

Discussion Paper No. 431

Wicksellian Cumulative Dynamics
under Interest-Rate and Quantity Closures

Ichiro Takahashi

October 2026



INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

Wicksellian Cumulative Dynamics under Interest-Rate and Quantity Closures

Ichiro Takahashi

September 30, 2026

Abstract

This paper examines how alternative monetary closures affect the stochastic stability of a Keynesian economy with decentralized investment adjustment. Under a quantity closure, movements in the capital stock induce endogenous changes in the market interest rate, which feed back into profitability and investment and provide a restoring force around the neutral capital stock. Under a pure interest-rate peg, this mechanism disappears: the money stock adjusts endogenously while the interest rate remains fixed, leaving the economy without endogenous restoring drift except at the knife-edge neutral rate. The resulting dynamics provide a stochastic formulation of the Wicksellian cumulative process. We then show that a state-dependent interest-rate rule can reconstruct the missing stabilizing feedback. For a corridor rule that adjusts the policy rate outside an inactive region around the neutral capital stock, we derive a critical feedback strength above which the opposing expected log-growth conditions at the two boundaries ensure stochastic stability. Restoring stochastic stability, however, does not make the two monetary closures equivalent. Under the baseline calibration, the stationary root-mean-square deviation of capital from its neutral level is 2.32 under the quantity closure, compared with 3.15 even under the strongest corridor feedback considered. The comparison indicates that a policy rule can reconstruct sufficient negative feedback to restore ergodicity without reproducing the tighter stationary concentration generated by continuous endogenous interest-rate adjustment. The analysis thereby links the choice of monetary closure not only to the existence of a stationary stochastic regime but also to its quantitative properties, and provides a microeconomic interpretation of stabilizing interest-rate feedback as a substitute for the endogenous adjustment mechanism removed by an interest-rate closure.

Keywords: Wicksellian cumulative process; monetary policy rules; interest-rate peg; endogenous money; stochastic stability; ergodicity.

1 Introduction

Monetary closure determines not only how nominal variables are determined but also whether stochastic macroeconomic dynamics possess an endogenous restoring force or exhibit cumulative divergence. This paper studies that distinction by comparing two monetary closures within the same decentralized investment economy: a quantity closure, under which the nominal money stock M is predetermined and the market interest rate r adjusts endogenously, and an interest-rate closure, under which the policy rate is predetermined and the money stock adjusts endogenously to accommodate money demand. The comparison raises a simple question with important dynamic implications: what becomes of the economy's internal stabilizing mechanism when r , rather than M , is the monetary-policy anchor?

The two closures have familiar antecedents in monetary theory. A predetermined nominal quantity of money belongs to the quantity-theoretic tradition associated with Fisher (1935), whereas the interest-rate closure naturally recalls Wicksell's analysis of the cumulative process (Wicksell, 1936). The connection drawn here is not an identification of the present quantity closure with Fisher's quantity theory, nor of the policy regime considered below with Wicksell's original monetary system. Rather, the distinction provides a useful organizing principle. Under the quantity closure, adjustment of the market interest rate can provide endogenous feedback when the state of the economy changes. Under an interest-rate peg, by contrast, the monetary authority supplies money elastically at the policy rate, so that $M^S = M^d$ in equilibrium, while the interest rate itself ceases to perform this adjusting role. The resulting contrast allows the stability implications of the monetary closure to be isolated.

To make this comparison transparent, the analysis deliberately considers a stationary macroeconomic environment. Technology and resource endowments, including labor, are fixed, and government and foreign sectors are abstracted from. There is therefore no exogenous trend in productive capacity and no systematic fiscal, external, or technological force shifting the economy over time. This allows stability or instability to be attributed to the endogenous adjustment mechanism associated with the monetary closure rather than to changes in the underlying macroeconomic environment.

The starting point is the quantity-closure economy analyzed in Takahashi (2026). With the nominal money stock predetermined, movements in the capital stock change the market-clearing interest rate. The resulting interest-rate movements affect firms' expected profitability and hence their probabilistic investment decisions. Around the neutral capital stock, this mechanism reverses the direction of drift: sufficiently low capital raises profitability and favors expansion, whereas sufficiently high capital lowers profitability and favors contraction. The quantity closure thus supplies an endogenous negative-feedback mechanism capable of supporting a stationary stochastic distribution.

Following Takahashi (2026), investment incentives are summarized by the profitability index ζ , which measures the proportional excess of desired over installed

capital, normalized by the depreciation rate; a higher ζ therefore indicates stronger incentives for capital expansion. The endogenous interest-rate response dampens investment incentives and contributes to macroeconomic stability. The benchmark mechanism is

$$K \uparrow \implies M^d \uparrow \implies r \uparrow \implies \zeta \downarrow \implies \text{restoring drift.}$$

The first question of the present paper is what happens when this monetary closure is replaced by a pure interest-rate peg while the underlying investment mechanism is left unchanged. Under the peg, the market rate is aligned with the policy rate, $r_t = \bar{r}$, and the money stock adjusts endogenously to satisfy money demand. The interest rate therefore no longer responds to movements in the capital stock. Since profitability and the probabilities governing investment adjustment— p_+ for an upward adjustment and p_- for a downward adjustment—depend on the interest rate, fixing $\bar{r}$ also fixes the direction of the resulting drift. The relevant causal chain becomes

$$r_t = \bar{r} \implies M^S = M^d \implies \zeta = \zeta(\bar{r}) \implies p_+, p_- \text{ independent of } K.$$

Consequently, under a pure interest-rate peg, the stabilizing force generated by endogenous interest-rate movements disappears. Except at the knife-edge neutral rate r^* , there is no endogenous tendency for the drift to reverse as the capital stock moves away from its neutral level. The Wicksellian gap $r^* - \bar{r}$ instead determines the direction in which economic activity tends to cumulate.

This observation leads to the second question: if an interest-rate peg removes the endogenous restoring mechanism, can a state-dependent policy-rate rule reconstruct it and thereby generate ergodic dynamics? The paper studies a simple corridor rule centered on the neutral configuration. Within an inactive region, the policy rate remains at its neutral value r^* . When the capital stock crosses the lower or upper boundary, the authority switches to the lower policy rate r_L or the higher policy rate r_H , respectively. The rule therefore introduces deliberately the state dependence that is absent under a pure peg: policy raises profitability when capital is sufficiently low and reduces it when capital is sufficiently high.

A central parameter of the analysis is the feedback-strength parameter Λ . It determines how far the intervention rates $r_L(\Lambda)$ and $r_H(\Lambda)$ lie from the neutral rate r^* and, through the equilibrium mapping between the interest rate and capital, determines the associated intervention thresholds $K_L(\Lambda)$ and $K_H(\Lambda)$. Hence the interest-rate width $r_H - r_L$ and the inactive capital-stock width $K_H - K_L$ are jointly determined rather than independent policy choices. This gives rise to a policy trade-off. Relatively weak rate adjustments are associated with a narrower inactive region and therefore more frequent intervention, whereas stronger adjustments permit a wider region of inaction but imply larger policy-rate movements when intervention occurs. The relevant issue is not merely whether feedback is present, but how strong it must be to restore the opposing forces required for stochastic stability.

The analysis establishes a critical feedback-strength result. Increasing Λ strengthens the expected upward force at the lower boundary and the expected downward

force at the upper boundary. Beyond a critical value, the corresponding expected log-growth conditions have the signs required for stochastic stability. Thus an interest-rate closure need not be intrinsically destabilizing: what is lost when the interest rate becomes the monetary anchor can, under suitable conditions, be reconstructed through systematic state-contingent policy feedback. The distinction is between a pure peg, which suppresses the endogenous rate response, and a feedback rule, which deliberately restores state dependence to the policy rate.

A third question then arises. If both the quantity closure and the feedback interest-rate closure can support ergodic dynamics, are the resulting stationary regimes qualitatively and quantitatively alike? The existence of an invariant distribution alone does not imply equivalence between the two closures. They differ in the form of their restoring forces: under the quantity closure the market interest rate adjusts continuously and endogenously with the state of the economy, whereas under the corridor rule the policy response is deliberately inactive over an interior region and becomes operative only after a boundary is crossed. The quantitative analysis shows that this distinction matters. Under the baseline calibration, the stationary root-mean-square deviation of capital from its neutral level is $\sigma_{K,Q}^* = 2.32$ under the quantity closure, compared with $\sigma_{K,R}^* = 3.15$ even under the strongest corridor feedback considered, $\Lambda = 1$. Thus, state-dependent interest-rate policy can reconstruct sufficient negative feedback to restore ergodicity without reproducing the tighter stationary concentration generated by continuous endogenous interest-rate adjustment.

The paper's contribution is therefore threefold. First, it develops a stochastic interpretation of the Wicksellian cumulative process in which the presence or absence of restoring drift depends fundamentally on the monetary closure. A pure interest-rate peg eliminates the endogenous interest-rate feedback present under the quantity closure and, except at the neutral rate, generates persistent directional dynamics. Second, it shows how state-dependent interest-rate policy can substitute for this missing endogenous mechanism and identifies the critical strength of feedback required for stochastic stability. Third, it shows that restoring ergodicity does not imply equivalence between monetary closures: under the baseline calibration, the feedback rate closure generates greater stationary capital-stock dispersion than the quantity closure even at the strongest feedback considered. The comparison thereby distinguishes between the existence of stabilizing feedback and the form in which that feedback operates.

The deliberately parsimonious monetary and financial structure is intended to isolate this mechanism. In richer environments, bank balance sheets, credit spreads, asset prices, portfolio adjustment, expectations, aggregate shocks, and other transmission channels may reinforce or offset the forces identified here. The present analysis provides a benchmark for studying such extensions by first establishing the stability consequences attributable to the monetary closure itself.

The feedback rule studied here is not a Taylor rule in the conventional sense: Taylor-type rules prescribe systematic movements in the policy rate in response to inflation and measures of real activity, whereas the present rule responds directly to

the capital stock. The connection is instead structural: both replace a fixed interest-rate setting with systematic state-contingent policy. In this sense, the analysis provides a complementary stability rationale for systematic interest-rate feedback.

The alternative monetary closures considered here connect the analysis to established traditions in monetary theory. Quantity-oriented approaches emphasize an exogenously controlled supply of liquidity and endogenous interest-rate adjustment (Friedman and Schwartz, 1963; Brunner and Meltzer, 1964, 1968), corresponding to the quantity closure adopted in Takahashi (2026). By contrast, the pure interest-rate peg is closely related to Wicksell’s pure-credit analysis, in which the interest rate is fixed while the quantity of money accommodates endogenously (Wicksell, 1936). The broader interest-rate-closure framework also shares with the endogenous-money tradition of Kaldor (1970) and Moore (1988) the view that the interest rate is policy determined while liquidity accommodates at that rate. The state-dependent threshold rule studied here builds on this monetary architecture by allowing the policy rate itself to respond to macroeconomic conditions.

The remainder of the paper is organized as follows. Section 2 describes the economic environment and the pure-credit institutional structure. Section 3 introduces the interest-rate closure and contrasts it with the quantity closure adopted in Takahashi (2026). Section 4 derives the transition structure under a pure interest-rate peg and shows how fixing the policy rate removes the state-dependent restoring drift. Section 5 introduces the threshold interest-rate rule and shows how state-dependent policy reconstructs the missing profitability feedback. Section 6 establishes the stochastic-stability properties of the resulting capital-stock process, including positive Harris recurrence, existence and uniqueness of the invariant distribution, aperiodicity, and V -geometric convergence. Section 7 examines the quantitative implications of alternative feedback strengths, comparing invariant distributions, first-passage risks, and the frequency of policy intervention and regime switching, and then compares stationary capital-stock dispersion under the feedback rate closure with the quantity-closure benchmark. Section 8 concludes by drawing together the implications of the monetary closure for Wicksellian cumulative dynamics and stochastic stability, and by interpreting systematic policy-rate feedback as a means of reconstructing the negative feedback removed by an interest-rate closure. The Appendix collects supplementary derivations and proofs, documents the numerical implementation, and provides information on the replication programs underlying the reported results.

2 Economic Environment

The present model builds on the institutional framework developed in Takahashi (2026), but simplifies its financial structure to isolate the consequences of interest-rate closure. Households hold money and equity claims, firms finance investment through bank credit, and commercial banks and the monetary authority are represented by a consolidated monetary sector. Fiscal activity is omitted, since taxation, government expenditure, and public debt play no role in the analysis. The essential features of

households, firms, and the monetary sector are summarized below.

2.1 Households

Households are homogeneous and passive in the short run. They consume C_t and supply labor N_t at the given nominal wage W .

Households hold money balances M together with ownership claims on firms, denoted by E . They receive wage income and firm dividends Div^F . For simplicity, household portfolio decisions are represented as an allocation between money balances and equity claims.

The consumption and money-demand functions are inherited from Chapters 3 and 4 of Takahashi (2026) and are taken as primitives in the present paper. The consumption specification combines a Keynesian propensity to consume out of current real income with a wealth effect from capital holdings. Money demand is obtained from a CES specification of transaction services provided by consumption and real balances, implying that real money demand is proportional to consumption and decreasing in the opportunity cost of holding money, $r/(1+r)$. Their microfoundations are not repeated here because the present analysis concerns the consequences of alternative monetary closures rather than household optimization.

$$C = c_y \frac{Y}{P} + c_w K, \quad 0 < c_y < 1, \ c_w > 0, \quad (2.1)$$

and

$$\frac{M^d}{P} = \lambda^{1-\beta} C \left(\frac{r}{1+r} \right)^{-\beta},$$

respectively, where Y/P denotes real income (equivalently, real output in aggregate equilibrium), while c_y and c_w denote the propensities to consume out of income and wealth, respectively; β is the elasticity of substitution between consumption and real balances; and λ scales real balances into transaction services. Equation (2.1) captures Keynes' insight regarding the marginal propensity to consume and, at the same time, allows the *symmetric quasi-steady state (SQSS)* equilibrium to be expressed entirely as a function of the capital stock K .

2.2 Firms

A large number of price-setting firms produce differentiated varieties using a common Cobb–Douglas technology. Cost minimization yields the standard Cobb–Douglas unit cost (real marginal cost):

$$mc(\omega, r + \delta) = \frac{\omega^{1-\alpha}(r + \delta)^\alpha}{(1 - \alpha)^{1-\alpha}\alpha^\alpha},$$

where $\omega := W/P$ denotes the real wage, $0 < \alpha < 1$ is the capital elasticity of output, and δ is the depreciation rate. Under Dixit–Stiglitz monopolistic competition with

elasticity $\eta > 1$, the optimal price is a constant markup over nominal marginal cost MC :

$$P = \frac{\eta}{\eta - 1} MC,$$

which yields the Price–Markup (PM) condition.

$$\left(\frac{\omega}{1 - \alpha}\right)^{1 - \alpha} \left(\frac{r + \delta}{\alpha}\right)^\alpha = \frac{\eta - 1}{\eta},$$

which is equivalent to

$$\omega = \theta_{\text{PM}} (r + \delta)^{-\gamma},$$

where $\gamma := \alpha/(1 - \alpha)$ and

$$\theta_{\text{PM}} := (1 - \alpha)\alpha^\gamma \left(\frac{\eta - 1}{\eta}\right)^{1/(1 - \alpha)}.$$

New investment ordered in period t becomes productive only in period $t + 1$, implying a one-period gestation lag. Consistent with the pure-credit framework, firms finance all investment expenditures through bank loans.

2.3 Monetary Sector

The financial system is represented by a consolidated monetary sector comprising commercial banks and the monetary authority. This consolidation reflects the pure-credit benchmark adopted in the paper: reserve constraints, interbank settlement frictions, and separate bank funding requirements are abstracted from. No separate government sector is introduced, since taxation, government expenditure, and public debt play no role in the analysis.

Firms finance investment through bank loans L , while the monetary sector supplies demand deposits M that serve as the means of payment for households and firms. In the absence of a separate reserve-funding layer, loans and deposits may therefore be represented on the consolidated monetary-sector balance sheet without distinguishing commercial-bank deposits from central-bank reserve liabilities.

The consolidation is purely an accounting simplification and does not eliminate the distinct institutional role of the monetary authority. Commercial banks provide credit and payment services, whereas the monetary authority determines the policy interest rate. The pure-credit assumption itself does not determine that rate. The monetary closure governing its determination is introduced in Sect. 3, where the pure interest-rate peg is specified as

$$r_t = \bar{r}.$$

Under this closure, the quantity of money adjusts endogenously to accommodate money demand at the policy rate.

The balance sheets reported below are institutional rather than accounting balance sheets. They indicate the ownership structure of financial claims and productive

Table 1: Balance Sheet of the Quasi-Steady-State Economy

Households		Firms		Consolidated monetary sector	
Assets	Liab./eq.	Assets	Liab./eq.	Assets	Liab./eq.
M	$\mathcal{W}$	K	L	L	M
E			E		

Note. Firm equity E is held by households and serves as the residual ownership claim on the firm sector. Its valuation dynamics play no independent role in the analysis.

assets. Questions concerning valuation, timing conventions, and the distinction between nominal and real magnitudes are introduced later only where they are required for the analysis.

Assumption 2.1 (Exponent-Matching Assumption (Benchmark Mapping))

Throughout the paper, we maintain the exponent-matching restriction

$$\beta = \frac{\alpha}{1 - \alpha}.$$

Under this restriction, the short-run equilibrium conditions admit a closed-form mapping between the interest rate and the capital stock under quantity closure.

The exponent-matching restriction is stronger than required for local stability of the underlying PM–LM equilibrium. As shown in Takahashi (2026), local stability requires only that the LM locus be steeper in absolute value than the PM locus at their intersection and does not depend on the equality $\beta = \alpha/(1 - \alpha)$. The restriction is maintained here for analytical convenience: it yields a closed-form mapping between the interest rate and the capital stock, which allows the quantity-closure benchmark and the policy thresholds introduced below to be expressed transparently in terms of K . The substantive stability mechanism studied in this paper therefore does not hinge on exponent matching.

The economic environment specified above determines the real and financial structure within which monetary policy operates, but it does not by itself determine the interest rate or the money stock. Their determination depends on the monetary closure imposed on the model. The next section introduces the interest-rate closure adopted in this paper and contrasts it with the quantity closure employed in Takahashi (2026).

3 Monetary Structure under Interest-Rate Closure

Table 2: Main Variables and Parameters

Symbol	Definition
Y	Nominal income (output)
P	Aggregate price level
C	Real consumption
K	Aggregate capital stock
N	Labor input
W	Nominal wage
$\omega = W/P$	Real wage
M, M^d	Money balances and money demand
r	Interest rate
δ	Depreciation rate
α	Capital elasticity in Cobb–Douglas production
η	Elasticity of substitution across differentiated goods
β	Elasticity of substitution between consumption and real balances
c_y, c_w	Propensities to consume out of income and wealth
λ	Scaling parameter for transaction services of real balances

3.1 From Quantity Closure to Interest-Rate Closure

Traditional monetarist approaches emphasized control of monetary aggregates through reserve management and the money-multiplier mechanism (Friedman and Schwartz, 1963; Brunner and Meltzer, 1964, 1968). In these approaches, the quantity of liquidity is fixed and the interest rate adjusts endogenously. Thus, the direction of causality runs primarily from money to economic activity ($M \rightarrow Y$), corresponding closely to the quantity closure adopted in Takahashi (2026).

By contrast, the present framework adopts an interest-rate closure in which the policy rate is fixed and liquidity is supplied elastically at that rate. This perspective is closely related to Wicksell's pure-credit analysis (Wicksell, 1936) and to the endogenous-money tradition subsequently developed by Kaldor (1970) and Moore (1988). Wicksell's pure-credit economy provides the natural benchmark for the pure interest-rate peg, under which the market rate is pegged while the quantity of money adjusts endogenously. The horizontalist approach, associated in particular with Moore, likewise treats the short-term interest rate as an administered policy variable and regards the quantity of reserves as accommodating the needs of the banking system. The present model adopts this common interest-rate-closure logic while abstracting from the additional banking frictions emphasized by the structuralist branch of Post-Keynesian monetary theory (see, e.g., Palley, 1994; Lavoie, 1996).

