^-_{1,m}(\omega) = \dfrac{\arg[a_1(i\omega)] + (2m-1)\pi - \theta_1(\omega)}{\omega} \\[2ex] \tau^-_{2,n} = \tau^-_{2,n}(\omega) = \dfrac{\arg[a_2(i\omega)] + (2n-1)\pi + \theta_2(\omega)}{\omega} \end{cases} \tag{9}$$

for $m = m_0^{\pm}, m_0^{\pm}+1, m_0^{\pm}+2, \ldots$, and $n = n_0^{\pm}, n_0^{\pm}+1, n_0^{\pm}+2, \ldots$, where $n_0^+, n_0^-, m_0^+, m_0^-$ ($n_0^+ \leq n_0^-$ and $m_0^+ \geq m_0^-$) are the smallest possible integers such that the corresponding $\tau_1^{n_0^+ +}, \tau_1^{n_0^- -}, \tau_2^{m_0^+ +}, \tau_2^{m_0^- -}$ values are nonnegative.

In the expressions (8) and (9), the terms $\arg[a_1(i\omega)]$ and $\arg[a_2(i\omega)]$ denote the argument of $a_1(i\omega)$ and $a_2(i\omega)$, respectively, and are given by

$$\arg[a_1(i\omega)] = \arg\left[-\frac{s\alpha}{i\omega}\right] = \arg\left[\frac{s\alpha}{\omega}i\right] = \frac{\pi}{2}$$

and

$$\arg[a_2(i\omega)] = \arg\left[\frac{\delta}{i\omega}\right] = \arg\left[-\frac{\delta}{\omega}i\right] = \frac{3\pi}{2},$$

while $\theta_1, \theta_2 \in [0, \pi]$ represent the internal angles of the triangle, and are determined by the law of cosines as follows,

$$\theta_1(\omega) = \cos^{-1}\left(\frac{1 + |a_1(i\omega)|^2 - |a_2(i\omega)|^2}{2\,|a_1(i\omega)|}\right) = \cos^{-1}\left(\frac{\omega^2 + s^2\alpha^2 - \delta^2}{2s\alpha\omega}\right)$$

and

$$\theta_2(\omega) = \cos^{-1}\left(\frac{1 + |a_2(i\omega)|^2 - |a_1(i\omega)|^2}{2\,|a_2(i\omega)|}\right) = \cos^{-1}\left(\frac{\omega^2 - s^2\alpha^2 + \delta^2}{2\delta\omega}\right).$$

The inequalities in (7) yield that the arccosines functions are well-defined being

$$-1 \le \frac{\omega^2 + s^2\alpha^2 - \delta^2}{2s\alpha\omega} \le 1 \text{ and } -1 \le \frac{\omega^2 - s^2\alpha^2 + \delta^2}{2\delta\omega} \le 1.$$

In conclusion, Eqs. (8) and (9) are given by

$$C^+(m,n): \begin{cases} \tau^+_{1,m}(\omega) = \dfrac{1}{\omega}\left[-\dfrac{\pi}{2} + 2m\pi + \cos^{-1}(A)\right] \\ \tau^+_{2,n}(\omega) = \dfrac{1}{\omega}\left[\dfrac{\pi}{2} + 2n\pi - \cos^{-1}(B)\right] \end{cases} \tag{10}$$

and

$$C^-(m,n): \begin{cases} \tau^-_{1,m}(\omega) = \dfrac{1}{\omega}\left[-\dfrac{\pi}{2} + 2m\pi - \cos^{-1}(A)\right] \\ \tau^-_{2,n}(\omega) = \dfrac{1}{\omega}\left[\dfrac{\pi}{2} + 2n\pi + \cos^{-1}(B)\right], \end{cases} \tag{11}$$

where

$$A = A(\omega) = \frac{\omega^2 + s^2\alpha^2 - \delta^2}{2s\alpha\omega}, \quad B = B(\omega) = \frac{\omega^2 - s^2\alpha^2 + \delta^2}{2\delta\omega} > 0.$$

Furthermore, noticing that $\cos^{-1}(A), \cos^{-1}(B) \in [0, \pi]$, one has $\omega\tau_1^+, \omega\tau_2^+ \in [-\pi/2, \pi/2]$ with $m = n = 0$.