This abstraction is deliberate. Once reserve scarcity at the level of individual banks, interbank funding, liquidity and leverage risk, asymmetric information, and balance-sheet propagation are introduced, monetary adjustment depends not only on the policy rate but also on the internal structure of the banking system. Those mechanisms are important, but incorporating them would require a substantially richer theory of bank balance sheets and reserve-market interaction. The purpose of the present paper is narrower: to isolate the dynamic consequences of replacing a quantity closure by an interest-rate closure and to determine whether state-contingent policy-rate feedback can restore the stabilizing force that disappears under a pure peg.

3.2 Relationship between Policy Rate and Money Stock

The Wicksellian cumulative process arises whenever the market rate of interest deviates from the natural rate. In a fractional-reserve banking system, a cumulative expansion of credit increases deposits and therefore raises the demand for reserves. As reserve positions become tighter, banks may respond by increasing lending rates, thereby limiting further credit expansion. Such reserve-market feedback is absent in the pure-credit limit considered here.

The pure-credit assumption should be viewed as a benchmark institutional arrangement rather than as the monetary closure itself. The closure adopted in the present paper is the interest-rate peg

$$r_t = \bar{r}.$$

A pure-credit economy provides a particularly transparent environment in which

this condition holds exactly. Because reserve supply adjusts elastically to reserve demand, reserve scarcity does not exert upward pressure on the market rate, which therefore remains aligned with the policy rate. Under the interest-rate peg, the money stock adjusts to satisfy money demand at $r = \bar{r}$. Using the SQSS relation $C = cK$, together with the PM condition $W/P = \theta_{\text{PM}}(\bar{r} + \delta)^{-\gamma}$, and substituting these expressions into the money-demand equation yields

$$M = M(K; \bar{r}) = \frac{Wc\lambda^{1-\beta}}{\theta_{\text{PM}}} \left(\frac{1 + \bar{r}}{\bar{r}} \right)^\beta (\bar{r} + \delta)^\gamma K. \quad (3.1)$$

Thus, for a given policy rate $\bar{r}$, the money stock accommodates the level of money demand associated with the capital stock K ; in particular, $M(K; \bar{r})$ is proportional to K . Hence, for a given policy rate $\bar{r}$, the money stock varies proportionally with the scale of economic activity, as measured by the capital stock (K), and adjusts endogenously to accommodate the demand for liquidity.

3.3 Natural Rate and Neutral Rate

The preceding discussion raises an immediate question: what object in the present framework corresponds to Wicksell's natural rate? In the framework of Takahashi (2026), fluctuations in K shift the LM locus under quantity closure. Under interest-rate closure, however, the interest rate is fixed at $\bar{r}$. Since the production technology is linearly homogeneous and consumption is proportional to income and wealth, the PM relation remains independent of the scale of activity. Consequently, once $r = \bar{r}$, the real wage ω is pinned down by the PM condition, as shown in Takahashi (2026, Sect. 4.4):

$$\left(\frac{\omega}{1 - \alpha} \right)^{1-\alpha} \left(\frac{\bar{r} + \delta}{\alpha} \right)^\alpha = \frac{\eta - 1}{\eta}.$$

Thus, with the user cost of capital $\bar{r} + \delta$ fixed and the real wage ω determined by the PM condition, the profitability conditions governing investment remain unchanged as K varies. Under quantity closure, the interest rate adjusts with the capital stock and thereby provides a feedback mechanism for profitability. Under a pure interest-rate peg, this adjustment is absent.

The model developed by Takahashi (2026) provides a natural counterpart to Wicksell's natural rate. In the present framework, this role is played by the neutral rate r^* , defined as the interest rate at which the expected capital-stock drift vanishes. In the quantity-closure model of Takahashi (2026), $r^* = r(K^*)$ and is characterized in Proposition 8.1 by

$$\frac{\alpha(c + \delta)}{r^* + \delta} \frac{\eta - 1}{\eta} = 1 + \delta \left(\varphi - \frac{1 + c_b}{2} \right), \quad \varphi = \frac{\alpha(\eta - 1)}{1 + \alpha(\eta - 1)}.$$

The neutral rate is not simply the physical marginal product of capital net of depreciation. Rather, it is the interest rate that, after accounting for markup pricing, depreciation, and the stochastic investment mechanism, makes expansionary and

contractionary investment strategies equally attractive. Equivalently, it satisfies the neutrality condition $p_+^{\text{raw}} = p_-^{\text{raw}}$. Thus, although r^* is defined within a different microeconomic environment, it performs the role analogous to Wicksell's natural rate: it is the benchmark against which the prevailing market rate determines the direction of cumulative adjustment.

When $\bar{r} = r^*$, the forces of upward and downward drift are exactly balanced, corresponding to the knife-edge case in which the cumulative process disappears. If $\bar{r} < r^*$, cumulative capital expansion begins; if $\bar{r} > r^*$, cumulative contraction occurs. Because the policy rate remains fixed, the profitability differential underlying either process is not automatically eliminated through interest-rate adjustment. Although the underlying state variables differ—prices in Wicksell's analysis and capital accumulation in the present framework—the logic of the cumulative process is closely analogous. In what follows, this intuition is developed formally.

3.4 Monetary Accommodation and Cumulative Dynamics

Equation (3.1) shows immediately why money becomes endogenous under the interest-rate closure. Since the policy rate is fixed at $\bar{r}$, any change in economic activity must be accommodated through an adjustment in the money stock. Consequently, fluctuations in capital stock induce proportional adjustments in the money stock rather than adjustments in the interest rate. The direction of adjustment is therefore reversed relative to the quantity-closure framework studied in Takahashi (2026). Under quantity closure, fluctuations in K are absorbed through movements in the interest rate. Under interest-rate closure, the same fluctuations are absorbed through movements in the money stock.

The fundamental stabilizing mechanism under quantity closure rests on the fact that the probability of upward drift, $p_+(K)$, decreases with the capital stock, whereas the probability of downward drift, $p_-(K)$, increases with it (Lemma 6.19 of Takahashi (2026)). Consequently, sufficiently large values of K generate an endogenous tendency toward contraction. This property ultimately derives from the fact that the elasticity of desired capital with respect to installed capital is less than unity,

$$\epsilon_{K^d, K} < 1,$$

as established in Appendix 6.B of Takahashi (2026). As the capital stock expands, the ratio K^d/K declines, reducing the profitability of further expansion and generating a restoring force.

Under interest-rate closure, this stabilizing mechanism no longer operates in the same way. An increase in capital accumulation is accompanied by an endogenous expansion of the money stock, which accommodates the higher demand for liquidity while keeping the policy rate fixed at $\bar{r}$. As a result, the profitability conditions governing investment are not automatically altered through movements in the interest rate. If the policy rate differs from the neutral rate r^* , the resulting gap $r^* - \bar{r}$ persists rather than being eliminated by market adjustment. Consequently, absent state-contingent policy feedback, the cumulative process contains no endogenous mechanism that restores the capital stock toward its neutral level.

3.5 Quantity Closure versus Interest-Rate Closure

The distinction between the two monetary closures can now be summarized as follows.

Table 3: Two monetary closures

Quantity Closure	Interest-Rate Closure
$M = \bar{M}$	$r = \bar{r}$
Money exogenous	Money endogenous
Interest rate adjusts	Money stock adjusts
$K \uparrow \Rightarrow r \uparrow$	$K \uparrow \Rightarrow M \uparrow$
Profitability differentials diminish	Profitability differentials persist
Endogenous restoring force	No restoring force under a pure peg

Thus, changing the monetary closure changes the feedback from the capital stock to investment profitability: quantity closure generates an endogenous restoring mechanism through interest-rate adjustment, whereas a pure interest-rate peg does not. The next section introduces the stochastic investment dynamics through which this distinction determines long-run stability.

4 Transition Structure under Interest-Rate Closure

4.1 Transition structure and notation

The preceding section showed that, under a pure interest-rate peg, monetary accommodation prevents changes in the capital stock from inducing the interest-rate adjustment that provides the restoring mechanism under quantity closure. Whether the capital-stock process nevertheless possesses an endogenous stabilizing force therefore depends on how investment profitability and the associated transition probabilities behave when the market rate is fixed at $\bar{r}$.

We begin by reformulating the stochastic capital-adjustment mechanism under interest-rate closure. The key causal link runs from the gap between the neutral and policy rates to investment profitability, from profitability to expansionary and contractionary transition probabilities, and ultimately to the drift of the capital stock.

The following table lists notations used in the rest of the paper.

Table 4: Key functions and thresholds

Object	Definition	Role
$r(K)$	$\frac{s(K)-(1+\delta)-\sqrt{[s(K)-(1+\delta)]^2-4\delta}}{2}, \quad s(K) := \left(\frac{\theta_{\text{PM}}}{\theta_{\text{LM}}K}\right)^{1/\beta}$	Quantity closure
$\zeta(r)$	$\zeta(r) = \frac{1}{\delta} \left[\frac{\alpha(c+\delta)}{r+\delta} \frac{\eta-1}{\eta} - 1 \right]$	Profitability
$p_+(\zeta)$	$\frac{1}{2} + v_b(\zeta - \varphi)$	Expansion probability
$p_-(\zeta)$	$\frac{1}{2} - v_b(\zeta + 1 + c_b - \varphi)$	Contraction probability
$h(r)$	$\frac{\delta}{2v_b} \left[p_+(r) \left(1 - \frac{p_+(r)}{2}\right) - p_-(r) \left(1 - \frac{p_-(r)}{2}\right) \right]$	Conditional per-capital drift
$f(K, r)$	$Kh(r)$	Conditional drift
ζ^*	$\varphi - \frac{1+c_b}{2}$	Neutral profitability

4.2 Profitability and Transition Probabilities

Following the investment-adjustment mechanism developed in Sec. 5.4 and Chap. 6 of Takahashi (2026), investment incentives are summarized by the profitability index ζ , defined as the proportional excess of desired capital over installed capital, normalized by the depreciation rate. A positive (negative) value of ζ therefore indicates that desired capital exceeds (falls short of) installed capital and favors expansionary (contractionary) investment.

Investment decisions follow a two-stage mutation-imitation process. Firms first experiment with expansionary or contractionary investment actions and subsequently imitate successful experiments. Because imitation favors strategies associated with higher profitability, small profitability differences can be amplified into aggregate expansionary or contractionary pressures.

When the policy rate coincides with the neutral rate, $\bar{r} = r^*$, the profitability index attains its neutral value, $\zeta(\bar{r}) = \zeta^*$, implying balanced expansionary and contractionary pressures. By the definition of ζ^* , this implies $p_+ = p_-$ in the unclipped region.

More generally, the Wicksellian gap $r^* - \bar{r}$ induces a profitability gap $\zeta(\bar{r}) - \zeta^*$, which in turn affects the balance between expansionary and contractionary transitions. The mechanism to be formalized below can therefore be summarized as

$$r^* - \bar{r} \implies \zeta(\bar{r}) - \zeta^* \implies p_+ - p_- \implies \text{capital-stock drift.}$$

4.3 State independence under a pure interest-rate peg

We now formalize the dependence of the transition probabilities on the policy rate. The imitation thresholds are

$$\hat{\varepsilon}^+(\zeta) = \max\{\varphi - \zeta, 0\}, \quad \hat{\varepsilon}^-(\zeta) = \max\{1 + c_b + \zeta - \varphi, 0\}.$$

The profitability index is written as $\zeta(K)$ because quantity closure makes the market interest rate an endogenous function of the capital stock. For the present purpose, it is useful to separate these two relationships. The profitability index can be written as the decreasing function of the market rate

$$\zeta(r) = \frac{1}{\delta} \left[\frac{\alpha(c + \delta)}{r + \delta} \frac{\eta - 1}{\eta} - 1 \right], \quad \zeta'(r) < 0.$$

Intuitively, the market interest rate enters the user cost of capital, $r + \delta$. A lower interest rate reduces the cost of capital and increases the attractiveness of expansionary investment, whereas a higher interest rate has the opposite effect. The profitability index therefore provides the channel through which interest-rate policy influences investment incentives.

Under quantity closure, substituting $r = r(K)$ gives the state-dependent index $\zeta(K) = \zeta(r(K))$. Under interest-rate closure, by contrast, the market rate is fixed at the policy rate, so that

$$\zeta(\bar{r}) = \frac{1}{\delta} \left[\frac{\alpha(c + \delta)}{\bar{r} + \delta} \frac{\eta - 1}{\eta} - 1 \right].$$

Thus, the profitability index becomes policy-determined rather than state-dependent. Define the probability imbalance

$$d(r) := p_+(\bar{r}) - p_-(\bar{r}) = v_b[2(\zeta(\bar{r}) - \varphi) + 1 + c_b],$$

in the unclipped region. Then under the peg the relevant imbalance is simply $d(\bar{r})$. Since $\zeta'(\bar{r}) < 0$, the probability imbalance is decreasing in the policy rate, $d'(r) = 2v_b\zeta'(r) < 0$, and satisfies $d(r^*) = 0$. Since $\zeta'(r) < 0$ and $d(r^*) = 0$, the sign of the probability imbalance is determined entirely by the Wicksellian gap:

$$\bar{r} < r^* \implies d(\bar{r}) > 0, \quad \bar{r} = r^* \implies d(\bar{r}) = 0, \quad \bar{r} > r^* \implies d(\bar{r}) < 0.$$

Most importantly, for a given policy rate $\bar{r}$, this sign is independent of the capital stock. The transition probabilities therefore do not adjust as K changes, and the state dependence that generated the restoring force under quantity closure disappears. This neutrality condition is the interest-rate counterpart of the neutral-capital condition under quantity closure.

4.4 Drift under a Pure Interest-Rate Peg

Define the conditional drift function by

$$f(K) := \mathbb{E}[\Delta K | K].$$

Under quantity closure, the proportional drift $h(K)$ is strictly decreasing in K (Takahashi, 2026, Lemma 6.19). This monotonicity provides the endogenous restoring force.

$$h(K) = \frac{\delta}{2v_b} \left[p_+(K) \left(1 - \frac{p_+(K)}{2} \right) - p_-(K) \left(1 - \frac{p_-(K)}{2} \right) \right]$$

is strictly decreasing in K . This monotonicity is the source of the endogenous restoring force.

Under the pure peg, the proportional drift is independent of K and can therefore be written as $h(\bar{r})$.

$$h(\bar{r}) = \frac{\delta}{2v_b} \left[p_+(\bar{r}) \left(1 - \frac{p_+(\bar{r})}{2} \right) - p_-(\bar{r}) \left(1 - \frac{p_-(\bar{r})}{2} \right) \right],$$

and the drift function becomes $f(K) = Kh(\bar{r})$.

Under the continuum-industry specification, the term $p(1 - p/2)$ represents the expected aggregate contribution of a mutation branch. Since the upward and downward contributions enter through the same strictly increasing function $g(p)$:

$$g(p) := p(1 - p/2), \quad g'(p) = 1 - p > 0, \quad \text{for } p \in [0, 1/2].$$

Then

$$h(\bar{r}) = \frac{\delta}{2v_b} [g(p_+) - g(p_-)].$$

Since $d(\bar{r})$ is decreasing in $\bar{r}$ and satisfies $d(r^*) = 0$, it follows that

$$\bar{r} < r^* \implies h(\bar{r}) > 0, \quad \bar{r} = r^* \implies h(\bar{r}) = 0, \quad \bar{r} > r^* \implies h(\bar{r}) < 0.$$

These observations yield the following result.

Proposition 4.1 (Direction of Drift under a Pure Interest-Rate Peg) *Suppose the economy operates under the interest-rate closure $r_t = \bar{r}$. Then the profitability index becomes policy-determined, $\zeta = \zeta(\bar{r})$, and the drift function satisfies*

$$f(K) = Kh(\bar{r}).$$

Moreover,

$$\begin{aligned} \bar{r} < r^* &\implies f(K) > 0, \\ \bar{r} = r^* &\implies f(K) = 0, \end{aligned}$$

and

$$\bar{r} > r^* \implies f(K) < 0$$

for all interior states K .

Proof. Under a pure interest-rate peg, $r_t = \bar{r}$, so that the profitability index $\zeta(\bar{r})$ and the transition probabilities $p_+(\bar{r})$ and $p_-(\bar{r})$ are independent of the capital stock.

Consider first the unclipped region. The probability imbalance is

$$d(r) := p_+(r) - p_-(r) = v_b [2(\zeta(r) - \varphi) + 1 + c_b].$$

Since

$$\zeta'(r) < 0,$$

we have

$$d'(r) = 2v_b \zeta'(r) < 0.$$

The interaction of these two channels determines realized policy activity and the resulting degree of stabilization. The mechanism can be summarized schematically as follows: By the definition of the neutral rate r^* ,

$$d(r^*) = 0.$$

It follows that

$$d(r) > 0 \quad \text{for } r < r^*, \quad d(r) < 0 \quad \text{for } r > r^*.$$

The proportional drift is

$$h(r) = \frac{\delta}{2v_b} \left[p_+(r) \left(1 - \frac{p_+(r)}{2} \right) - p_-(r) \left(1 - \frac{p_-(r)}{2} \right) \right].$$

In the unclipped region, the expression in brackets can be written as

$$(p_+(r) - p_-(r)) \left[1 - \frac{p_+(r) + p_-(r)}{2} \right].$$

Because $0 \leq p_+(r), p_-(r) \leq 1/2$, the second factor is strictly positive. Hence, in the unclipped region,

$$\text{sgn } h(r) = \text{sgn } d(r).$$

It remains to verify that clipping does not alter this sign relation. At sufficiently high interest rates, the expansion probability reaches its lower clipping bound, $p_+(r) = 0$. The proportional drift then becomes

$$h(r) = -\frac{\delta}{2v_b} p_-(r) \left(1 - \frac{p_-(r)}{2} \right) < 0.$$

Conversely, at sufficiently low interest rates, the contraction probability reaches its lower clipping bound, $p_-(r) = 0$, and

$$h(r) = \frac{\delta}{2v_b} p_+(r) \left(1 - \frac{p_+(r)}{2} \right) > 0.$$

Thus clipping preserves the direction of drift established in the unclipped region.

Evaluating these relations at the pegged rate $r = \bar{r}$ therefore gives

$$h(\bar{r}) > 0 \quad \text{if } \bar{r} < r^*, \quad h(\bar{r}) = 0 \quad \text{if } \bar{r} = r^*, \quad h(\bar{r}) < 0 \quad \text{if } \bar{r} > r^*.$$

Since $\bar{r}$ is fixed under the peg, the sign of the drift is independent of K . Hence there is no endogenous reversal of drift as the capital stock changes. $\square$

Proposition 4.1 provides a stochastic counterpart to Wicksell's cumulative-process argument. When the policy rate lies below the neutral rate, expansionary transitions dominate contractionary transitions, generating positive drift throughout the interior of the state space. Conversely, when the policy rate exceeds the neutral rate, contractionary transitions dominate.

Corollary 4.2 (Absence of Interior Restoring Drift) *Suppose the economy operates under a pure interest-rate peg with $\bar{r} \neq r^*$. Then the drift has the same sign at every interior state. Hence no interior capital level exists at which the drift reverses sign.*

Proof. Immediate from Proposition 4.1. $\square$

At the knife-edge peg $\bar{r} = r^*$, the drift vanishes at every interior state rather than changing sign around a particular capital level; hence the process still possesses no restoring drift toward K^* .