Theorem 1 *Let n be fixed. Then the segments of $C^+(m,n)$ and $C^-(m,n)$ form a continuous curve as m increases.*

Proof. In order to understand the possible configurations of stability crossing curves of our model, we analyse the behavior of $C^+(m,n)$ and $C^-(m,n)$ at the initial and end points of Ω. Since

$$\theta_1(-s\alpha + \delta) = \cos^{-1}(-1) = \pi, \quad \theta_2(-s\alpha + \delta) = \cos^{-1}(1) = 0$$

and

$$\theta_1(s\alpha + \delta) = \cos^{-1}(1) = 0, \quad \theta_2(s\alpha + \delta) = \cos^{-1}(1) = 0,$$

(10) gives that the initial and end points of $C^+(m,n)$ are

$$I^+(m,n) = \left(\frac{1}{-s\alpha + \delta}\left(\frac{3\pi}{2} + (2m-1)\pi\right), \frac{1}{-s\alpha + \delta}\left(\frac{3\pi}{2} + (2n-1)\pi\right)\right)$$

and

$$E^{+}(m,n) = \left(\frac{1}{s\alpha+\delta} \left(\frac{\pi}{2} + (2m-1)\pi \right), \frac{1}{s\alpha+\delta} \left(\frac{3\pi}{2} + (2n-1)\pi \right) \right).$$

As well, it follows from (11) that the initial and end points of $C^{-}(m,n)$ are

$$I^{-}(m,n) = \left(\frac{1}{-s\alpha+\delta} \left(\frac{\pi}{2} + (2m-1)\pi - \pi \right), \frac{1}{-s\alpha+\delta} \left(\frac{3\pi}{2} + (2n-1)\pi \right) \right)$$

and

$$E^{-}(m,n) = \left(\frac{1}{s\alpha+\delta} \left(\frac{\pi}{2} + (2m-1)\pi \right), \frac{1}{s\alpha+\delta} \left(\frac{3\pi}{2} + (2n-1)\pi \right) \right).$$

Since $I^{+}(m,n) = I^{-}(m+1,n)$ and $E^{+}(m,n) = E^{-}(m,n)$, we arrive at the conclusion that the curves $C^{+}(m,n)$ and $C^{-}(m,n)$ are connected at the endpoint of Ω, while $C^{+}(m,n)$ and $C^{-}(m+1,n)$ are connected at the initial point of Ω. ■

After having determined the stability crossing curves corresponding to Ω, we will discuss the direction in which the solutions of (5) cross the imaginary axis as (τ_1, τ_2) deviates from a curve in T and find the directions of crossing as one moves along the curve. Before doing this, we examine the case with $\tau_1 = 0$ in (5).

Proposition 1 *Let $\tau_1 = 0$. Then there exists $\tau_2^0 > 0$ such that the equilibrium k_* of* (1) *is locally asymptotically stable when $\tau_2 \in [0, \tau_2^0)$ and unstable when $\tau_2 > \tau_2^0$. Furthermore, Eq.* (1) *undergoes a Hopf bifurcation at k_* when $\tau_2 = \tau_2^0$.*

Proof. To determine the stability of the system, we need first to find the roots $\lambda = i\omega$, $\omega > 0$ of the characteristic equation when $\tau_1 = 0$. Letting $\lambda = i\omega$, $\omega > 0$, in $1 + a_1(\lambda) + a_2(\lambda)e^{-\lambda\tau_2} = 0$, and separating real and imaginary parts leads to

$$\omega = \delta \sin \omega\tau_2, \qquad s\alpha = \delta \cos \omega\tau_2. \tag{12}$$

Adding the squares of these equations implies ω to be a root of $\omega^2 = \delta^2 - s^2\alpha^2$. Thus, we get the unique positive root

$$\omega_0 = \sqrt{\delta^2 - s^2\alpha^2}.$$

Hence, we obtain from (12) the critical value,

$$\tau_2^0 = \frac{1}{\omega_0} \tan^{-1} \left(\frac{\omega_0}{s\alpha} \right).$$

Selecting τ_2 as the bifurcation parameter**,** we consider λ as function of τ_2**:** $\lambda = \lambda(\tau_2)$**.** Differentiating the characteristic equation with respect to τ_2, we obtain

$$\left[1 - (s\alpha - \lambda)\tau_2\right] \frac{d\lambda}{d\tau_2} = \lambda (s\alpha - \lambda).$$

It is immediate to check that $i\omega_0$ is a simple root. In fact, if it were a repeated root, then $\lambda = i\omega$ would satisfy both equations

$$\lambda - s\alpha + \delta e^{-\lambda\tau_2} = 0 \text{ and } 1 + \delta e^{-\lambda\tau_2}(-\tau_2) = 0$$

implying that $0 = 1 - \tau_2(s\alpha - \lambda)$, which is impossible if λ is purely imaginary.

Next, we have

$$sign\left[\frac{d\,(\text{Re}\lambda)}{d\tau_2}\right]_{\lambda=i\omega_0,\ \tau_2=\tau_2^0} = sign\left[\text{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\lambda=i\omega_0,\ \tau_2=\tau_2^0}\right] = sign\left[\frac{1}{\omega_0^2+s^2\alpha^2}\right] > 0,$$

which completes the proof. ■

For discussing the direction of crossing, we need some definitions. We call the direction of the stability switch curve with increasing ω the *positive direction*, while the region on the left hand side, as we head in the positive direction of the curve, is called the *region on the left*, denoted by **L**. The region on the right hand side is called the *region on the right*, denoted as **R**.

Gu et al. (2005) proved (see Proposition 6.1) that if $\lambda = i\omega$ is a simple solution of Eq. (5), as (τ_1, τ_2) moves from **R** to **L** of the corresponding curve in $\mathcal{T}$, then a pair of solutions of (5) cross the imaginary axis to the right if $Q > 0$, where

$$Q = \text{Im}\left[a_1(i\omega)a_2(-i\omega)e^{i\omega(\tau_2-\tau_1)}\right], \tag{13}$$

and the crossing is in the opposite direction if $Q < 0$.

Theorem 2 *As (τ_1, τ_2) moves from $\boldsymbol{R}$ to $\boldsymbol{L}$, we have*

1) *a stability loss if $\omega < \delta - s\alpha$ on $C^+(m,n)$ or if $\omega > \delta - s\alpha$ on $C^-(m,n)$;*

2) *a stability gain if $\omega > \delta - s\alpha$ on $C^+(m,n)$ or if $\omega < \delta - s\alpha$ on $C^-(m,n)$.*

Proof. In the first part of the proof we show by contradiction that the root $\lambda = i\omega$ is simple. Suppose $\lambda = i\omega$ is a root of (5) which is repeated. Then, the derivative of (5) with respect to λ evaluated at $\lambda = i\omega$ must also be zero, and we have the following two equations

$$\begin{cases} i\omega - s\alpha e^{-i\omega\tau_1} + \delta e^{-i\omega\tau_2} = 0, \\ 1 + s\alpha\tau_1 e^{-i\omega\tau_1} - \delta\tau_2 e^{-i\omega\tau_2} = 0. \end{cases} \tag{14}$$

From (14), we get

$$e^{-i\omega\tau_1} = \frac{1+i\omega\tau_2}{s\alpha(\tau_2-\tau_1)}, \qquad e^{-i\omega\tau_2} = \frac{1+i\omega\tau_1}{\delta(\tau_2-\tau_1)}. \tag{15}$$