4.5 Long-Run Implications of a Pure Peg

The absence of restoring drift gives the pure peg its cumulative character. If sustained capital expansion or contraction eventually activates the wage-adjustment mechanisms studied under quantity closure, the real cumulative process may spill over into nominal wage and price adjustment. If cumulative capital expansion or contraction eventually activates the wage-adjustment mechanisms studied in the quantity-closure model, the real cumulative process may spill over into nominal wage and price adjustment. In this sense, the capital-stock dynamics studied here provide a real-side analogue of the Wicksellian cumulative process, with nominal wage and price adjustment arising once the fixed-wage regime ceases to be maintained.

5 Policy-Induced Restoration of Stability

5.1 Lost profitability feedback under a pure peg

Under quantity closure, write $\zeta(K) \equiv \zeta(r(K))$. The interest rate acts as a scarcity signal that reflects the evolving capital-labor ratio. As the capital-labor ratio rises, diminishing marginal productivity of capital reduces the profitability of additional capital. Through the PM-LM interaction, this change is reflected in the equilibrium interest rate $r(K)$. Consequently, the profitability index $\zeta(K)$ declines with capital accumulation. The resulting profitability index determines the expansionary and contractionary transition probabilities:

$$\begin{aligned} p_+(K) &= \text{clip}[0.5 + v_b(\zeta(r(K)) - \varphi); 0, 0.5], \\ p_-(K) &= \text{clip}[0.5 - v_b(\zeta(r(K)) - \varphi + 1 + c_b); 0, 0.5], \end{aligned}$$

where $\varphi := P^S/P^K = \alpha(\eta - 1)/(\alpha(\eta - 1) + 1)$, and c_b denotes the behavioral investment wedge (Lemma 6.5). Economically, φ measures the replacement cost of capital relative to its prospective value, and therefore acts as an investment hurdle: a larger φ lowers the expansionary probability p_+ and raises the contractionary probability p_- .

Moreover, on any interval over which both branch probabilities remain interior and unclipped, Lemma 6.19 implies that the expected per-unit drift $h(K)$ is strictly

decreasing in K^1 . Thus, the resulting decline in profitability lowers $p_+(K)$ and raises $p_-(K)$, eventually reversing the sign of drift. These chains of dynamics can be expressed as:

$$K \uparrow \implies r(K) \uparrow \implies \zeta(K) \downarrow \implies p_+(K) - p_-(K) \downarrow \implies h(K) \downarrow.$$

Under an interest-rate peg, this feedback mechanism is severed. Since the policy rate is fixed, $r_t = \bar{r}$, the profitability index becomes policy-determined: $\zeta = \zeta(\bar{r})$. As a result, changes in the capital stock no longer generate the profitability feedback required for an interior restoring force.

5.2 The threshold interest-rate rule

The preceding analysis suggests a natural way to restore the missing stabilizing feedback: allow the policy rate to respond to changes in the underlying state of the economy. Since the capital stock summarizes the economy's profitability conditions, a natural choice is to let the policy rate depend on the current capital stock: $r_t = r(K_t)$. The essential requirement is that the policy rate respond monotonically to the capital stock, assigning higher rates to states with greater capital accumulation. For a smooth feedback rule, this condition can be written as $r'(K) > 0$. This feedback allows changes in resource scarcity to be transmitted once again to the profitability index through the interest rate.

As a simple illustration, consider the linear policy rule

$$r_t = r^* + \phi_r(K_t - K^*), \quad \phi_r > 0,$$

where r^* is the neutral interest rate and K^* is the corresponding neutral capital stock.

Under this rule, higher capital accumulation induces a higher policy rate and lower profitability, thereby recreating the stabilizing feedback mechanism absent under the pure interest-rate peg. Thus, a positively sloped state-dependent rule would in principle restore the missing feedback.

Unlike a conventional Taylor rule, which reacts to inflation and measures of real activity, the present policy rule responds directly to the aggregate capital stock. The capital stock is the fundamental state variable governing investment profitability and the stochastic evolution of the economy, making it a natural target for feedback within the present framework. The central question addressed by this paper is therefore: if the objective is to stabilize capital accumulation directly, what is the simplest interest-rate feedback rule capable of restoring stochastic stability?

To answer this question, we replace the smooth feedback illustrated above by its simplest discrete counterpart: a three-regime threshold rule that maintains the neutral rate around K^* and switches to corrective rates when capital crosses either of two intervention thresholds.

¹ $h(K) = \delta/4[2(\zeta(K) - \varphi) + 1 + c_b][1 + v_b(1 + c_b)]$

The linear specification illustrates the required direction of feedback, but it is not analyzed further here. We instead consider the piecewise-constant rule

$$r(K) = \begin{cases} r_L, & K \leq K_L, \\ r^*, & K_L < K < K_H, \\ r_H, & K \geq K_H. \end{cases} \quad (5.1)$$

The policy rates are chosen strictly inside the zero-clipping thresholds,

$$r_{q0} < r_L < r^* < r_H < r_{p0},$$

where $p_-(r_{q0}) = 0$ and $p_+(r_{p0}) = 0$. Hence the contractionary branch remains active under r_L , while the expansionary branch remains active under r_H .

The corresponding capital thresholds are defined using the quantity-closure benchmark mapping $r(K)$:

$$r(K_L) = r_L, \quad r(K_H) = r_H,$$

with

$$r(K_{q0}) = r_{q0}, \quad r(K_{p0}) = r_{p0}.$$

Since $r(K)$ is increasing over the relevant branch, the interest-rate ordering implies

$$K_{q0} < K_L < K^* < K_H < K_{p0}.$$

The quantity-closure mapping is used here only to associate the policy-rate thresholds with economically corresponding capital states; the actual policy schedule is the piecewise-constant rule in (5.1).

5.3 Policy regimes and the stability corridor

The policy rule (5.1) implements

$$\begin{aligned} K \leq K_L : \quad & \bar{r} = r_L, & h(r_L) > 0, \\ K_L < K < K_H : \quad & \bar{r} = r^*, & h(r^*) = 0, \\ K \geq K_H : \quad & \bar{r} = r_H, & h(r_H) < 0. \end{aligned}$$

Inside the corridor,

$$K_L < K_t < K_H,$$

the threshold rule (5.1) sets

$$\bar{r} = r^*, \quad \zeta = \zeta^*.$$

By the proof of Proposition 6.20, the expected per-unit drift is

$$h(K) = \mathbb{E} \left[\frac{K_{t+1} - K_t}{K_t} \middle| K_t = K \right] = \frac{\delta}{4} [2(\zeta(\bar{r}(K)) - \varphi) + 1 + c_b] [1 + v_b(1 + c_b)].$$

By the definition of neutrality, $\zeta^* = \varphi - (1 + c_b)/2$, and hence $2(\zeta^* - \varphi) + 1 + c_b = 0$. Therefore, when the policy rate is r^* ,

$$\bar{r}(K) = r^* \implies \zeta(\bar{r}(K)) = \zeta^* \implies h(K) = 0.$$

Moreover, since

$$p_+(K) - p_-(K) = v_b [2(\zeta^* - \varphi) + 1 + c_b],$$

neutrality also implies

$$p_+(K) = p_-(K).$$

Consequently,

$$f(K) = \mathbb{E}[K_{t+1} - K_t \mid K_t = K] = Kh(K) = 0.$$

Hence $h(K) = f(K) = 0$ for $K_L < K < K_H$. We refer to $[K_L, K_H]$ as the stability corridor because policy maintains neutrality in its interior while activating corrective inward drift once either boundary is crossed.

The ordering $r_L < r^* < r_H$ determines the direction of corrective drift, while the distances of r_L and r_H from r^* determine its strength. Since r^* is defined by $h(r^*) = 0$, and $h(r)$ is strictly decreasing over the relevant unclipped region, the ordering

$$r_L < r^* < r_H$$

implies

$$h(r_L) > 0, \quad h(r^*) = 0, \quad h(r_H) < 0.$$

Thus, we may equivalently write

$$h(r_L) - h(r_H) = |h(r_L)| + |h(r_H)|.$$

Hence $h(r_L) - h(r_H)$ provides a natural measure of the combined strength of the inward corrective drift on the two sides of the stability corridor. By Proposition 6.20, on the interior, unclipped region, the per-unit drift can be written as

$$h(r) = \frac{\delta}{4} [2(\zeta(r) - \varphi) + 1 + c_b] [1 + v_b(1 + c_b)]. \quad (5.2)$$

Evaluating (5.2) at the two policy rates and taking their difference gives the following result.

Lemma 5.1 (Corrective-drift differential) *On the relevant interior, unclipped region,*

$$h(r_L) - h(r_H) = \frac{\delta}{2} [1 + v_b(1 + c_b)] [\zeta(r_L) - \zeta(r_H)],$$

and hence a larger profitability differential generates a larger combined inward-drift strength.

Thus, the corrective-drift differential is proportional to the profitability differential generated by the two policy regimes. Since $\zeta(r)$ is strictly decreasing in r , moving r_L downward and r_H upward increases the profitability differential $\zeta(r_L) - \zeta(r_H)$, and therefore strengthens the combined inward drift. The mechanism can be summarized as

$$\zeta(r_L) - \zeta(r_H) \uparrow \implies h(r_L) - h(r_H) \uparrow. \quad (5.3)$$

The quantity-closure regime supplies a continuously state-dependent stabilizing signal, whereas the threshold rule responds only after an intervention threshold has been crossed. The threshold rule therefore cannot replicate the continuously preventive character of the quantity-closure mechanism.

The threshold rule therefore entails a natural trade-off between the timing and intensity of policy intervention. When r_L and r_H are chosen close to r^* , the corresponding thresholds K_L and K_H lie close to K^* , so policy intervention begins relatively early. The associated profitability differentials are small, however, and the resulting corrective drift is weak. Conversely, choosing r_L and r_H farther from r^* moves the intervention thresholds farther from K^* , allowing the capital stock to travel farther before policy responds, but generates stronger inward drift once a threshold is crossed. The threshold rule thus trades earlier but weaker intervention against later but stronger intervention.

The threshold rule is reminiscent of the target-zone literature (Krugman, 1991), in that policy maintains a neutral regime within a prescribed corridor and switches regimes when the state reaches its intervention thresholds. The objective here, however, is different. Rather than stabilizing an asset price, the policy rule reconstructs the state-dependent profitability feedback that arises endogenously under quantity closure.

5.4 Feedback-strength parameter Λ

To parameterize the timing–intensity trade-off identified above, we introduce the feedback-strength parameter Λ .

$$r_L(\Lambda) = r^* - \Lambda(r^* - r_{q0}), \quad r_H(\Lambda) = r^* + \Lambda(r_{p0} - r^*), \quad (5.4)$$

where $0 \leq \Lambda \leq 1$. For $0 < \Lambda < 1$, the policy rates lie strictly inside the zero-clipping thresholds. The limiting cases $\Lambda = 0$ and $\Lambda = 1$ are useful benchmark cases: the former collapses to the neutral pure peg $r_t = r^*$, whereas the latter places the policy rates exactly at the zero-clipping thresholds. We see that the policy-rate spread is proportional to Λ :

$$r_H(\Lambda) - r_L(\Lambda) = \Lambda(r_{p0} - r_{q0}),$$

thus

$$\frac{d}{d\Lambda} [r_H(\Lambda) - r_L(\Lambda)] = r_{p0} - r_{q0} > 0.$$

We refer to Λ as the feedback-strength parameter because it monotonically increases the policy-rate spread and hence the magnitude of the corrective drift once policy is activated.

At $\Lambda = 0$,

$$r_L(0) = r_H(0) = r^*, \quad K_L(0) = K_H(0) = K^*,$$

so the distinction among the three policy regimes disappears and the rule collapses to the neutral pure peg $r_t = r^*$.

At $\Lambda = 1$,

$$r_L(1) = r_{q0}, \quad r_H(1) = r_{p0},$$

and therefore

$$K_L(1) = K_{q0}, \quad K_H(1) = K_{p0}.$$

Thus $\Lambda = 1$ uses the full admissible range bounded by the zero-clipping thresholds. Intermediate Λ values continuously parameterize the strength of the policy response.

$$\frac{dr_L}{d\Lambda} = -(r^* - r_{q0}) < 0, \quad \frac{dr_H}{d\Lambda} = r_{p0} - r^* > 0.$$

Thus, in view of (5.3), the effect of stronger feedback can be summarized as

$$\Lambda \uparrow \implies r_L \downarrow, r_H \uparrow \implies \zeta(r_L) - \zeta(r_H) \uparrow \implies h(r_L) - h(r_H) \uparrow.$$

So the first effect of increasing Λ is stronger corrective action. Stronger feedback, however, also changes the capital-stock thresholds at which corrective policy is activated.

Figure 1 illustrates the intervention trade-off generated by the feedback parameter Λ . A larger Λ increases the distance between the intervention rates r_L and r_H , thereby increasing the magnitude of the policy-rate adjustment when intervention occurs. At the same time, because the upward-sloping equilibrium relation $r(K)$ maps these policy rates into the corresponding capital thresholds K_L and K_H , a larger Λ widens the inactive region $K_H - K_L$. Intervention is therefore triggered less readily: stronger feedback implies larger policy adjustments conditional on intervention, but a wider range of capital stocks over which the policy rate remains at its neutral value r^* . Conversely, weaker feedback produces smaller policy-rate adjustments but a narrower inactive region, so that intervention is triggered more readily. Thus, feedback strength governs a trade-off between the intensive margin of policy intervention—the magnitude of the rate adjustment—and its extensive margin—how readily intervention is triggered.

The relationship illustrated in Fig. 1 can be stated formally in terms of the dependence of the intervention thresholds on feedback strength.

Proposition 5.2 (Feedback strength and corridor width) *Let $K_L(\Lambda) = K(r_L(\Lambda))$ and $K_H(\Lambda) = K(r_H(\Lambda))$, where $r_L(\Lambda)$ and $r_H(\Lambda)$ are given by (5.4). Suppose that $K'(r) > 0$ over the relevant branch. Then*

$$\frac{dK_L(\Lambda)}{d\Lambda} = -K'(r_L(\Lambda))(r^* - r_{q0}) < 0,$$

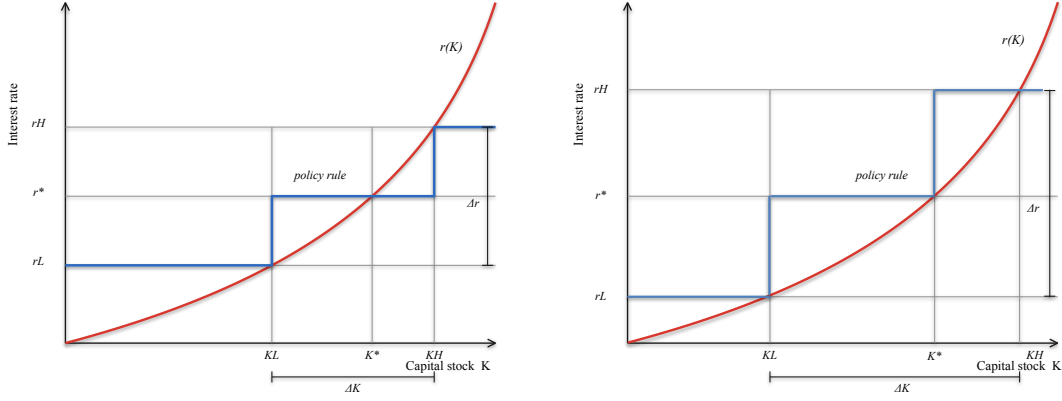


Figure 1: Feedback strength and the policy corridor

The left panel illustrates a smaller feedback strength, with a smaller policy-rate gap and a narrower inactive region, whereas the right panel illustrates a larger feedback strength, with a larger policy-rate gap and a wider inactive region. The upward-sloping curve $r(K)$ is the equilibrium interest-rate mapping, while the step function represents the policy rule. In both panels, $r(K^*) = r^*$.

whereas

$$\frac{dK_H(\Lambda)}{d\Lambda} = K'(r_H(\Lambda))(r_{p0} - r^*) > 0.$$

Consequently,

$$\frac{d}{d\Lambda} [K_H(\Lambda) - K_L(\Lambda)] > 0.$$

Thus, an increase in Λ widens the stability corridor by moving K_L downward and K_H upward from K^* .

In view of the threshold ordering established above, for $0 < \Lambda < 1$,

$$0 < K_H(\Lambda) - K_L(\Lambda) < K_{p0} - K_{q0}.$$

Moreover,

$$K^* - K_L(\Lambda) \uparrow, \quad K_H(\Lambda) - K^* \uparrow,$$

so a larger excursion from K^* is required before corrective policy is activated on either side.

Proposition 5.2, together with Lemma 5.1, reveals the timing–intensity trade-off inherent in the threshold rule. An increase in Λ moves the intervention thresholds farther from K^* , so that a larger movement in the capital stock is required before corrective policy is triggered. At the same time, once a threshold is crossed, the larger policy-rate differential generates stronger inward drift. Hence an increase in Λ trades later intervention for stronger corrective action, whereas a decrease in Λ trades earlier intervention for weaker corrective action.

The next section establishes the conditions under which the threshold rule restores stochastic stability.

6 Stochastic Stability under the Threshold Rule

We now examine whether the threshold interest-rate rule generates sufficient restoring forces to stabilize the capital-stock process in the long run. We first formulate the stochastic investment mechanism as a Markov transition kernel and then establish conditions for the existence, uniqueness, and geometric stability of an invariant distribution.

6.1 Stochastic investment kernel

The stochastic investment mechanism follows Chapter 6 of Takahashi (2026). For completeness, we summarize only those elements needed to construct the transition kernel and establish the stochastic-stability results below.

6.1.1 Conditional jump-size distributions

Before introducing the conditional jump-size distribution, it is useful to distinguish the two sources of randomness underlying the investment dynamics. The first operates at the macroeconomic level through the aggregate sentiment variable ν_t , which determines whether the economy enters an upward or downward investment regime. The second operates at the industry level through heterogeneous idiosyncratic activations, represented by independent draws $U \sim \text{Unif}[0, 1]$. These two mechanisms play fundamentally different roles. Aggregate sentiment selects the direction of adjustment, whereas industry-level heterogeneity determines how many industries respond once that direction has been selected.

Conditional on an upward (or downward) regime, industries do not adjust simultaneously. Instead, investment is activated sequentially across heterogeneous industries according to their idiosyncratic thresholds. Consequently, the aggregate investment response is not characterized by a single jump size but by conditional distributions of upward and downward jump magnitudes, denoted by ε^+ and ε^- . Figure 2 illustrates the distributional structure of the upward jump variable ε^+ ; the downward case is defined analogously. The corresponding transition kernel, inherited from the stochastic investment mechanism developed in Chapter 6 of the monograph, is summarized below and forms the analytical basis for the critical feedback-strength theorem.

6.1.2 State-transition maps and Markov kernel

The capital-stock process is modeled as a time-homogeneous Markov chain on the state space $\mathcal{K} = (0, \infty)$, equipped with the Borel σ -algebra $\mathcal{B}(\mathcal{K})$.

Definition of $\Gamma_{\pm}$ Given the current capital stock $K \in \mathcal{K}$, the next-period state, $K_{t+1} = M_t K_t$, takes one of three forms,

$$K' \in \{K(1 + \varepsilon^+), K(1 - \varepsilon^-), K\},$$

corresponding to upward adjustment, downward adjustment, and no adjustment, respectively. The associated branch probabilities are $p_+(K)$, $p_-(K)$, and $p_0(K) = 1 - p_+(K) - p_-(K)$.

For each conditional branch, let

$$\varepsilon_{\pm}(K, j), \quad j \in [\vartheta_{\pm}(K), 1],$$

denote the quantile function of the conditional uniform distribution. The corresponding state-transition maps are

$$G_+(K, j) = K(1 + \varepsilon_+(K, j)), \quad G_-(K, j) = K(1 - \varepsilon_-(K, j)).$$

Conditional on selecting an upward (respectively downward) branch, the jump magnitudes ε^+ and ε^- are assumed to follow state-dependent uniform distributions,

$$\varepsilon^+ \sim \text{Unif}[\underline{\alpha}_1(K), \alpha_2], \quad \varepsilon^- \sim \text{Unif}[\underline{\beta}_1(K), \beta_2].$$

The corresponding conditional probability measures are denoted by $\theta_+(\cdot; K)$ and $\theta_-(\cdot; K)$.