Separating real and imaginary parts in (15), and then comparing, we obtain

$$\sin\omega\tau_1 = -\omega\tau_2\cos\omega\tau_1, \qquad \sin\omega\tau_2 = -\omega\tau_1\cos\omega\tau_2.$$

Consequently,

$$\tan\omega\tau_1 = -\omega\tau_2, \qquad \tan\omega\tau_2 = -\omega\tau_1. \tag{16}$$

If we are on $C^+(m,n)$, we notice from (10) that, being $\cos^{-1}(A), \cos^{-1}(B) \in [0,\pi]$, one has that with $m = n = 0$, $\omega\tau_1^+, \omega\tau_2^+ \in [-\pi/2, \pi/2]$. Using (16), and recalling that the tangent function is an odd function, we arrive at the identity

$$\tan\omega\tau_1^+ = \tan^{-1}\omega\tau_1^+.$$

On the other hand, a graphical inspection shows that this two functions do not have zero intersection when $\omega\tau_1^+ \in (-\pi/2, \pi/2)$. The proof when we are on $C^-(m,n)$ is similar. In conclusion, we have shown that the root $\lambda = i\omega$ is simple. In this second part of the proof we find the conditions for $Q > 0$ (stability loss) and for $Q < 0$ (stability gain). From (13),we have

$$Q = \mathrm{Im}\left\{-\frac{s\alpha\delta}{\omega^2}\left[\cos\omega(\tau_2 - \tau_1) + i\sin\omega(\tau_2 - \tau_1)\right]\right\} = -\frac{s\alpha\delta}{\omega^2}\sin\omega(\tau_2^+ - \tau_1^+). \tag{17}$$

On $C^+(m,n)$, it is

$$\omega(\tau_2^+ - \tau_1^+) = \left[2(n-m)+1\right]\pi - \left[\cos^{-1}(A) + \cos^{-1}(B)\right]. \tag{18}$$

Therefore, from (17) and (18), using the angle-sum and angle-difference identities for trigonometric functions, i.e. $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$ and $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$, it follows

$$Q = -\frac{s\alpha\delta}{\omega^2}\sin\left[\cos^{-1}(A) + \cos^{-1}(B)\right] = \frac{s\alpha\delta}{\omega^2}\left\{-B\sqrt{1-A^2} - A\sqrt{1-B^2}\right\}.$$

On $C^+(m,n)$, one has

$$sign\,(Q) = sign\left(-B\sqrt{1-A^2} - A\sqrt{1-B^2}\right).$$

Since $B > 0$, and so $-B < 0$, we see that $Q < 0$ if $A \geq 0$, i.e. if $\omega \geq \sqrt{\delta^2 - s^2\alpha^2}$. Let $A < 0$, i.e. $\omega < \sqrt{\delta^2 - s^2\alpha^2}$. Then $Q < 0$ if $B > -A$, i.e. if $\omega > \delta - s\alpha$. Since $\sqrt{\delta^2 - s^2\alpha^2} > \delta - s\alpha$, we have that $Q < 0$ if $\omega > \delta - s\alpha$. As well, we have that $Q > 0$ for $\omega < \delta - s\alpha$.

Similarly, on $C^-(m,n)$, one has

$$\omega(\tau_2^- - \tau_1^-) = \left[2(n-m)+1\right]\pi + \left[\cos^{-1}(A) + \cos^{-1}(B)\right],$$

yielding

$$Q = \frac{s\alpha\delta}{\omega^2}\sin\left[\cos^{-1}(A) + \cos^{-1}(B)\right] = \frac{s\alpha\delta}{\omega^2}\left\{B\sqrt{1-A^2} + A\sqrt{1-B^2}\right\}.$$

Hence,

$$sign\,(Q) = sign\left(B\sqrt{1-A^2} + A\sqrt{1-B^2}\right) \text{ on } C^-(m,n).$$