As discussed in the preceding subsection, these state-dependent uniform distributions arise from the cross-sectional aggregation of heterogeneous industry-level investment responses and are inherited from the stochastic investment mechanism developed in the monograph.

Transition operator P The corresponding conditional jump distributions induce the following pushforward measures:

$$\Gamma_+(K, A) = \theta_+(\{e > 0 : K(1 + e) \in A\}; K), \quad \Gamma_-(K, A) = \theta_-(\{e > 0 : K(1 - e) \in A\}; K),$$

for every Borel set $A \in \mathcal{B}(\mathcal{K})$.

The Markov transition kernel is therefore

$$P(K, A) = p_+(K)\Gamma_+(K, A) + p_-(K)\Gamma_-(K, A) + p_0(K)\delta_K(A),$$

where $p_0(K) = 1 - p_+(K) - p_-(K)$, and $\delta_K(A) = \mathbf{1}_{\{K \in A\}}$.

The associated Markov transition operator acts on bounded measurable functions $f : \mathcal{K} \rightarrow \mathbb{R}$ according to

$$\begin{aligned} (Pf)(K) &= (1 - p_+(K) - p_-(K))f(K) \\ &\quad + p_+(K) \int_{\vartheta_+(K)}^1 f(G_+(K, j)) \frac{dj}{1 - \vartheta_+(K)} \\ &\quad + p_-(K) \int_{\vartheta_-(K)}^1 f(G_-(K, j)) \frac{dj}{1 - \vartheta_-(K)}. \end{aligned}$$

Equivalently,

$$(Pf)(K) = \int_{\mathcal{K}} f(K') P(K, dK'),$$

which is the integral representation of the transition operator induced by the Markov kernel P .

6.2 Expected logarithmic growth

Because the capital-stock process is multiplicative, expected logarithmic growth provides a natural measure of local persistence near the lower boundary and identifies the feedback strength at which the lower-regime growth tendency changes sign. Positive recurrence, however, requires control of both tails of the state space. We therefore first derive the logarithmic threshold and then establish stochastic stability using the two-sided Foster–Lyapunov function $V(K) = K + K^{-1}$.

Define the one-step gross capital multiplier as $M_t := K_{t+1}/K_t$. The expected logarithmic growth rate conditional on the policy rate r is then

$$m(r) := \mathbb{E}[\log M_t \mid r_t = r].$$

6.2.1 Derivation of the expected logarithmic growth rate

Let

$$a := \frac{\delta}{2v_b}.$$

Under a given policy rate, the gross multiplier M has the three-branch distribution

$$M = \begin{cases} M_+, & \text{with probability } p, \\ M_-, & \text{with probability } q, \\ 1, & \text{with probability } 1 - p - q, \end{cases}$$

where

$$M_+ \sim \text{Unif}[1 + a(1 - p), 1 + a]$$

and

$$M_- \sim \text{Unif}[1 - a, 1 - a(1 - q)].$$

Therefore, by the law of total expectation,

$$\mathbb{E}[\log M] = p \mathbb{E}[\log M_+ \mid \text{up}] + q \mathbb{E}[\log M_- \mid \text{down}] + (1 - p - q) \log 1.$$

Since $\log 1 = 0$,

$$\mathbb{E}[\log M] = p \mathbb{E}[\log M_+ \mid \text{up}] + q \mathbb{E}[\log M_- \mid \text{down}].$$

Now the upward interval has length

$$(1 + a) - [1 + a(1 - p)] = ap.$$

Hence its conditional density is $1/(ap)$. Let

$$F(m) := m \log m - m, \quad m > 0.$$

Since $F'(m) = \log m$, the unconditional contribution of the upward branch to expected logarithmic growth is

$$\begin{aligned} p \mathbb{E}[\log(M_+) \mid \text{up}] &= \frac{1}{a} \int_{1+a(1-p)}^{1+a} \log m \, dm \\ &= \frac{F(1+a) - F(1+a(1-p))}{a}. \end{aligned}$$

Similarly, the unconditional contribution of the downward branch is

$$\begin{aligned} q \mathbb{E}[\log(M_-) \mid \text{down}] &= \frac{1}{a} \int_{1-a}^{1-a(1-q)} \log m \, dm \\ &= \frac{F(1-a(1-q)) - F(1-a)}{a}. \end{aligned}$$

Define the expected-log-growth aggregator

$$\begin{aligned} \mathcal{M}(p, q) &:= p \mathbb{E}[\log(M_+) \mid \text{up}] + q \mathbb{E}[\log(M_-) \mid \text{down}] \\ &= \frac{F(1+a) - F(1+a(1-p))}{a} + \frac{F(1-a(1-q)) - F(1-a)}{a}. \end{aligned}$$

The expected logarithmic growth rate associated with a policy rate r is therefore

$$m(r) = \mathcal{M}(p(r), q(r)).$$

Finally, under the threshold policy rule,

$$m_L(\Lambda) := \mathcal{M}(p(r_L(\Lambda)), q(r_L(\Lambda))),$$

and

$$m_H(\Lambda) := \mathcal{M}(p(r_H(\Lambda)), q(r_H(\Lambda))).$$

The factors p and q convert the conditional upward- and downward-branch expectations into their unconditional contributions. Their sum, $\mathcal{M}(p, q)$, is therefore the unconditional expected logarithmic growth rate of the gross multiplier. Accordingly,

$$m_L(\Lambda) = \mathbb{E}[\log M_L(\Lambda)], \quad m_H(\Lambda) = \mathbb{E}[\log M_H(\Lambda)].$$

6.3 Logarithmic critical feedback strength

6.3.1 Existence and uniqueness of $\Lambda_c^{\log}$

Recall the policy rule:

$$r(K) = \begin{cases} r_L, & K \leq K_L, \\ r^*, & K_L < K < K_H, \\ r_H, & K \geq K_H. \end{cases}$$

We compare policies of differing strength by introducing a parameter $\Lambda \in [0, 1]$. The case $\Lambda = 1$ gives the hard-corridor rule; $0 < \Lambda < 1$ gives an interior soft-corridor rule; and $\Lambda = 0$ gives the pure peg $r_L = r_H = r^*$.

At the hard-corridor endpoint $\Lambda = 1$,

$$r_L = r_{q0}, \quad r_H = r_{p0},$$

and hence

$$q(r_L) = 0, \quad p(r_H) = 0.$$

Thus, below K_L , capital can only remain unchanged or rise; above K_H , it can only remain unchanged or fall. The outer policy regimes create one-sided transitions, not merely inward expected drift. Under the hard-corridor rule, the one-sided transition restrictions confine the capital-stock process to a compact interval.

Following the support argument developed in Chapter 6 of Takahashi (2026), denote the corresponding lower and upper bounds by $\underline{K}$ and $\overline{K}$. Thus the invariant distribution, when it exists, has support contained in $[\underline{K}, \overline{K}]$. The wage-adjustment thresholds K_D and K_F derived in Chapter 7 of the monograph are assumed to lie outside this interval,

$$K_D < \underline{K} < \overline{K} < K_F,$$

so that the wage-revision mechanism remains nonbinding throughout the relevant stochastic state space.²

The hard-corridor rule therefore describes a central bank that chooses sufficiently forceful boundary rates to eliminate further outward movement. This specification may therefore be characterized as a support-preserving threshold rule, or hard-corridor rule. By contrast, interior specification,

$$r_{q0} < r_L < r^* < r_H < r_{p0},$$

is a soft-corridor rule. It makes outward transitions less likely but does not eliminate them.

Stability therefore relies on inward expected drift outside the corridor, rather than on one-sided transition restrictions. The corridor rates are

$$\begin{aligned} r_L(\Lambda) &= r^* - \Lambda(r^* - r_{q0}) \\ r_H(\Lambda) &= r^* + \Lambda(r_{p0} - r^*) \end{aligned}$$

Therefore,

$$r'_L(\Lambda) = -(r^* - r_{q0}) < 0, \quad r'_H(\Lambda) = r_{p0} - r^* > 0.$$

As Λ becomes smaller, the corridor $[K_L, K_H]$ becomes narrower, intervention is activated earlier, and, most importantly, the positive restoring drift in the lower regime becomes weaker because r_L approaches r^* . Consequently, although policy intervention begins earlier, its weaker restoring force allows the stochastic process to wander farther toward the origin.

²The explicit formulas for K_D and K_F are not required here; only their location outside the stochastic support matters. See Appendix 7.A and Proposition 7.2 of Takahashi (2026).

Theorem 6.1 (Critical feedback-strength theorem) *Suppose $0 < \delta/(2v_b) < 1$, the branch probabilities remain in the unclipped interior for $\Lambda \in [0, 1)$, with the endpoint probabilities defined continuously at $\Lambda = 1$, and $\zeta'(r) < 0$. Then $m_L(\Lambda)$ is continuous and strictly increasing in Λ , whereas $m_H(\Lambda)$ is continuous and strictly decreasing. Moreover,*

$$m_L(0) = m_H(0) < 0, \quad m_L(1) > 0.$$

Consequently, there exists a unique $\Lambda_c^{\log} \in (0, 1)$ such that

$$m_L(\Lambda_c^{\log}) = 0.$$

For $\Lambda > \Lambda_c^{\log}$, the logarithmic drift is positive in the lower regime and negative in the upper regime, whereas for $\Lambda < \Lambda_c^{\log}$ the lower-regime logarithmic drift remains negative.

Proof. Write $\mathcal{M}(p, q)$ for the expected logarithmic growth rate as a function of the two branch probabilities. Its partial derivatives are

$$\mathcal{M}_p = \log(1 + a(1 - p)) > 0, \quad \mathcal{M}_q = \log(1 - a(1 - q)) < 0.$$

In the unclipped interior,

$$p'(r) = v_b \zeta'(r) < 0, \quad q'(r) = -v_b \zeta'(r) > 0.$$

Hence, by the chain rule,

$$\begin{aligned} m'_L(\Lambda) &= r'_L(\Lambda) [\mathcal{M}_p p'(r_L) + \mathcal{M}_q q'(r_L)] \\ &= r'_L(\Lambda) v_b \zeta'(r_L) [\log(1 + a(1 - p_L)) - \log(1 - a(1 - q_L))]. \end{aligned}$$

Since $r'_L(\Lambda) < 0$, $\zeta'(r_L) < 0$, and the expression in brackets is positive, it follows that

$$m'_L(\Lambda) > 0.$$

Similarly,

$$m'_H(\Lambda) = r'_H(\Lambda) v_b \zeta'(r_H) [\log(1 + a(1 - p_H)) - \log(1 - a(1 - q_H))] < 0,$$

because $r'_H(\Lambda) > 0$. Thus m_L is strictly increasing and m_H is strictly decreasing. Continuity follows from the continuity of the policy rates, the branch probabilities, and $\mathcal{M}$.

At $\Lambda = 0$, both policy rates coincide with the neutral rate,

$$r_L(0) = r_H(0) = r^*.$$

By neutrality,

$$\mathbb{E}[M \mid r^*] = 1.$$

Since M is nondegenerate, strict Jensen's inequality gives

$$m_L(0) = m_H(0) = \mathbb{E}[\log M \mid r^*] < \log \mathbb{E}[M \mid r^*] = 0.$$

At $\Lambda = 1$, $r_L(1) = r_{q0}$ and hence $q(r_{q0}) = 0$. The gross multiplier is therefore equal to one or strictly greater than one, with positive probability of the latter. Consequently,

$$m_L(1) > 0.$$

Since m_L is continuous, $m_L(0) < 0$, and $m_L(1) > 0$, the Intermediate Value Theorem yields a $\Lambda_c^{\log} \in (0, 1)$ such that

$$m_L(\Lambda_c^{\log}) = 0.$$

Strict monotonicity of m_L makes this root unique and implies

$$m_L(\Lambda) < 0 \quad \text{for} \quad \Lambda < \Lambda_c^{\log}, \quad m_L(\Lambda) > 0 \quad \text{for} \quad \Lambda > \Lambda_c^{\log}.$$

Moreover, since $m_H(0) < 0$ and m_H is strictly decreasing,

$$m_H(\Lambda) < 0 \quad \text{for all } \Lambda > 0.$$

The stated result follows. □

The strict inequality $m_L(0) = m_H(0) < 0$ highlights the distinction between neutrality in levels and neutrality in logarithms. Although the neutral rate r^* implies zero expected level drift, multiplicative risk generates negative expected logarithmic growth through Jensen's inequality. Consequently, the lower policy regime must be sufficiently expansionary to offset this effect before it generates positive logarithmic drift away from the lower tail.

6.3.2 Economic interpretation

The signs $m'_L(\Lambda) > 0$ and $m'_H(\Lambda) < 0$ show that stronger feedback reinforces the restoring forces in both tails. As Λ increases, the lower policy rate $r_L(\Lambda)$ falls, raising profitability and strengthening the upward investment response when capital is low. Conversely, the upper policy rate $r_H(\Lambda)$ rises, reducing profitability and strengthening the downward investment response when capital is high. Hence stronger feedback moves the two tail log drifts in opposite but stabilizing directions.

The critical value $\Lambda_c^{\log}$ marks the point at which the lower-tail log drift changes sign. For $\Lambda > \Lambda_c^{\log}$, expected logarithmic growth is positive in the lower regime and negative in the upper regime, so policy generates inward log drift in both tails.

This condition, however, is not by itself sufficient for positive recurrence. For $\Lambda < 1$, downward transitions remain possible even in the low-rate regime, so positive lower-tail logarithmic drift does not imply a pathwise positive lower bound. The stochastic-stability analysis therefore proceeds on $(0, \infty)$ and establishes the stronger two-sided Foster–Lyapunov condition developed below.

6.4 Transition-kernel regularity

For regime $j \in \{L, *, H\}$, the upward gross multiplier has support

$$M_{j,+} \in \left[1 + \frac{\delta}{2v_b}(1 - p_j), 1 + \frac{\delta}{2v_b} \right].$$

Figure 2 illustrates how the regime-dependent supports generate a common subinterval on which the conditional log-increment densities are uniformly bounded away from zero.

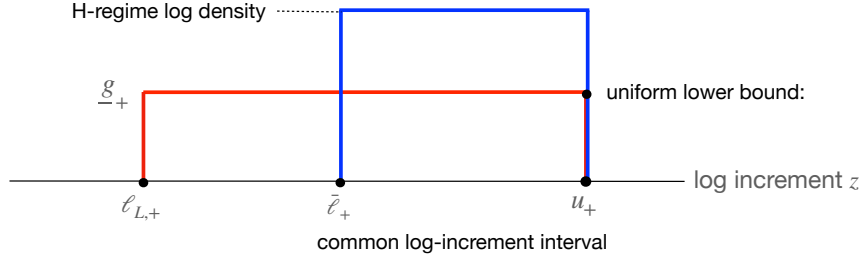


Figure 2: Regime-common support and uniform density bound for the upward log increment

The figure illustrates the conditional densities of the upward log increment $z = \log M$ in the two extreme policy regimes, L and H . Since $p_H \leq p_* \leq p_L$, the neutral-regime support lies between the two illustrated cases. The interval $[\bar{\ell}_+, u_+]$ is contained in the upward log-increment support of every regime. The horizontal line $\underline{g}_+$ anticipates the regime-uniform density lower bound established below. The analogous construction for the downward branch is omitted.

Accordingly, define

$$\ell_{j,+} := \log \left(1 + \frac{\delta}{2v_b}(1 - p_j) \right), \quad u_+ := \log \left(1 + \frac{\delta}{2v_b} \right),$$

so that the upward log-support is

$$[\ell_{j,+}, u_+], \quad 0 < \ell_{j,+} < u_+.$$

Only its lower endpoint depends on the policy regime.

Analogously, the downward gross multiplier has support

$$M_{j,-} \in \left[1 - \frac{\delta}{2v_b}, 1 - \frac{\delta}{2v_b}(1 - q_j) \right].$$

Define

$$\ell_- := \log \left(1 - \frac{\delta}{2v_b} \right), \quad u_{j,-} := \log \left(1 - \frac{\delta}{2v_b}(1 - q_j) \right).$$

The downward log-support is therefore

$$[\ell_-, u_{j,-}], \quad \ell_- < u_{j,-} < 0,$$

where only the upper endpoint depends on the policy regime.

Lemma 6.2 (Regime-uniform branch subintervals) *Define*

$$\bar{\ell}_+ := \max_{j \in \{L, *, H\}} \ell_{j,+}, \quad \bar{u}_- := \min_{j \in \{L, *, H\}} u_{j,-}.$$

*Then, for every regime $j \in \{L, *, H\}$,*

$$[\bar{\ell}_+, u_+] \subseteq [\ell_{j,+}, u_+], \quad (6.1)$$

$$[\ell_-, \bar{u}_-] \subseteq [\ell_-, u_{j,-}]. \quad (6.2)$$

Proof. By definition,

$$\ell_{j,+} \leq \bar{\ell}_+ \quad \text{and} \quad \bar{u}_- \leq u_{j,-}$$

for every $j \in \{L, *, H\}$. Since u_+ and ℓ_- are regime-independent, the two stated interval inclusions follow immediately. $\square$

Lemma 6.3 (Open-set accessibility) *Assume that the upward and downward branches have strictly positive probabilities $p_j, q_j > 0$ for each regime $j \in \{L, *, H\}$. For every initial state $K_0 > 0$ and every nonempty open target interval $J \subset (0, \infty)$, there exist integers $m, n \geq 0$, a finite ordering of these $m + n$ branch choices, and an open subinterval of multiplier realizations such that the resulting K_{m+n} lies in J .*

Proof. Let $K_0 > 0$, and let $J \subset (0, \infty)$ be a nonempty open target interval. Choose $K_y \in J$, and write

$$x_0 = \log K_0, \quad y = \log K_y.$$

Since J is open, we may choose $\eta_y > 0$ sufficiently small that

$$J_x := (y - \eta_y, y + \eta_y)$$

satisfies

$$\exp(J_x) \subset J.$$

Likewise, choose $\eta_0 > 0$ and define

$$U_x := (x_0 - \eta_0, x_0 + \eta_0), \quad U_0 := \exp(U_x).$$

We prove the stronger local statement that the same finite construction works uniformly for every initial log-state $x \in U_x$. It is therefore sufficient to show that

$$J_x \subset \bigcap_{x \in U_x} I_{m,n}^0(x).$$

From inclusion relationships (6.1) and (6.2), the regime-uniform inner log-multiplier intervals are

$$[\bar{\ell}_+, u_+] \quad \text{and} \quad [\ell_-, \bar{u}_-],$$

with

$$0 < \bar{\ell}_+ < u_+, \quad \ell_- < \bar{u}_- < 0.$$

The two quantities controlling the width of the log-support are exactly

$$d_+ := u_+ - \bar{\ell}_+ > 0,$$

and

$$d_- := \bar{u}_- - \ell_- > 0.$$

The common attainable log-state interval generated from x after m upward and n downward moves is

$$I_{m,n}^0(x) = [x + m\bar{\ell}_+ + n\ell_-, x + mu_+ + n\bar{u}_-]. \quad (6.3)$$

Its width is therefore

$$|I_{m,n}^0(x)| = md_+ + nd_-.$$

Define

$$A_{m,n} := m\bar{\ell}_+ + n\ell_-, \quad B_{m,n} := mu_+ + n\bar{u}_-.$$

Then the midpoint of the attainable interval is expressed as:

$$C_{m,n} := \frac{A_{m,n} + B_{m,n}}{2} = mc_+ + nc_-,$$

where

$$c_+ := \frac{\bar{\ell}_+ + u_+}{2} > 0, \quad \text{and} \quad c_- := \frac{\ell_- + \bar{u}_-}{2} < 0.$$

Let $D = y - x_0$ denote the displacement between the centers of the target and initial neighborhoods in log coordinates; D may have either sign. We also define e by $e := \eta_0 + \eta_y$.

Then to ensure

$$J_x \subset I_{m,n}^0(x) \quad \text{for every } x \in U_x,$$

it follows from (6.3) that the pair of integers (m, n) must satisfy:

$$\underbrace{x_0 + \eta_0 + m\bar{\ell}_+ + n\ell_-}_{\text{largest possible left endpoint}} < \underbrace{y - \eta_y}_{\text{left target edge}} < \underbrace{y + \eta_y}_{\text{right target edge}} < \underbrace{x_0 - \eta_0 + mu_+ + n\bar{u}_-}_{\text{smallest possible right endpoint}},$$

which is equivalent to

$$m\bar{\ell}_+ + n\ell_- < y - x_0 - \eta_y - \eta_0, \quad y - x_0 + \eta_y + \eta_0 < mu_+ + n\bar{u}_-.$$

Or, equivalently,

$$A_{m,n} + e < D < B_{m,n} - e.$$

A simple calculation shows that this is equivalent to

$$|D - C_{m,n}| < \frac{md_+ + nd_-}{2} - e.$$

The midpoint and half-width of the attainable interval are therefore

$$C_{m,n} = mc_+ + nc_-, \quad H_{m,n} := \frac{B_{m,n} - A_{m,n}}{2} = \frac{md_+ + nd_-}{2},$$

respectively. Thus $C_{m,n}$ determines its location, while $H_{m,n}$ determines its half-width.
Define

$$r_0 := -\frac{c_-}{c_+} = -\frac{\ell_- + \bar{u}_-}{\ell_+ + u_+} > 0, \quad m_n := \text{round}\left(r_0 n + \frac{D}{c_+}\right),$$

which immediately gives

$$\left| m_n - r_0 n - \frac{D}{c_+} \right| \leq \frac{1}{2}.$$

Since $r_0 > 0$, we have $m_n \geq 0$ for all sufficiently large n . Therefore the error in the midpoint is bounded

$$|C_{m_n,n} - D| = c_+ \left| m_n - r_0 n - \frac{D}{c_+} \right| \leq \frac{c_+}{2} = O(1).$$

Meanwhile, the half-width is

$$H_{m_n,n} = \frac{m_n d_+ + n d_-}{2} \sim \frac{n}{2}(r_0 d_+ + d_-) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Hence choose finite n sufficiently large that

$$H_{m_n,n} - e > \frac{c_+}{2}.$$

It follows immediately that

$$|D - C_{m_n,n}| \leq \frac{c_+}{2} < H_{m_n,n} - e,$$

which is the required containment condition.

The construction therefore yields finite m_n, n for which the same common attainable interval contains J_x for every $x \in U$. Fix any ordering E of the m_n upward and n downward branch choices. Since $p_j, q_j > 0$ in every regime, E has strictly positive probability. This construction is unaffected by policy-regime crossings, and any fixed ordering of the m_n upward and n downward branch choices is admissible. Moreover, because the conditional multiplier distributions assign positive probability to every nonempty open subinterval of the regime-uniform branch supports, the set of multiplier realizations leading into J_x has strictly positive conditional probability. Hence, from every $K \in U_0$, the process reaches J after $m_n + n$ steps with positive probability. In particular, the asserted open-set accessibility follows. $\square$

Figure 3 illustrates the geometry underlying the construction: admissible integer pairs (m, n) are obtained as lattice points lying within a band around the linear relation that matches the target displacement.

6.5 Continuous component and the T -chain property

The regime-switching policy introduces a technical complication for the analysis of the transition kernel. Because the upward and downward transition probabilities change

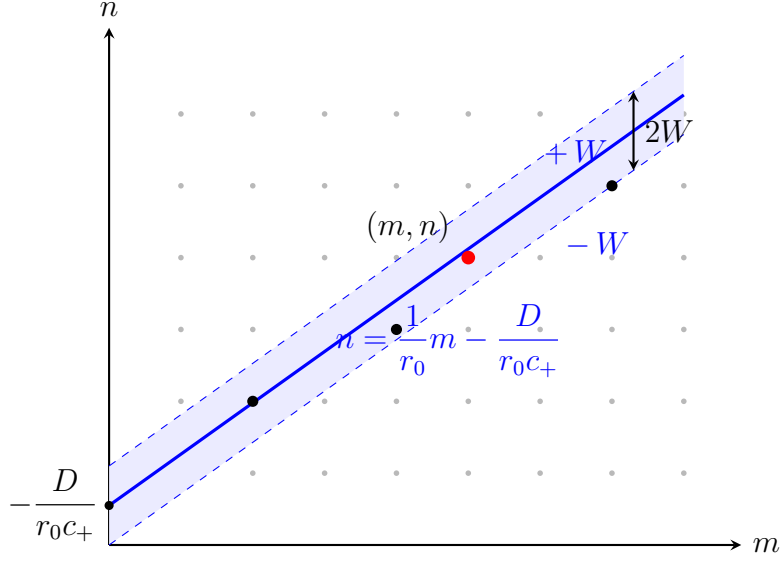


Figure 3: Admissible integer pairs in the accessibility construction.

The midpoint condition $mc_+ + nc_- \approx D$ defines a linear relation between the numbers of upward and downward moves. Since m and n must be integers, admissible pairs correspond to lattice points lying within a band around this line. As the numbers of moves increase, the half-width of the common attainable interval grows linearly, while the rounding error in its midpoint remains bounded. Hence finite admissible integer pairs can always be found.

discontinuously when the state crosses a policy threshold, the one-step kernel need not possess the continuity properties that would permit a direct application of standard compact-set arguments. This discontinuity does not, however, preclude the process from possessing a sufficiently regular component over a finite number of transitions. We therefore establish that the process is a T -chain. The T -chain property requires a sampled transition kernel to admit a nontrivial continuous component and thus provides the appropriate regularity notion when the original transition kernel is not itself continuous. In the present model, this continuous component is generated by the continuous variation of multiplier realizations over common regime-uniform subintervals, together with the finite-step accessibility established above.

Lemma 6.4 (Uniform one-step log-density bounds) *Suppose $\Lambda \in (\Lambda_c, 1)$, so that the upward and downward branch probabilities are strictly positive in every policy regime. Then there exist constants*

$$\underline{g}_+ > 0, \quad \underline{g}_- > 0,$$

*such that, for every regime $j \in \{L, *, H\}$,*

$$g_{j,+}(z) \geq \underline{g}_+ \quad \text{for all } z \in [\bar{\ell}_+, u_+],$$

and

$$g_{j,-}(z) \geq \underline{g}_- \quad \text{for all } z \in [\ell_-, \bar{u}_-].$$

Proof. First, we show that the one-step upward and downward conditional branches have strictly positive densities on common regime-uniform subintervals. For the gross multiplier U , write the upward support in regime j as

$$U \in [a_{j,+}, b_+],$$

and the downward support as

$$U \in [a_-, b_{j,-}].$$

The corresponding common gross-multiplier subintervals are

$$[\bar{a}_+, b_+] \quad \text{and} \quad [a_-, \bar{b}_-].$$

Now define the log increment

$$Z = \log U.$$

For the upward branch, let

$$\ell_{j,+} := \log a_{j,+}, \quad u_+ := \log b_+, \quad \bar{\ell}_+ := \log \bar{a}_+.$$

Thus the common upward log interval is

$$[\bar{\ell}_+, u_+].$$

Similarly, for the downward branch, let

$$\ell_- := \log a_-, \quad u_{j,-} := \log b_{j,-}, \quad \bar{u}_- := \log \bar{b}_-,$$

so that the common downward log interval is

$$[\ell_-, \bar{u}_-].$$

Conditional on an upward branch in regime j , the density of U is

$$f_{j,+}(u) = \frac{2v_b}{\delta p_j}.$$

Hence the density of $Z = \log U$ is

$$g_{j,+}(z) = f_{j,+}(e^z) \frac{de^z}{dz} = e^z \frac{2v_b}{\delta p_j}.$$

On the common upward log interval $z \in [\bar{\ell}_+, u_+]$, we have

$$e^z \geq \bar{a}_+, \quad p_j \leq p_L,$$

and therefore

$$g_{j,+}(z) \geq \frac{2v_b \bar{a}_+}{\delta p_L} =: \underline{g}_+ > 0.$$

Analogously, conditional on a downward branch,

$$g_{j,-}(z) = e^z \frac{2v_b}{\delta q_j}.$$

On the common downward log interval $z \in [\ell_-, \bar{u}_-]$, we have

$$e^z \geq a_-, \quad q_j \leq q_H,$$

so

$$g_{j,-}(z) \geq \frac{2v_b a_-}{\delta q_H} =: \underline{g}_- > 0.$$

□

Lemma 6.5 (T-chain property and petite compact sets) *Suppose $\Lambda \in (\Lambda_c, 1)$. Then the log-capital process $\{X_t\}$ is a T-chain. Together with Lemma 6.3, this implies that the chain is ψ -irreducible and that every compact subset of $\mathbb{R}$ is petite.*

Proof. See Appendix B. □

Since the transformation $K \mapsto X = \log K$ is a homeomorphism from $(0, \infty)$ onto $\mathbb{R}$, the corresponding properties carry over to the capital-stock chain. Thus the K -process is ψ -irreducible and a T-chain, and every compact subset of $(0, \infty)$ is petite.

6.6 Foster–Lyapunov stability

6.6.1 Two-sided Lyapunov drift

To apply the Foster – Lyapunov criterion, we work with the Lyapunov function

$$V(K) = K + \frac{1}{K}.$$

It controls both tails:

$$V(K) \rightarrow \infty \quad \text{as } K \rightarrow \infty,$$

and

$$V(K) \rightarrow \infty \quad \text{as } K \downarrow 0.$$

More generally, replacing $V(K) = K + K^{-1}$ by $V_\chi(K) = K + \chi K^{-1}$ for any $\chi > 0$ leaves the critical feedback strength unchanged, since the lower-tail drift condition depends on the sign of $\mathbb{E}[M_L^{-1}] - 1$ rather than on the positive coefficient χ .

Under policy regime $j \in \{L, *, H\}$, define

$$A_j(\Lambda) := \mathbb{E}[M_j(\Lambda)] - 1, \quad B_j(\Lambda) := \mathbb{E}[M_j(\Lambda)^{-1}] - 1.$$

For K in regime j ,

$$PV(K) - V(K) = A_j(\Lambda)K + \frac{B_j(\Lambda)}{K}.$$

Lemma 6.6 (Negative upper-tail drift) *For every $\Lambda > 0$, the high-rate regime satisfies $A_H(\Lambda) = \mathbb{E}[M_H] - 1 < 0$.*

Proof. For every $\Lambda > 0$, we have $r_H(\Lambda) > r^*$. Since $\zeta(r)$ is strictly decreasing in r , $\zeta(r_H) < \zeta(r^*) = \zeta^*$.

The difference between the upward- and downward-transition probabilities is

$$p_H - q_H = v_b[2\zeta(r_H) + 1 + c_b - 2\varphi] = 2v_b[\zeta(r_H) - \zeta^*] < 0.$$

Hence, $p_H < q_H$.

Define $g(x) = x(1 - x/2)$. Since $g'(x) = 1 - x > 0$ for $x \in [0, 1/2]$, g is strictly increasing over the admissible range of p_H and q_H . Therefore,

$$A_H = \frac{\delta}{2v_b}[g(p_H) - g(q_H)] < 0.$$

□

6.6.2 The Lyapunov stability threshold

Lemma 6.7 (Lyapunov critical feedback strength) *Let*

$$G_L(\Lambda) := \mathbb{E}[M_L(\Lambda)^{-1}], \quad \Lambda \in [0, 1],$$

where

$$a = \frac{\delta}{2v_b} \in (0, 1).$$

Suppose that, for $\Lambda \in [0, 1)$, the lower-policy probabilities remain in the unclipped region, so that

$$p_L(\Lambda) + q_L(\Lambda) = 1 - v_b(1 + c_b),$$

with

$$p'_L(\Lambda) > 0, \quad q'_L(\Lambda) < 0.$$

Suppose further that

$$m_L(0) = \mathbb{E}[\log M_L(0)] < 0,$$

and that at the hard-corridor endpoint $\Lambda = 1$,

$$q_L(1) = 0, \quad p_L(1) > 0.$$

Then $G_L(\Lambda)$ is continuous and strictly decreasing on $[0, 1]$, satisfies

$$G_L(0) > 1 \quad \text{and} \quad G_L(1) < 1,$$

and hence there exists a unique

$$\Lambda_c^V \in (0, 1)$$

such that

$$\mathbb{E}[M_L(\Lambda_c^V)^{-1}] = 1.$$

Moreover,

$$\mathbb{E}[M_L(\Lambda)^{-1}] \begin{cases} > 1, & 0 \leq \Lambda < \Lambda_c^V, \\ = 1, & \Lambda = \Lambda_c^V, \\ < 1, & \Lambda_c^V < \Lambda \leq 1. \end{cases}$$

Proof. Conditional on an upward transition,

$$M_{L,+} \sim \text{Unif}[1 + a(1 - p_L), 1 + a],$$

whereas, conditional on a downward transition,

$$M_{L,-} \sim \text{Unif}[1 - a, 1 - a(1 - q_L)].$$

The remaining probability $1 - p_L - q_L$ is assigned to the no-adjustment outcome $M_L = 1$. Therefore,

$$G_L(\Lambda) = 1 - p_L - q_L + \frac{1}{a} \log \frac{1 + a}{1 + a(1 - p_L)} + \frac{1}{a} \log \frac{1 - a(1 - q_L)}{1 - a}.$$

Continuity of $p_L(\Lambda)$ and $q_L(\Lambda)$ implies continuity of $G_L(\Lambda)$.

In the unclipped region,

$$p_L + q_L = 1 - v_b(1 + c_b)$$

is constant. Hence q_L may be regarded as a decreasing affine function of p_L . Differentiating G_L with respect to p_L gives

$$\frac{dG_L}{dp_L} = \frac{1}{1 + a(1 - p_L)} - \frac{1}{1 - a(1 - q_L)}.$$

Since

$$1 + a(1 - p_L) > 1 \quad \text{and} \quad 1 - a(1 - q_L) < 1,$$

it follows that

$$\frac{dG_L}{dp_L} < 0.$$

Moreover,

$$p'(r) = v_b \zeta'(r) < 0$$

and

$$r'_L(\Lambda) = -(r^* - r_{q0}) < 0,$$

so that

$$p'_L(\Lambda) = p'(r_L(\Lambda))r'_L(\Lambda) > 0.$$

Consequently,

$$G'_L(\Lambda) < 0$$

throughout the unclipped region. By continuity, G_L is strictly decreasing on $[0, 1]$.

At $\Lambda = 0$, Jensen's inequality yields

$$\log G_L(0) = \log \mathbb{E}[M_L(0)^{-1}] \geq \mathbb{E}[\log M_L(0)^{-1}] = -m_L(0).$$

Since $m_L(0) < 0$,

$$\log G_L(0) > 0,$$

and therefore

$$G_L(0) > 1.$$

At $\Lambda = 1$, the lower-policy rate reaches r_{q0} , so $q_L(1) = 0$. Hence the multiplier is either equal to one or belongs to the upward branch, on which $M_{L,+} > 1$ almost surely. Since $p_L(1) > 0$,

$$\mathbb{E}[M_L(1)^{-1}] < 1,$$

and therefore

$$G_L(1) < 1.$$

Thus G_L is continuous on $[0, 1]$ and satisfies

$$G_L(0) > 1 > G_L(1).$$

The intermediate value theorem therefore guarantees the existence of some $\Lambda_c^V \in (0, 1)$ such that

$$G_L(\Lambda_c^V) = 1.$$

Strict monotonicity of G_L implies that this solution is unique. The stated inequalities on either side of Λ_c^V follow immediately. $\square$

6.6.3 Relation between the two critical thresholds

Corollary 6.8 (Threshold ordering) *Suppose that $M_L(\Lambda_c^{\log})$ is nondegenerate. Then*

$$\Lambda_c^V > \Lambda_c^{\log}.$$

Proof. At $\Lambda = \Lambda_c^{\log}$,

$$\mathbb{E}[\log M_L] = 0.$$

By the strict form of Jensen's inequality,

$$\log \mathbb{E}[M_L^{-1}] > \mathbb{E}[\log M_L^{-1}] = -\mathbb{E}[\log M_L] = 0.$$

Hence

$$\mathbb{E}[M_L^{-1}] > 1$$

at $\Lambda = \Lambda_c^{\log}$. Since $\mathbb{E}[M_L(\Lambda)^{-1}]$ is strictly decreasing in Λ and equals one uniquely at Λ_c^V , it follows that $\Lambda_c^V > \Lambda_c^{\log}$. $\square$

The logarithmic threshold $\Lambda_c^{\log}$ therefore marks the onset of positive expected logarithmic growth in the lower regime, whereas the stronger threshold Λ_c^V guarantees

the inverse-moment condition required for the Foster–Lyapunov argument³. Since the remainder of the stochastic-stability analysis concerns the latter condition, we henceforth write

$$\Lambda_c := \Lambda_c^V.$$

6.7 Invariant distribution and geometric convergence

6.7.1 Geometric Foster–Lyapunov drift

Lemma 6.9 (Geometric Foster–Lyapunov drift) *Let*

$$V(K) = K + \frac{1}{K}.$$

For each fixed $\Lambda \in (\Lambda_c, 1)$,

$$\lim_{K \downarrow 0} \frac{PV(K)}{V(K)} = \mathbb{E}[M_L(\Lambda)^{-1}] < 1$$

and

$$\lim_{K \rightarrow \infty} \frac{PV(K)}{V(K)} = \mathbb{E}[M_H(\Lambda)] < 1.$$

Consequently, there exist $\rho_\Lambda \in (0, 1)$, $b_\Lambda < \infty$, and a compact interval

$$C_\Lambda = [\underline{K}_\Lambda, \overline{K}_\Lambda] \subset (0, \infty)$$

such that

$$PV(K) \leq \rho_\Lambda V(K) + b_\Lambda \mathbf{1}_{C_\Lambda}(K), \quad K > 0.$$

Proof. In any fixed regime j ,

$$PV(K) = \mathbb{E} \left[M_j K + \frac{1}{M_j K} \right] = K \mathbb{E}[M_j] + \frac{1}{K} \mathbb{E}[M_j^{-1}].$$

Hence,

$$\frac{PV(K)}{V(K)} = \frac{K^2 \mathbb{E}[M_j] + \mathbb{E}[M_j^{-1}]}{K^2 + 1}.$$

For sufficiently small K , we are in the low-rate regime L . Thus

$$\frac{PV(K)}{V(K)} = \frac{K^2 \mathbb{E}[M_L] + \mathbb{E}[M_L^{-1}]}{K^2 + 1}.$$

As $K \rightarrow 0$,

$$\lim_{K \downarrow 0} \frac{PV(K)}{V(K)} = \mathbb{E}[M_L^{-1}].$$

³Under the current baseline parameter setting, `policy_stability_thresholds.py` calculates $\Lambda_c^{\log} = 0.02187$, and $\Lambda_c^V = 0.04361$.