Proceeding as before, we derive $Q < 0$ if $\omega < \delta - s\alpha$ and $Q > 0$ if $\omega > \delta - s\alpha$. ∎

3 Numerical Simulations

We numerically confirm the analytical results obtained above by assuming the Cobb-Douglas production function

$$f(k) = Ak^\beta$$

with $\beta \in (0,1)$ and $A = 1$. Since the equilibrium per capita k_* solves $s(k_*)^\beta = \delta k_*$, the marginal product at the equilibrium point satisfies the following relation,

$$\beta\,(k_*)^{\beta-1} = \frac{\beta\delta}{s}(=\alpha)$$

implying that $s\alpha = \beta\delta$ at the equilibrium point.

In the first numerical example, we specify the parameter values as follows:

$$s = 0.3,\ \beta = 0.5 \text{ and } \delta = 0.1.$$

Figure 1(A) shows the stability switching curve in which the red segments are described by $C^{-}(m,0)$ for $m = 1,2,3,4$ and the left most blue segment by $C^{+}(0,0)$ and the other blue segments by $C^{+}(m,0)$ for $m = 1,2,3$. Notice that m is a horizontal-shift parameter and n is a vertical-shift parameter. The curve divides the non-negative (τ_1,τ_2) plane into two regions. The equilibrium point is stable in the region including the origin[2] and unstable in the other region. We immediately observe the two results that Bianca et al. (2013) have already shown in their Proposition 1 and the first half of Theorem 3,

(1) the equilibrium point is always stable for $\tau_2 = 0$ because it is locally asymptotically stable for $\tau_1 = 0$ and no stability switch occurs for any $\tau_1 > 0$ as the stability switching curve does not intersect the horizontal axis.

(2) for $\tau_1 = 0$, the equilibrium point is locally asymptotically stable for $\tau_2 < \tau_2^0$, loses stability at $\tau_2 = \tau_2^0$ and unstable for $\tau_2 > \tau_2^0$ where $\tau_2^0 \simeq 12.09$ is the intersection of the most left blue curve with the vertical axis as shown in Figure 1(A). Our Proposition 1 formally proves this observation.

We can observe further results. First notice that since the blue curves take U-shaped form, it has a minimum value denoted by τ_2^m, which is obtained by differentiating $\tau_{2,m}(\omega)$ with respect to ω and solving the resultant expression being equal to zero.

(3) for $\tau_2 < \tau_2^m$, any $\tau_1 \geq 0$ is *harmless* implying that the equilibrium point is locally asymptotically stable where $\tau_2^m \simeq 9.35$.

We now turn attention to Figure 1(B) that is an enlargement of the shaded rectangular region in Figure 1(A). According to Theorem 1, both $C^{+}(0,0)$ and $C^{-}(1,0)$ start but in the opposite direction at the green point where $I^{+}(0,0) = I^{-}(1,0)$ holds and both $C^{+}(1,0)$ and $C^{-}(1,0)$ finally arrive at the yellow point where $E^{+}(1,0) = E^{-}(1,0)$ holds. As shown by arrows, the point of (τ_1,τ_2) on the red curve moves from the green point to the yellow point as the value of ω increases. By the same token, the point on the lower blue curve moves forward to the yellow point and the point on the upper blue curve moves away from the yellow point as the value of ω increases. As seen in Figure 1(B), the vertical line standing at $\bar{\tau}_1 = (\tau_1^m + \tau_1^M)/2$ crosses the stability switching curve three time at points a, b and c. The τ_2-value of point a is calculated as follows. Solving $\tau_{1,1}^{+}(\omega) = \bar{\tau}_1$ gives $\omega_a \simeq 0.145$ that is, in turn, substituted into $\tau_{2,0}^{-}(\omega)$ to determine $\tau_2^a \simeq 9.56$. In the same way, we have $\omega_b \simeq 0.092$ and $\tau_2^b \simeq 22.63$ at point b and $\omega_c \simeq 0.051$ and $\tau_2^c \simeq 33.14$ at point c. At each point, the stability switch occurs according to Theorem 2:

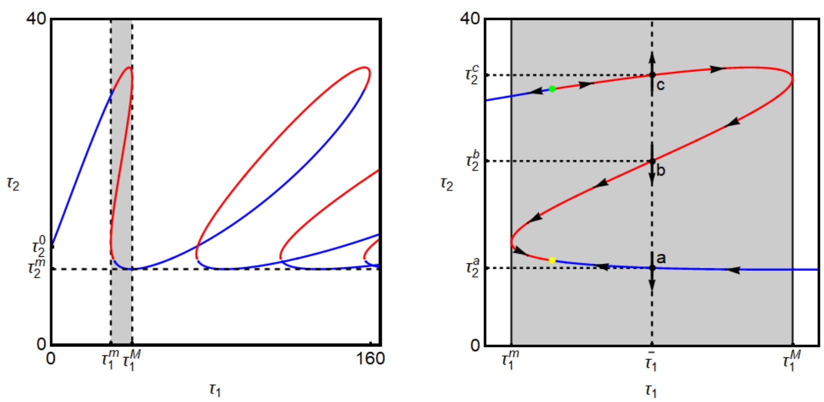
(4-a) Since the lower blue curve is described by $C^{+}(1,0)$ and passes through point a from right to left in the positive direction, the **R**-region is above the curve and the **L**-region is below. The specified values of the parameters lead to $\delta - s\alpha = \delta - \beta\delta = 0.05 < \omega_a$, the value of ω at point a. Theorem 2 (1) implies that as (τ_1,τ_2) moves downward from **R** to **L** along the vertical line passing though point a, the stability is gained.

[2] We have already confirmed that k_* is stable in case of absence of delays (i.e., $\tau_1 = \tau_2 = 0$).

(4-b) The red curve is described by $C^-(1,0)$. With the positive direction in the neighborhood of point b, the pair of the delay moves right to left along the downward part of the red curve as the arrows exhibit. The **R**-region is above the curve and the **L**-region is below. Hence, as in the same way as above, Theorem 2 (1) indicates that the stability is lost when the pair of the delays moves downward from **R** to **L** along the vertical line passing through point b at which $\omega_b > \delta - s\alpha$ on $C^-(1,0)$.

(4-c) The positive direction is reversed in the neighborhood of point c so that the **R**-region is below the positive sloping part of the red curve and the **L**-region is above. Theorem 2 (1) also indicates that the stability is lost when the pair of the delay moves upward from **R** to **L** along the vertical line passing thorough point c at which $\omega_c > \delta - s\alpha$ on $C^-(1,0)$.

The last results are summarized as follows: when the value of τ_2 increases along the vertical line at $\tau_1 \in (\tau_1^m, \tau_1^M)$, the switch from stability to instability occurs at the first intersection, the stability is regained at the second intersection and the stability is lost again at the third intersection. No stability switch occurs for further increase of τ_2.



(A) Stability switching curve (B) Enlargement

Figure 1. Stability switching curve

We now visit the numerical examples considered by Bianca et al. (2013) and confirm their results in our way. We examine their first and second examples together that have the following parameter values

$$s = 0.11,\ \beta = 0.1 \text{ and } \delta = 0.8$$

and two sets of the delay combination (τ_1, τ_2),

$$A = (1, 2) \text{ and } B = (3, 2).$$

The stability switching curve is illustrated in Figure 2(A) in which the blue and red curves are described by $C^+(m,0)$ and $C^-(m,0)$ for $m = 1, 2$. Let us examine dynamics at point A. For