And by the definition of the Lyapunov threshold,

$$\Lambda > \Lambda_c^V \implies \mathbb{E}[M_L^{-1}] < 1.$$

For sufficiently large K , we are in the high-rate regime H . Hence

$$\frac{PV(K)}{V(K)} = \frac{K^2 \mathbb{E}[M_H] + \mathbb{E}[M_H^{-1}]}{K^2 + 1}.$$

Dividing numerator and denominator by K^2 , or simply taking the limit,

$$\lim_{K \rightarrow \infty} \frac{PV(K)}{V(K)} = \mathbb{E}[M_H],$$

which is strictly less than unity by Lemma 6.6. By the two strict inequalities, choose $\rho_\Lambda < 1$ larger than both limiting values. There then exist $0 < \underline{K}_\Lambda < K_L$ and $\overline{K}_\Lambda > K_H$ such that

$$PV(K) \leq \rho_\Lambda V(K)$$

outside $C_\Lambda = [\underline{K}_\Lambda, \overline{K}_\Lambda]$. On the compact set C_Λ , define an appropriate finite b_Λ , yielding the stated inequality. $\square$

Figure 4 provides a numerical illustration of the Foster–Lyapunov construction for the representative case $\Lambda = 0.50$. The figure is intended to clarify the geometry of the drift argument rather than to provide a comparative numerical exercise; the latter is undertaken in Sect. 7.2.

6.7.2 Establishing positive Harris recurrence

Theorem 6.10 (Positive Harris recurrence) *For each fixed $\Lambda \in (\Lambda_c, 1)$, the capital-stock chain is positive Harris recurrent.*

Proof. By Lemma 6.9,

$$PV(K) - V(K) \leq -(1 - \rho_\Lambda)V(K), \quad K \notin C_\Lambda.$$

Since

$$V(K) = K + \frac{1}{K} \geq 2,$$

it follows that

$$PV(K) - V(K) \leq -2(1 - \rho_\Lambda) =: -\varepsilon_\Lambda < 0, \quad K \notin C_\Lambda.$$

Because C_Λ is bounded away from both zero and infinity, V is bounded on C_Λ . Moreover, since the supports of the multipliers $M_{j,-}$ and $M_{j,+}$ are bounded away from zero and above by finite constants, PV is also bounded on C_Λ . Hence one can choose a finite constant b_Λ satisfying

$$b_\Lambda \geq \sup_{K \in C_\Lambda} \{PV(K) - V(K) + \varepsilon_\Lambda\}.$$

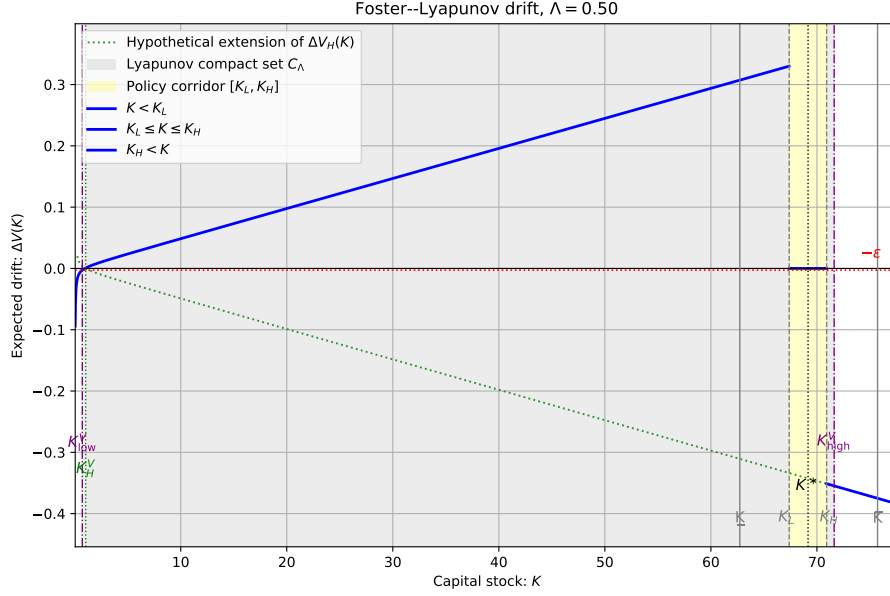


Figure 4: Foster–Lyapunov drift under the feedback interest-rate rule

The solid curve shows the conditional drift $\Delta V_\Lambda(K)$ of the Lyapunov function. The vertical lines indicate the relevant capital-stock thresholds, while the shaded region denotes the compact set $C_\Lambda = [\underline{K}_\Lambda, \bar{K}_\Lambda]$. The horizontal dashed line at $-\varepsilon_\Lambda$ represents a strictly negative uniform drift bound. Outside the small set, the drift is bounded above by $-\varepsilon_\Lambda < 0$, illustrating the restoring force required by the Foster–Lyapunov criterion.

Therefore,

$$PV(K) \leq V(K) - \varepsilon_\Lambda + b_\Lambda \mathbf{1}_{C_\Lambda}(K), \quad K \in \mathcal{K}.$$

By Lemma 6.5, the compact set C_Λ is petite. Rescaling V by the positive constant ε_Λ^{-1} converts the drift inequality into condition (V2) of Meyn and Tweedie (1993, Theorem 11.3.4), with unit negative drift outside C_Λ . Hence, the hypotheses of that theorem are satisfied, and the chain is positive Harris recurrent. $\square$

6.7.3 Unique invariant distribution

Corollary 6.11 (Existence and uniqueness of the invariant distribution) *For each fixed $\Lambda \in (\Lambda_c, 1)$, the capital-stock chain admits a unique invariant probability distribution π_Λ .*

Proof. By Theorem 6.10, the chain is positive Harris recurrent. Since it is also ψ -irreducible, positive Harris recurrence implies the existence of an invariant probability measure. Moreover, the invariant probability measure of a positive Harris recurrent ψ -irreducible chain is unique. Hence there exists a unique invariant probability distribution π_Λ . $\square$

6.7.4 Establishing aperiodicity

Lemma 6.12 (Aperiodicity) *For each fixed $\Lambda \in (\Lambda_c, 1)$, the capital-stock chain is aperiodic.*

Proof. Choose a nonempty compact interval

$$D \subset (K_L, K_H)$$

strictly inside the neutral policy region. By the local minorization established in Appendix B, there exist an integer $N \geq 1$, a constant $\epsilon > 0$, and a nontrivial finite measure ν such that

$$P^N(K, A) \geq \epsilon \nu(A), \quad K \in D,$$

for every Borel set A . Thus D is N -small.

For every $K \in D$, the process is in the neutral regime and the no-adjustment probability satisfies

$$s_* = 1 - p_* - q_* > 0.$$

Hence

$$P(K, \{K\}) \geq s_*.$$

Using the stay realization in the first step and then the N -step minorization gives

$$\begin{aligned} P^{N+1}(K, A) &= \int P^N(K', A) P(K, dK') \\ &\geq P(K, \{K\}) P^N(K, A) \\ &\geq s_* \epsilon \nu(A), \quad K \in D. \end{aligned}$$

Therefore the same set D admits minorization at both step lengths N and $N + 1$. Since

$$\gcd(N, N + 1) = 1,$$

the ψ -irreducible chain has period one and is therefore aperiodic. $\square$

6.7.5 Geometric ergodicity

Theorem 6.13 (V-geometric convergence) *For each fixed $\Lambda \in (\Lambda_c, 1)$, there exist constants $r_\Lambda > 1$ and $R_\Lambda < \infty$ such that*

$$\sum_{n=0}^{\infty} r_\Lambda^n \|P^n(K, \cdot) - \pi_\Lambda\|_V \leq R_\Lambda V(K), \quad K > 0.$$

Proof. By Lemma 6.5, the chain is ψ -irreducible and compact sets are petite. By Lemma 6.12, it is aperiodic. Moreover, Lemma 6.9 gives

$$PV(K) \leq \rho_\Lambda V(K) + b_\Lambda \mathbf{1}_{C_\Lambda}(K), \quad \rho_\Lambda < 1,$$

for a petite compact set C_Λ . Hence the geometric Foster–Lyapunov criterion of Meyn and Tweedie (1993, Theorem 15.0.1) applies.

Since

$$V(K) = K + \frac{1}{K} < \infty \quad \text{for every } K > 0,$$

we have $S_V = (0, \infty)$. Therefore there exist $r_\Lambda > 1$ and $R_\Lambda < \infty$ such that

$$\sum_{n=0}^{\infty} r_\Lambda^n \|P^n(K, \cdot) - \pi_\Lambda\|_V \leq R_\Lambda V(K), \quad K > 0.$$

□

7 Numerical Results

This section provides the numerical counterpart to the stability analysis of Sect. 6 and examines its quantitative implications. Sections 7.1–7.2 describe the numerical implementation and evaluate the critical feedback strengths and Foster–Lyapunov conditions. Section 7.3 examines how the resulting transition mechanism is reflected in the invariant distribution of the capital stock. Section 7.4 complements these long-run results with finite-horizon first-passage risks, while Sect. 7.5 examines how frequently stabilization under the threshold rule requires actual policy-rate adjustment.

7.1 Calibration and numerical kernel

Table 5 presents the numerical parameter values of the baseline model, which are identical with those used in Takahashi (2026). The parameter $c = c_w/(1 - c_y)$ summarizes the consumption side of effective demand. Although it is constructed from the underlying consumption propensities, its role in the present model differs from that of a conventional saving parameter: a higher c strengthens effective demand and utilization and thereby supports output and capital formation. The baseline values $\alpha = 1/3$ and $\beta = 1/2$ imply $\gamma := \alpha/(1 - \alpha) = \beta$, consistent with the exponent-matching assumption maintained throughout the paper.

In particular, the stochastic investment process implies the support bounds

$$\underline{K} = \left(1 - \frac{\delta}{2v_b}\right) K_{q0}, \quad \overline{K} = \left(1 + \frac{\delta}{2v_b}\right) K_{p0}.$$

The wider interval $[K_D, K_F]$ defines the fixed-wage region within which the wage-adjustment mechanism remains inactive. Since the stochastic support is contained in the narrower interval $[\underline{K}, \overline{K}] \subset (K_D, K_F)$, the stationary dynamics generated by the interest-rate feedback rule remain within the fixed-wage region. These bounds are derived quantities rather than independently calibrated parameters. For numerical implementation, the computational state space is chosen to contain $[\underline{K}, \overline{K}]$ with an additional margin controlled by the enlargement parameter ρ . The resulting grid endpoints are numerical truncation bounds and should not be interpreted as boundaries of the theoretical state space $\mathcal{K} = (0, \infty)$.

Table 5: Baseline model and numerical parameter values

Parameter	Description	Baseline value
<i>Model parameters</i>		
α	Capital income share	1/3
η	Competitive price elasticity	7.0
δ	Depreciation rate	0.06
β	Consumption-money substitution elasticity	1/2
γ	$\alpha/(1 - \alpha)$	1/2
c	Effective-demand parameter, $c = c_w/(1 - c_y)$	0.175
$v_b^{(a)}$	Stickiness in revising transition probabilities	0.7
$M^{(b)}$	Benchmark money stock ^b	1000
W	Nominal wage	25.0
λ	Transactions-demand coefficient	0.1
$\bar{N}^{(c)}$	Labor endowment	12.0
<i>Numerical parameters</i>		
ρ	Computational state-space enlargement parameter	0.4
N_{base}	Number of intervals in the baseline capital grid	1000
ε_{tol}	Numerical tolerance	10^{-12}

^(a) The parameter v_b is defined as $v_b = c_b/V$, where V denotes the sentiment shock amplitude (half-range) of the aggregate economy-wide sentiment shock.

^(b) Here, M denotes the benchmark money stock used solely to construct the quantity-closure mapping $K(r)$; it is not an exogenous money stock under the interest-rate closure studied in this paper.

^(c) The labor endowment is chosen so that the wage-adjustment thresholds lie outside the recurrent region, $K_D < \underline{K} < \bar{K} < K_F$. Nominal wages therefore remain fixed over the region relevant for the stationary comparison.

For each grid point, the full conditional distributions of the upward and downward capital multipliers derived in Sect. 6 are mapped directly onto the capital grid. Conditional on an upward or downward adjustment, the destination capital stock is uniformly distributed over the corresponding state-dependent interval. Probability mass is allocated across all grid cells intersecting that interval in proportion to their overlap lengths, yielding a finite-state approximation to the full transition kernel.

Together with the probability of no adjustment, this procedure yields a finite-state transition matrix P_Λ whose rows sum to one. The numerical invariant distribution π_Λ is obtained from

$$P_\Lambda^\top \pi_\Lambda \simeq \pi_\Lambda, \quad \sum_i \pi_{\Lambda,i} = 1.$$

The numerical resolutions are chosen sufficiently fine to provide a stable approximation to the reported results.

As a check on the finite-state approximation, we recompute the invariant distribution after progressively enlarging the computational state space while holding the remaining numerical settings fixed. The resulting distributions are essentially unchanged over the economically relevant region, indicating that the reported results are not driven by the choice of numerical truncation bounds. Table 9 reports the corresponding robustness measures.

7.2 Numerical stability thresholds

This subsection is the numerical counterpart of the stability theory in Section 6. Its central question is: How strong must feedback be to generate the required restoring drift?

The case $\Lambda = 1$ is included as a limiting hard-corridor benchmark. Although the Lyapunov tail inequalities remain satisfied at this endpoint, the irreducibility and T-chain arguments underlying the positive-Harris-recurrence and geometric-ergodicity results in Sect. 6 are established for $\Lambda \in (\Lambda_c, 1)$ and do not directly cover $\Lambda = 1$. The numerical results for $\Lambda = 1$ should therefore be interpreted separately from the interior soft-corridor cases.

Critical feedback strengths

The logarithmic threshold $\Lambda_c^{\log}$ is defined by the condition

$$\mathbb{E}[\log M_L(\Lambda_c^{\log})] = 0,$$

whereas the Foster–Lyapunov threshold Λ_c^V is defined by

$$\mathbb{E}[M_L(\Lambda_c^V)^{-1}] = 1.$$

For each value of the feedback-strength parameter Λ , these parameters determine the policy thresholds K_L and K_H , the regime-specific transition probabilities, and the conditional distributions of the gross capital multiplier.

The numerical results confirm the analytical ordering

$$\Lambda_c^{\log} < \Lambda_c^V,$$

showing that the Foster–Lyapunov condition requires a stronger feedback rule than the weaker condition of nonnegative expected logarithmic growth.

Under the baseline calibration, the logarithmic threshold is $\Lambda_c^{\log} = 0.021817$, while the stronger Foster–Lyapunov threshold is $\Lambda_c^V = 0.043610$.

The numerical gap between the two thresholds is economically meaningful. The condition $\Lambda > \Lambda_c^{\log}$ guarantees positive expected logarithmic growth in the lower regime, but this alone is insufficient for stochastic stability. Geometric ergodicity requires the stronger inverse-moment condition $\Lambda > \Lambda_c^V$, which ensures the two-sided Foster–Lyapunov drift established in Section 6.

Figure 5 provides the numerical counterpart of Theorem 6.1. Under the baseline calibration, it traces the expected logarithmic growth rates $m_L(\Lambda)$ and $m_H(\Lambda)$ as feedback strength varies. Consistent with the theorem, $m_L(\Lambda)$ is strictly increasing, whereas $m_H(\Lambda)$ is strictly decreasing. The lower-regime curve crosses zero at the unique critical value $\Lambda_c^{\log}$, beyond which the expected logarithmic growth rates have the stabilizing sign pattern $m_L(\Lambda) > 0 > m_H(\Lambda)$.

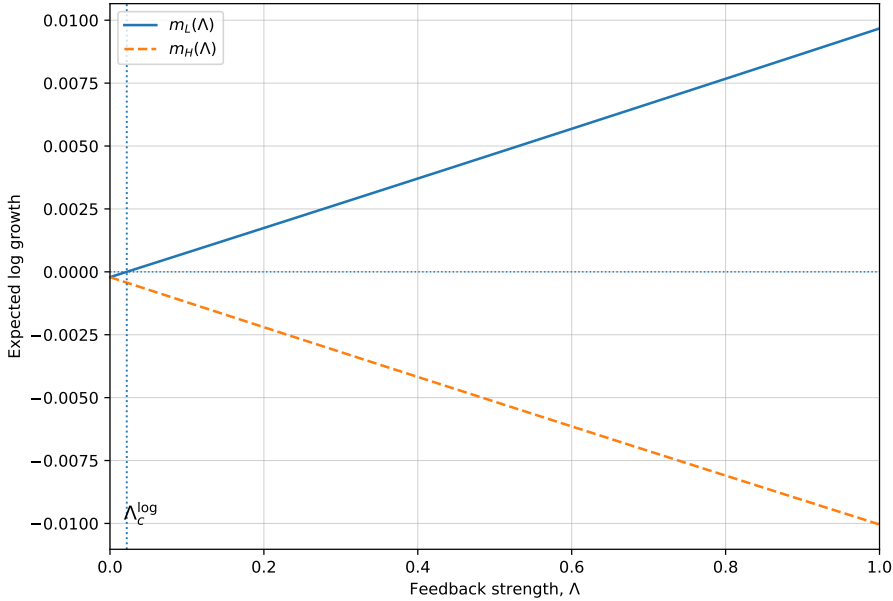


Figure 5: Numerical evaluation of the expected logarithmic growth rates

Expected logarithmic growth rates in the lower and upper policy regimes as functions of the feedback-strength parameter $\Lambda \in [0, 1]$. Here, $m_L(\Lambda) = \mathbb{E}[\log(K_{t+1}/K_t) \mid r_t = r_L(\Lambda)]$ and $m_H(\Lambda) = \mathbb{E}[\log(K_{t+1}/K_t) \mid r_t = r_H(\Lambda)]$. The lower-regime rate is strictly increasing and crosses zero at the unique critical value $\Lambda_c^{\log}$, whereas the upper-regime rate remains negative and decreases monotonically. Hence, for $\Lambda > \Lambda_c^{\log}$, expected logarithmic growth is positive in the lower regime and negative in the upper regime, yielding restoring pressure from both tails. The curves are numerical evaluations of the analytical log-growth formulas under the baseline calibration.

The Taylor-rule literature emphasizes not just responding to economic conditions, but how strongly the policy rate responds. In this respect, the critical-feedback result shares with the broader policy-rule literature the principle that stabilization depends not merely on whether the policy rate responds to economic conditions, but also on whether that response is sufficiently strong.

Foster–Lyapunov drift coefficients

To evaluate the Foster–Lyapunov conditions numerically, define

$$A_j(\Lambda) = \mathbb{E}[M_j(\Lambda)] - 1, \quad B_j(\Lambda) = \mathbb{E}[M_j(\Lambda)^{-1}] - 1, \quad j \in \{L, M, H\}.$$

Table 6 reports these coefficients for alternative feedback strengths. The two coefficients relevant for tail stability are B_L in the lower regime and A_H in the upper regime. As shown in Lemma 6.9,

$$B_L(\Lambda) < 0 \iff \mathbb{E}[M_L(\Lambda)^{-1}] < 1,$$

and

$$A_H(\Lambda) < 0 \iff \mathbb{E}[M_H(\Lambda)] < 1.$$

For every reported interior feedback strength, both inequalities hold. The numerical results therefore confirm the two tail moment conditions underlying the geometric Foster–Lyapunov drift. The remaining coefficients are reported for completeness but are not required for the tail-drift argument.

Table 6: Foster–Lyapunov drift coefficients under alternative feedback strengths

Λ	A_L	B_L	A_*	B_*	A_H	B_H
0.25	0.00244	−0.00202	0	0.00043	−0.00248	0.00291
0.50	0.00490	−0.00449	0	0.00043	−0.00495	0.00537
0.75	0.00737	−0.00698	0	0.00043	−0.00741	0.00781
1.00	0.00985	−0.00950	0	0.00043	−0.00985	0.01024

Note. The reported values are rounded to five decimal places. The computed value of A_* is of order 10^{-16} because of floating-point roundoff and is therefore reported as zero. As Λ increases, the magnitudes of A_L and A_H increase approximately proportionally, whereas B_* remains constant because the neutral policy rate r^* is independent of the corridor width.