$\tau_1 = 1$, we can calculate the corresponding value of τ_2 to be on the stability switching curve by two steps. At the first step we solve $\tau^+_{1,0}(\omega) = 1$ to obtain $\omega_1 \simeq 0.744$. At the second step this ω_1 is substituted into $\tau^-_{1,0}(\omega_1) \simeq 2.013$. This threshold value is slightly larger than 2. Thus time trajectories under the A-specification converges to the equilibrium point as shown by Bianca et al. We now move to point B. Although we would not calculate the threshold value of τ_2 on the stability switching curve under the B-specification, it is apparent that point B is located under the stability switching curve and thus the equilibrium is also locally asymptotically stable under the B-specification. We have two more results. One is that τ_1 becomes harmless when $\tau_2 < \tau_2^m \simeq 1.751$ and the other is that stability loss and gain can occur repeatedly as the stability switching curve is wave-shaped. In particular, when the value of τ_1 is increased along the horizontal line at $\tau_2 = 2$, the stability is gained whenever the horizontal line crosses the positive sloping blue curve whereas the stability is lost whenever the horizontal line crosses the negative sloping red curve.

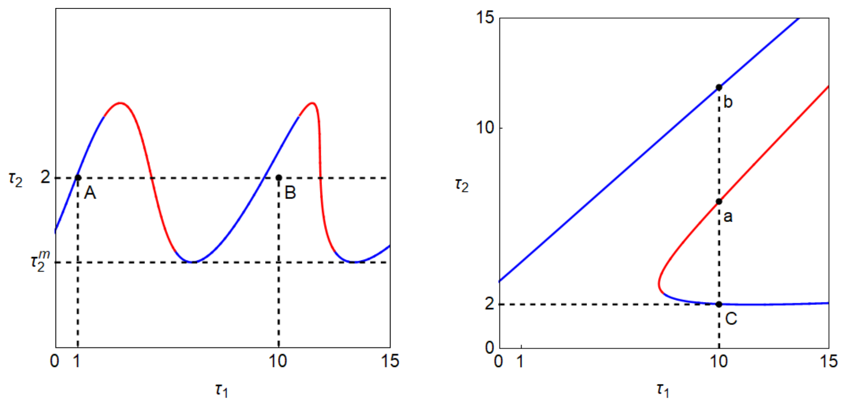
In their third example, the parameter values are changed to

$$s = 0.41,\ \alpha = 0.8 \text{ and } \delta = 0.35$$

and the combination of delays is selected

$$C = (10, 2)$$

The corresponding stability switching curves are illustrated by the blue curve and blue-red curve. For $\tau_1 = 10$, the corresponding τ_2-value of $C^+(1, 0)$ is approximately 2.018, slightly larger than 2. Therefore point C is actually located in the stable region below the stability switching curve. In consequence, although it takes much longer time to arrive at the equilibrium point, as point C is very close to the switching curve, the equilibrium point is locally asymptotically stable, oscillations can occurs for a long time. In this example, the multiple stability loss and gain can occur if τ_1 is fixed and τ_2 increases as indicated by the multiple intersection of the dotted vertical line at $\tau_1 = 10$ with the stability switching curve. In fact, if we increase the value of τ_2 along the dotted vertical line at $\tau_1 = 10$, we have three intersections, stability is lost at the first intersection at $\tau_2 \simeq 2.018$, regained at crossing point a with the red curve and finally lost again at crossing point b with the upper blue curve.



(A) (B)

Figure 2. Examples of Bianca et al. (2013)

4 Concluding Remarks

Stability of the traditional neoclassical model of Solow and Swan were examined under the assumption that there were two distinct delays, one is the production delay and the other is the depreciation delay. Confirming that the neoclassical growth model is locally asymptotically stable without time delays, we demonstrate that it could generate qualitatively different dynamics once the delays were introduced:

(1) a delay can be harmless meaning that the delays do not affect dynamics at all.

(2) confirmation of the instability effect of the delay in the neoclassical growth model.

(3) In the two delay case, we can have the repetition of stability loss and gain, implying that the delay not only stabilizes macrodynamics but also can destabilize it.

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