Drift under alternative feedback strengths

The preceding moment calculations evaluate the tail inequalities underlying the Foster–Lyapunov argument of Sect. 6. Figure 6 complements these asymptotic calculations by plotting the one-step Lyapunov drift,

$$\Delta V_\Lambda(K) = P_\Lambda V(K) - V(K),$$

over the state space. As feedback strength increases, the drift outside the central region becomes more strongly directed toward the interior. The figure therefore provides a numerical visualization of the restoring mechanism underlying the geometric Foster–Lyapunov drift established in Lemma 6.9.

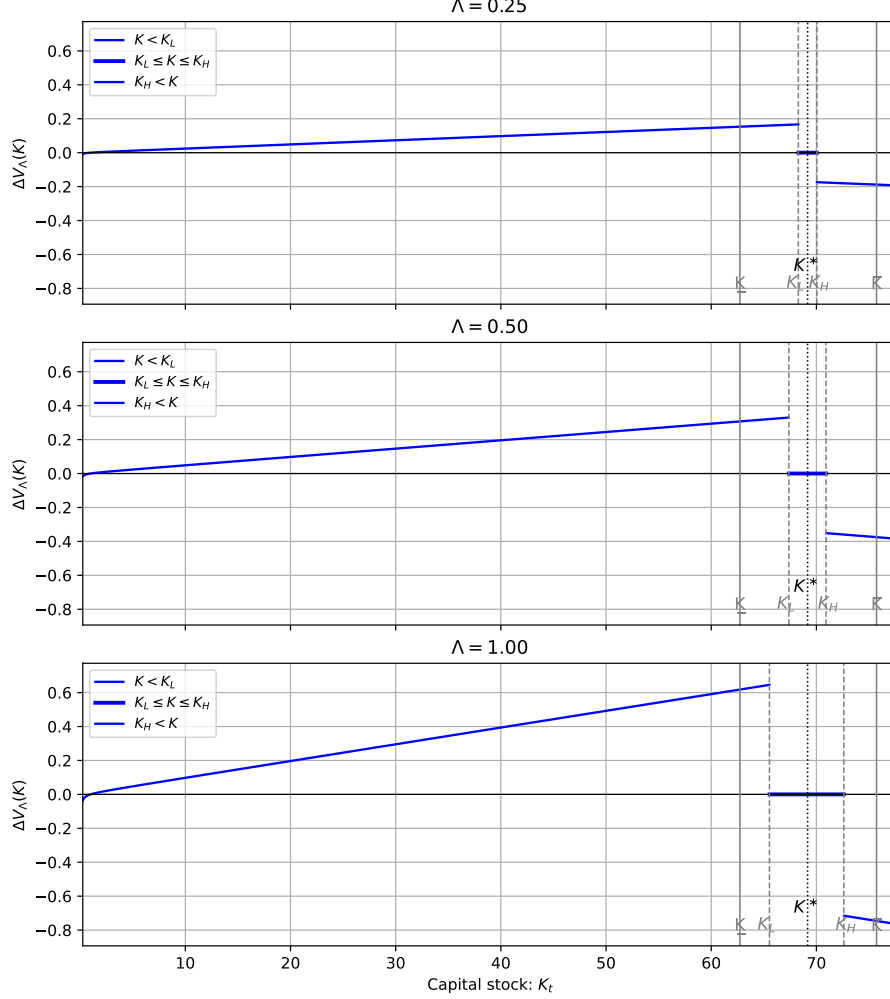


Figure 6: Foster–Lyapunov drift under alternative feedback strengths.

The three panels plot the one-step Foster–Lyapunov drift $\Delta V_\Lambda(K) = P_\Lambda V(K) - V(K)$ for $\Lambda = 0.25, 0.50$, and 1.00 , respectively, using $V(K) = K + K^{-1}$. The vertical lines indicate the benchmark capital levels and the policy thresholds K_L and K_H . Within the policy corridor, the interest rate remains at r^* ; below K_L it is set at r_L , and above K_H it is set at r_H . As Λ increases, the lower- and upper-regime drifts become more strongly directed toward the interior. For $\Lambda = 1$, the policy thresholds coincide with the corresponding clipping thresholds.

Having established how stronger feedback reinforces the inward Lyapunov drift, we next examine how this stabilizing mechanism is reflected in the invariant distribution of the capital stock.

7.3 Invariant distributions

Figure 7 relates the state-dependent transition mechanism to the resulting invariant distribution under alternative feedback strengths. As Λ increases, the lower and upper policy regimes generate stronger restoring forces toward the central region. The corresponding invariant distribution therefore becomes more concentrated around K^* , with less probability mass in the tails. Stronger feedback thus affects not only the conditions for stochastic stability but also the long-run dispersion of the capital stock around its stationary benchmark.

7.4 First-passage risks

To complement the long-run analysis, this subsection examines first-passage risks relative to the lower and upper benchmark bounds $\underline{K}$ and $\overline{K}$. Let L denote first passage through the lower bound and U first passage through the upper bound. We report the corresponding probabilities P_L and P_U , together with the conditional mean first-passage times $E[\tau | L]$ and $E[\tau | U]$.

For each feedback strength Λ , 10,000 sample paths are simulated from K^* over a horizon of 500 periods. A realization is classified as a lower passage (L) if it first reaches $K \leq \underline{K}$, as an upper passage (U) if it first reaches $K \geq \overline{K}$, and as a survivor (S) if neither benchmark is reached within the simulation horizon.

Let P_L , P_U , and P_S denote the corresponding Monte Carlo probabilities. Conditional mean first-passage times are denoted by $E[\tau | L]$ and $E[\tau | U]$.

Table 7 reveals a clear effect of stronger feedback on the timing of economically significant excursions. Under weak feedback, virtually every simulated path reaches one of the two benchmark levels within 500 periods. At $\Lambda = 0.1$, for example, the survival probability is zero and the conditional mean passage times are only about 39 periods in either direction. As feedback becomes stronger, first passage is progressively delayed. At $\Lambda = 0.5$, the conditional mean passage times have risen to approximately 88 periods, and at $\Lambda = 0.8$ they exceed 140 periods.

The effect becomes particularly pronounced near the upper end of the admissible feedback range. At $\Lambda = 0.9$, the probability of remaining between $\underline{K}$ and $\overline{K}$ throughout the 500-period horizon rises to 0.1708, while the conditional mean passage times increase to approximately 188 periods. Thus, stronger feedback not only reinforces the inward drift identified in the preceding subsections but also delays large finite-horizon excursions from the central region.

The lower- and upper-passage probabilities themselves need not vary monotonically with Λ . This is natural because they are competing first-passage events: changes in feedback affect both the likelihood and the timing of reaching either benchmark. The more robust indication of increasing finite-horizon protection is therefore the rise in survival probability together with the substantial increase in conditional passage times, rather than monotonicity of P_L or P_U separately.

In the limiting case $\Lambda = 1$, none of the 10,000 simulated paths reaches either benchmark within 500 periods. This endpoint should be interpreted separately from the interior cases: it corresponds to the maximal feedback specification, under which

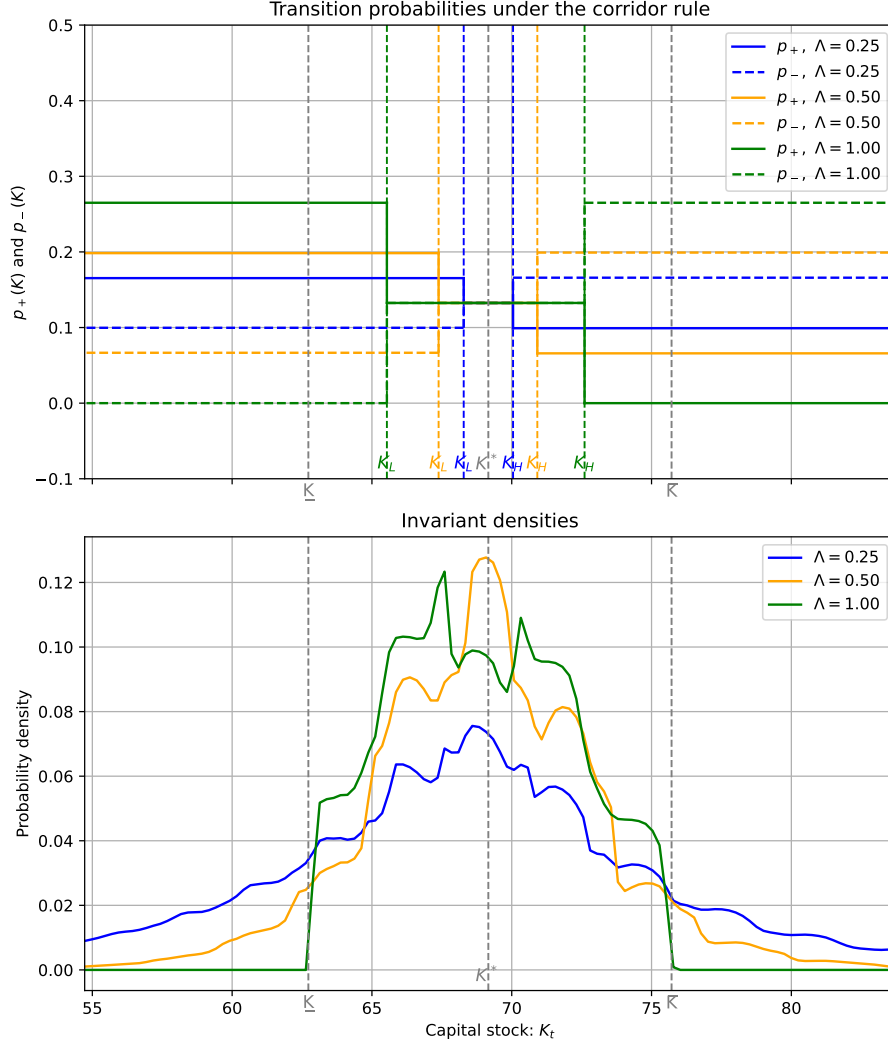


Figure 7: Transition probabilities and invariant distributions under alternative feedback strengths Λ

The upper panel shows the state-dependent transition probabilities under alternative feedback strengths, while the lower panel shows the corresponding invariant densities. The outer dashed lines indicate the stochastic support bounds $\underline{K}$ and $\bar{K}$. For $\Lambda = 1$, the policy thresholds coincide with the clipping thresholds, $K_L = K_{q0}$ and $K_H = K_{p0}$; accordingly, only the labels K_L and K_H are shown. The wage-adjustment thresholds K_D and K_F lie outside the stochastic support and are omitted from the figure.

Table 7: First-passage risks under alternative feedback strengths

Λ	P_L	P_U	P_S	$E[\tau \mid L]$	$E[\tau \mid U]$
0.1	0.4951	0.5049	0.0000	39.33	38.81
0.2	0.5099	0.4901	0.0000	47.30	46.70
0.3	0.5242	0.4757	0.0001	58.46	58.75
0.4	0.5298	0.4699	0.0003	73.93	72.16
0.5	0.5353	0.4633	0.0014	88.04	88.72
0.6	0.5104	0.4839	0.0057	103.57	103.74
0.7	0.4978	0.4916	0.0106	124.43	119.76
0.8	0.4954	0.4701	0.0345	144.66	142.21
0.9	0.4314	0.3978	0.1708	188.34	186.19
1.0	0.0000	0.0000	1.0000	—	—

Note: Each row is based on 10,000 simulated paths starting from K^* , with a horizon of 500 periods. P_L and P_U are the probabilities of first reaching $\underline{K}$ and $\overline{K}$, respectively, while P_S is the probability of reaching neither benchmark within the simulation horizon. Conditional mean passage times are reported only when the corresponding passage occurs.

the policy thresholds attain their limiting admissible positions. The interior results $\Lambda < 1$ are therefore more informative about the gradual effect of increasing feedback strength.

These first-passage calculations complement, rather than duplicate, the stochastic-stability results of Sect. 6. Positive recurrence and the existence of an invariant distribution are long-run properties and do not imply that the process remains within any prescribed finite interval over a finite horizon. The first-passage experiment instead measures the extent to which stronger feedback reduces the finite-horizon exposure of the economy to large excursions. In this sense, it provides a quantitative measure of the protective effect of policy feedback that is not captured by the invariant distribution alone.

7.5 Policy adjustment frequency

The preceding analysis shows that sufficiently strong state-dependent feedback can stabilize the capital process. This does not imply that the policy rate must be adjusted continuously. We therefore conclude the numerical analysis by examining the operation of the threshold rule along representative sample paths and then quantifying its long-run intervention and switching frequencies from the invariant distribution.

Figure 8 illustrates how the threshold policy operates along representative sample paths under alternative feedback strengths. The horizontal policy thresholds K_L and K_H delimit the region in which the policy rate remains at r^* . Excursions below K_L or above K_H activate the lower or upper policy rate, respectively. The figure therefore provides a sample-path illustration of the policy mechanism whose adjustment frequency is quantified below.

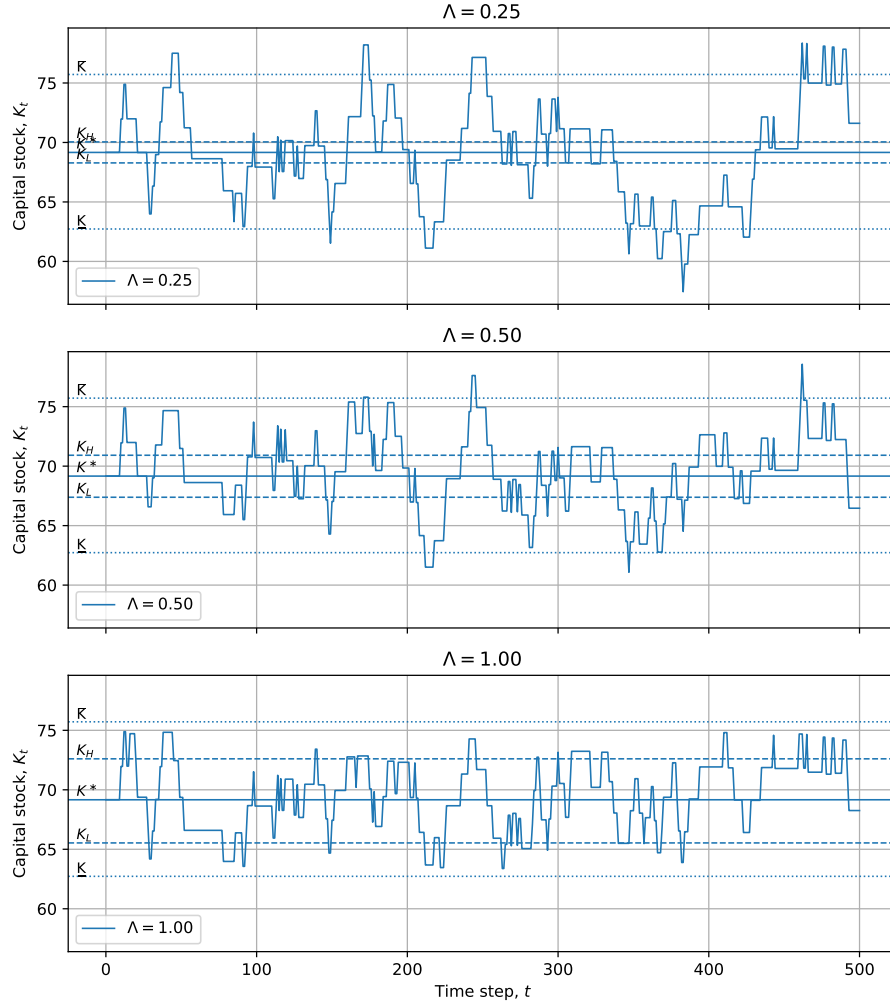


Figure 8: Representative capital-stock trajectories and policy thresholds under alternative feedback strengths

Each panel displays a representative realization of the stochastic capital-stock process for one value of the feedback-strength parameter Λ . All trajectories are generated from the same initial condition and random seed. The solid horizontal line indicates the steady-state capital stock K^* , while the dashed lines denote the corresponding policy thresholds K_L and K_H . The policy rate remains at r^* while the capital stock lies between these thresholds; crossings of the lower or upper threshold activate r_L or r_H , respectively.

Let $F_*(\Lambda)$ denote the stationary fraction of time for which the policy rate remains at its neutral value,

$$F_*(\Lambda) = \Pr_{\pi_\Lambda}(r_t = r^*) = \sum_{K_L \leq K_j \leq K_H} \pi_{\Lambda,j}.$$

The corresponding intervention frequency is

$$F_{\text{int}}(\Lambda) = 1 - F_*(\Lambda) = \Pr_{\pi_\Lambda}(r_t \neq r^*).$$

This measure should be distinguished from the stationary switching frequency

$$F_{\text{switch}}(\Lambda) = \Pr_{\pi_\Lambda}(r_{t+1} \neq r_t),$$

which measures the frequency of actual policy-rate revisions. A policy may therefore intervene for a substantial fraction of time while switching relatively infrequently if intervention episodes persist for several periods.

Table 8: Policy activity and stationary capital-stock dispersion under alternative monetary closures and feedback strengths

Closure / Λ	Δr	ΔK	Inactive	Intervention	Switching	σ_K^*
Quantity closure	—	—	—	—	—	2.32
Rate closure, 0.25	0.00037	1.77	12.3	87.7	8.6	8.11
Rate closure, 0.50	0.00075	3.53	36.4	63.6	15.7	4.29
Rate closure, 0.60	0.00090	4.24	45.3	54.7	16.1	3.73
Rate closure, 0.75	0.00112	5.30	57.2	42.8	15.7	3.32
Rate closure, 1.00	0.00150	7.07	70.1	29.9	14.7	3.15

Notes: The stationary statistics are computed using a capital-stock grid with 2,000 points. σ_K^* denotes the stationary root-mean-square deviation of the capital stock from the neutral level K^* . Inactive, intervention, and switching frequencies are reported as percentages and apply only to the feedback rate closure. The four benchmark feedback strengths are supplemented by $\Lambda = 0.60$, at which the switching frequency reaches its maximum on the finer numerical grid.

The switching frequency $F_{\text{switch}}(\Lambda)$ displays a non-monotonic pattern, increasing at lower feedback strengths, reaching its maximum on the finer numerical grid at $\Lambda = 0.60$, and declining thereafter. This reflects two opposing effects of stronger feedback. At first, stronger corrective movements generate more frequent transitions back toward the neutral policy regime. At higher feedback strengths, however, the widening inactive region and improved stabilization reduce the frequency with which the intervention boundaries are reached, causing policy-rate switching to decline.

The stationary results reveal two competing consequences of stronger feedback. A larger feedback parameter increases the size of policy adjustments and thereby

strengthens corrective action, but it also widens the inactive region and reduces the frequency of boundary encounters. The interaction of these two channels determines realized policy activity and the resulting degree of stabilization. The mechanism can be summarized schematically as follows:

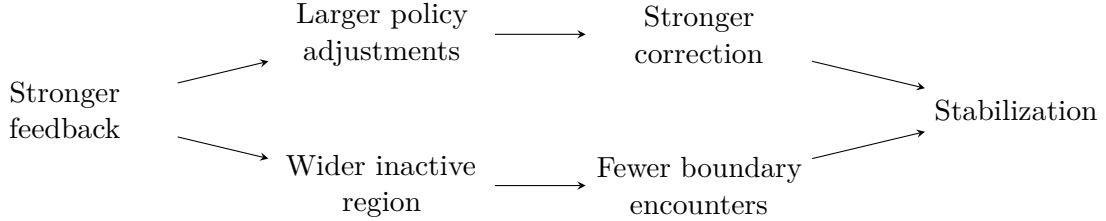


Table 8 reveals a notable feature of the threshold rule. As feedback strength increases, the stationary intervention frequency declines substantially even as capital-stock dispersion falls. Stronger feedback therefore does not stabilize the economy by keeping policy active more often. Rather, larger corrective adjustments, together with the widening inactive region, allow the economy to spend a greater fraction of time at the neutral policy rate while remaining more tightly concentrated around K^* . Switching frequency is non-monotonic: it initially rises as stronger correction generates more frequent transitions between intervention and neutral regimes, reaches a maximum around $\Lambda = 0.60$, and subsequently declines as the widening inactive region and improved stabilization reduce boundary encounters.

7.6 Comparison with the quantity-closure benchmark

Under the baseline calibration, the quantity closure generates a substantially more concentrated invariant distribution than the feedback rate closure. The stationary root-mean-square deviation from the neutral capital stock is $\sigma_{K,Q}^* = 2.32$, compared with $\sigma_{K,R}^* = 3.15$ under the strongest corridor feedback considered, $\Lambda = 1$. As shown in Table 8, the difference persists even though increasing Λ sharply reduces stationary capital-stock dispersion under the rate closure. This difference reflects the nature of the stabilizing feedback. Under the quantity closure, the market interest rate adjusts continuously with the capital stock, so that deviations from K^* immediately affect profitability and investment behavior. Under the rate closure, by contrast, the policy rate remains at r^* throughout the inactive region and adjusts only when a policy threshold is crossed. The corridor rule can therefore restore ergodicity without reproducing the tighter stationary concentration generated by continuous endogenous interest-rate adjustment.

The comparison at $\Lambda = 1$ is particularly informative. At this value, the intervention rates reach the corresponding clipping benchmarks, $r_L = r_{q0}$ and $r_H = r_{p0}$, with $K_L = K_{q0}$ and $K_H = K_{p0}$. The remaining difference in stationary dispersion therefore does not arise simply because the corridor uses insufficiently strong intervention rates. Rather, the two closures differ in the way monetary feedback operates between these benchmark points. Under the quantity closure, $r(K)$, and hence the investment probabilities, vary continuously with the state of the economy. Under

the corridor rule, the policy rate remains fixed at r^* throughout the inactive region and changes discretely at the intervention thresholds.

As Fig. 5 illustrates, this discontinuity is also reflected in the shape of the invariant distribution: under strong feedback, probability mass tends to concentrate just inside the intervention thresholds, producing local dents near K_L and K_H . The resulting multi-peaked distribution provides a visible counterpart to the greater stationary dispersion generated by the rate closure. Thus, stronger corridor feedback should not be interpreted simply as approaching the continuous feedback generated by the quantity closure: as Λ increases, corrective forces outside the inactive region become stronger, but the threshold character of the adjustment simultaneously becomes more pronounced.

Thus, stronger corridor feedback should not be interpreted simply as approaching the continuous feedback generated by the quantity closure: as Λ increases, corrective forces outside the inactive region become stronger, but the threshold character of the adjustment simultaneously becomes more pronounced.

8 Conclusion

This paper has examined how alternative monetary closures affect the stochastic stability of a Keynesian economy with decentralized investment adjustment. The central distinction is between a quantity closure, under which the nominal money stock is predetermined, and an interest-rate closure, under which the monetary authority accommodates money demand at the policy rate. Although the underlying real economy and the behavioral mechanism governing investment are unchanged, the two closures generate fundamentally different adjustment dynamics.

Under the quantity closure, variations in the capital stock induce movements in the market interest rate. These endogenous interest-rate movements feed back into expected profitability and hence into firms' investment decisions, providing a restoring force around the neutral capital stock. The monetary closure therefore contributes directly to the mean-reverting mechanism that supports a stationary distribution of capital.

The pure interest-rate peg removes this mechanism. When the policy rate is held fixed, the money stock adjusts endogenously to accommodate money demand, while the interest rate no longer responds to movements in the capital stock. Consequently, the profitability index and the associated investment probabilities remain fixed at values determined by the pegged rate. Except at the knife-edge neutral rate, the resulting stochastic dynamics possess a persistent directional tendency rather than a restoring drift. In this sense, the familiar Wicksellian cumulative process appears here in stochastic form: fixing the market rate away from its neutral value permits deviations in economic activity to cumulate rather than generating an endogenous force that reverses them.

The analysis also shows that abandoning a quantity closure need not imply abandoning stochastic stability. A state-dependent interest-rate rule can restore the missing feedback. The corridor rule studied in this paper leaves the policy

rate unchanged within an inactive region around the neutral capital stock but adjusts it once the economy crosses either boundary. Below the corridor, the lower policy rate raises expected profitability and strengthens upward adjustment; above the corridor, the higher policy rate lowers expected profitability and strengthens downward adjustment. The policy rule thereby recreates, through deliberate interest-rate feedback, the restoring force that arises endogenously under the quantity closure.

The critical feedback-strength result makes this mechanism precise. As the strength of policy intervention increases, the expected log-growth rate at the lower boundary rises, whereas its counterpart at the upper boundary falls. Stronger feedback therefore pushes the two boundary drifts in opposite directions—upward at the lower boundary and downward at the upper boundary. Once feedback exceeds the critical strength, these opposing forces satisfy the drift conditions required for stochastic stability. The result thus establishes a direct connection between the strength of monetary feedback and the existence of a stable long-run stochastic regime. In this respect, the critical-feedback result is related to a broader principle in the monetary-policy-rule literature: stabilization depends not only on whether the policy rate responds systematically to economic conditions, but also on the strength of that response. Taylor-type rules (Taylor, 1993) embody this principle through sufficiently strong interest-rate responses to inflation and real activity. The present result is distinct in both its state variable and its stability criterion, but provides a complementary mechanism through which sufficiently strong policy-rate feedback supports macroeconomic stability.

The quantitative comparison also shows that restoring stochastic stability does not make the two monetary closures equivalent. Under the baseline calibration, the stationary root-mean-square deviation of capital from its neutral level is $\sigma_{K,Q}^* = 2.32$ under the quantity closure, compared with $\sigma_{K,R}^* = 3.15$ under the strongest corridor feedback considered, $\Lambda = 1$. The difference is consistent with the distinct forms of monetary feedback under the two closures. Under the quantity closure, the market interest rate and hence investment incentives vary continuously with the capital stock. Under the corridor rule, by contrast, the policy rate remains at r^* throughout the inactive region and changes discretely only when an intervention threshold is crossed. As reflected in the multi-peaked invariant distribution under strong feedback, the corridor can therefore reconstruct the opposing forces required for ergodicity without reproducing the tighter stationary concentration generated by continuous endogenous interest-rate adjustment.

These findings suggest a microeconomic interpretation of interest-rate feedback rules. In the present framework, policy feedback is not imposed merely as a device for smoothing fluctuations around an independently specified equilibrium. Rather, it substitutes for an endogenous stabilizing mechanism that disappears when the monetary authority pegs the interest rate and accommodates the quantity of money. The stabilizing role of an interest-rate rule can therefore be understood as a consequence of the monetary closure itself: once the market rate ceases to respond endogenously to the state of the economy, systematic policy-rate adjustment can reconstruct the missing negative feedback. The quantitative comparison nevertheless shows that

reconstructing enough feedback to obtain stochastic stability is not the same as reproducing the stationary regime generated by endogenous monetary adjustment.

The analysis deliberately adopts a simple three-rate corridor rule and a parsimonious monetary and financial structure in order to isolate the feedback mechanism associated with the monetary closure. Two extensions follow naturally from the results. First, the quantitative difference between the quantity closure and the corridor rule raises the question of whether finer state-dependent interest-rate adjustment can more closely reproduce the continuous endogenous feedback generated under the quantity closure. A sequence of multi-step rules, converging toward a smooth feedback rule, would make it possible to examine whether the corresponding invariant distributions converge toward that obtained under continuous interest-rate adjustment. Second, introducing aggregate shocks would allow the analysis to distinguish stabilization around a given neutral configuration from policy adjustment to changes in the configuration itself. For example, shocks to c would alter profitability and the associated neutral conditions, raising the question of whether the policy rate and intervention thresholds should respond endogenously to such shifts. These extensions would provide a natural next step toward richer environments with bank balance sheets, credit spreads, asset prices, portfolio adjustment, and expectations.

The present results provide a benchmark for such extensions by identifying the stability consequences of the monetary closure before these additional mechanisms are introduced. The fundamental question is therefore not whether a particular monetary quantity is literally fixed, but whether the monetary architecture supplies, removes, or deliberately reconstructs the feedback between the state of the economy, the interest rate, profitability, and investment. From this perspective, the Wicksellian cumulative process and interest-rate stabilization can be viewed as two sides of the same underlying stability problem.

Declaration of Generative AI and AI-Assisted Technologies

During the preparation of this work, the author used **ChatGPT 5.5** to refine the text’s clarity and logical flow, and to assist in debugging and optimizing the programming code. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the final manuscript and technical outputs.

Appendix A: Numerical verification of stability and computational robustness

A.1 State-space truncation robustness

The purpose of this robustness check is not to establish a new economic result but to verify that the computed invariant distribution is insensitive to the artificial

Table 9: Robustness of the invariant distribution to state-space enlargement

ρ	Stationary exit	Left endpoint mass	Right endpoint mass	$E[K]$	$SD(K)$
0.2	2.660×10^{-3}	5.639×10^{-3}	4.487×10^{-3}	68.552	7.133
0.4	2.221×10^{-4}	2.537×10^{-4}	5.917×10^{-4}	68.493	8.082
0.6	2.833×10^{-5}	3.304×10^{-6}	1.044×10^{-4}	68.532	8.273
0.8	5.961×10^{-6}	1.997×10^{-9}	2.263×10^{-5}	68.542	8.317

Note. The table reports the stationary one-step exit probability, the invariant probability masses at the left and right endpoints of the numerical state space, and the mean and standard deviation of the invariant capital-stock distribution for alternative values of the truncation parameter ρ . Results are reported for $\Lambda = 0.25$.

truncation of the numerical state space. The truncation parameter ρ determines the computational interval

$$[\underline{K}(1 - \rho), \overline{K}(1 + \rho)].$$

Increasing ρ enlarges the numerical state space while leaving the underlying stochastic process unchanged.

Table 9 reports the results for $\Lambda = 0.25$, the weakest of the four feedback strengths considered in the main numerical analysis. As the numerical state space is enlarged, the stationary one-step exit probability and the invariant probability masses at the artificial endpoints decline rapidly, while the mean and standard deviation of the invariant distribution stabilize.

The same qualitative convergence pattern is obtained for the other interior feedback specifications, $\Lambda = 0.50$ and 0.75 . These additional results are omitted for brevity. The limiting case $\Lambda = 1$ is qualitatively different: the stationary one-step exit probability is identically zero for all reported values of ρ , while the two endpoint masses coincide. This reflects the special structure of the hard-corridor benchmark and is therefore treated separately from the interior truncation-robustness results.

Appendix B: Proof of the T-chain property

Proof. Fix an arbitrary state $x_0 = \log K_0$ and a neighborhood

$$U_x = (x_0 - \eta_0, x_0 + \eta_0).$$

Choose a nonempty target open interval

$$J = (y - \eta_y, y + \eta_y)$$

inside a region where the transition mechanism is regular.

By Lemma 6.3, we may select finite counts m, n with $N = m + n \geq 2$, together with a fixed ordering E of those branch choices, such that the associated common

attainable interval contains J uniformly for $x \in U_x$. Since E contains m upward and n downward branch choices, irrespective of the policy regimes encountered along the path,

$$\Pr_x(E) \geq \underline{p}^m \underline{q}^n =: c_E > 0, \quad x \in U_x.$$

Conditional on the prescribed branch sequence E , let the log increments be $Z_1, \dots, Z_N$. Their sum is

$$S_N = Z_1 + \dots + Z_N, \quad X_N = x + S_N.$$

Let I_i denote the common log interval associated with the i th branch choice:

$$I_i = \begin{cases} [\bar{\ell}_+, u_+], & \text{if the } i\text{th branch is upward,} \\ [\ell_-, \bar{u}_-], & \text{if the } i\text{th branch is downward.} \end{cases}$$

By Lemma 6.4, at each step the conditional density of Z_i , given the preceding history and the prescribed branch sequence E , is bounded below by $\underline{g}_+$ or $\underline{g}_-$ on the corresponding common log interval, uniformly over all policy regimes. Hence, by the chain rule for conditional densities,

$$f_{E,x}(z_1, \dots, z_N) \geq \underline{g}_+^m \underline{g}_-^n \prod_{i=1}^N \mathbb{1}_{I_i}(z_i).$$

Define

$$h_N(s) := \mathbb{1}_{I_1} * \dots * \mathbb{1}_{I_N}(s).$$

The function $h_N(s)$ measures the volume of admissible increment configurations whose sum equals s . Combining the joint-density lower bound above with convolution therefore gives

$$g_{E,x}^{S_N}(s) \geq \underline{g}_+^m \underline{g}_-^n h_N(s).$$

Moreover, h_N is continuous and strictly positive on the interior of the Minkowski sum

$$I_1 + \dots + I_N = [A_{m,n}, B_{m,n}].$$

Choose a nonempty open interval $J_0 \Subset J$ such that

$$\mathcal{C} := \{z - x : x \in \bar{U}_x, z \in \bar{J}_0\} \Subset (A_{m,n}, B_{m,n}).$$

Since h_N is continuous and strictly positive on $\mathcal{C}$,

$$\kappa := \min_{s \in \mathcal{C}} h_N(s) > 0.$$

Hence the conditional density of X_N given E satisfies

$$g_{E,x}^{X_N}(z) = g_{E,x}^{S_N}(z - x) \geq \underline{g}_+^m \underline{g}_-^n \kappa =: \underline{g}_E > 0, \quad x \in \bar{U}_x, \quad z \in \bar{J}_0.$$

Therefore, for every Borel set A ,

$$\begin{aligned} P_X^N(x, A) &\geq \Pr_x(E) \Pr_x(X_N \in A \mid E) \\ &\geq c_E \int_{A \cap J_0} g_{E,x}^{X_N}(y) dy \\ &\geq c_E \underline{g}_E \text{Leb}(A \cap J_0), \quad x \in U_x. \end{aligned}$$

Thus the desired local minorization holds with

$$c := c_E \underline{g}_E.$$

Next, choose a continuous function $\chi : \mathbb{R} \rightarrow [0, 1]$ such that

$$\chi(x_0) > 0, \quad \text{supp } \chi \subset U_x,$$

and define

$$T_{x_0}(x, A) := c \chi(x) \text{Leb}(A \cap J_0).$$

Since the local minorization holds for every $x \in U_x$ and χ vanishes outside U_x ,

$$P_X^N(x, A) \geq T_{x_0}(x, A), \quad x \in \mathbb{R}.$$

For every Borel set A , the map

$$x \mapsto T_{x_0}(x, A)$$

is continuous, hence lower semicontinuous. Moreover,

$$T_{x_0}(x_0, \mathbb{R}) = c \chi(x_0) \text{Leb}(J_0) > 0.$$

The required number of steps N may depend on the starting neighborhood. To accommodate this variation, let $w = \{w_r\}_{r=0}^{\infty}$ be a probability distribution on $\mathbb{N}_0$ satisfying $w_r > 0$ for every $r \in \mathbb{N}_0$, so that $\sum_{r=0}^{\infty} w_r = 1$.

Define the sampled kernel

$$P_w := \sum_{r \geq 0} w_r P_X^r.$$

If the local construction at x_0 uses $N = N(x_0)$, then $w_N > 0$ and

$$P_w(x, A) \geq w_N P_X^N(x, A) \geq w_N T_{x_0}(x, A).$$

Define

$$\tilde{T}_{x_0}(x, A) := w_N T_{x_0}(x, A).$$

For every Borel set A , the map $x \mapsto \tilde{T}_{x_0}(x, A)$ is continuous, and

$$P_w(x, A) \geq \tilde{T}_{x_0}(x, A), \quad x \in \mathbb{R}.$$

Moreover,

$$\tilde{T}_{x_0}(x_0, \mathbb{R}) = w_N c \chi(x_0) \text{Leb}(J_0) > 0.$$

Hence P_w admits a nontrivial continuous component at x_0 . Since $x_0 \in \mathbb{R}$ was arbitrary, Proposition 6.2.4 of Meyn and Tweedie (1993) implies that $\{X_t\}$ is a T-chain.

Moreover, Lemma 6.3 establishes that every nonempty open interval is accessible from every initial state. Let $O \subset \mathbb{R}$ be any nonempty open set, and let $L(x, O)$ denote the probability, starting from x , of ever reaching O . Since every nonempty open set $O \subset \mathbb{R}$ contains a nonempty open interval J , Lemma 6.3 implies that, for every $x \in \mathbb{R}$, there exists a finite $n = n(x, J)$ such that

$$P_X^n(x, O) \geq P_X^n(x, J) > 0.$$

Hence $L(x, O) > 0$ for every $x \in \mathbb{R}$ and every nonempty open set O . Since $\{X_t\}$ is a T-chain, Theorem 6.0.1(i) of Meyn and Tweedie (1993) therefore implies that $\{X_t\}$ is ψ -irreducible. Finally, Theorem 6.0.1(ii) implies that every compact subset of $\mathbb{R}$ is petite. $\square$

References

- Brunner K, Meltzer AH (1964) Some further investigations of demand and supply functions for money. *The Journal of Finance* 19(2):240–283
- Brunner K, Meltzer AH (1968) Liquidity traps for money, bank credit, and interest rates. *Journal of Political Economy* 76(1):1–37
- Fisher I (1935) 100% Money: Designed to Keep Checking Banks 100% Liquid; to Prevent Inflation and Deflation; Largely to Cure or Prevent Depressions; and to Wipe Out Much of the National Debt. Adelphi Company, New York, URL <https://books.google.com/books?id=00tZLbBN1BUC>, revised edition
- Friedman M, Schwartz AJ (1963) *A Monetary History of the United States, 1867–1960*. Princeton University Press, Princeton, NJ
- Kaldor N (1970) The new monetarism. *Lloyds Bank Review* 97(1):18
- Krugman PR (1991) Target zones and exchange rate dynamics. *Quarterly Journal of Economics* 106(3):669–682, DOI 10.2307/2937922
- Lavoie M (1996) Horizontalism, structuralism, liquidity preference and the principle of increasing risk. *Scottish Journal of Political Economy* 43(3):275–300, DOI 10.1111/j.1467-9485.1996.tb00846.x
- Meyn SP, Tweedie RL (1993) *Markov Chains and Stochastic Stability*. Communications and Control Engineering, Springer-Verlag, London

- Moore BJ (1988) Horizontalists and verticalists: The Macroeconomics of Credit Money. Cambridge University Press
- Palley TI (1994) Competing views of the money supply process: Theory and evidence. *Metroeconomica* 45(1):67–88, DOI 10.1111/j.1467-999X.1994.tb00013.x
- Takahashi I (2026) Microfoundations of Keynesian economics: An Evolutionary approach to stabilizing unstable Economy. Springer Nature Singapore Pte Ltd., DOI 10.1007/978-981-95-9390-3
- Taylor JB (1993) Discretion versus policy rules in practice. *Carnegie-Rochester Conference Series on Public Policy* 39:195–214, DOI 10.1016/0167-2231(93)90009-L
- Wicksell K (1936) Interest and Prices: A Study of the Causes Regulating the Value of Money. Macmillan for the Royal Economic Society, London, translated by R. F. Kahn; introduction by Bertil Ohlin. First published in German (1898) as *Geldzins und Güterpreise*

中央大学経済研究所
(INSTITUTE OF ECONOMIC RESEARCH, CHUO UNIVERSITY)
代表者 阿部 顕三 (Director: Kenzo Abe)
〒192-0393 東京都八王子市東中野 742-1
(742-1 Higashi-nakano, Hachioji, Tokyo 192-0393 JAPAN)
TEL: 042-674-3271 +81 42 674 3271
FAX: 042-674-3278 +81 42 674 3278
E-mail: keizaiken-grp@g.chuo-u.ac.jp
URL: <https://www.chuo-u.ac.jp/research/institutes/economic/>