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Dynamic Engineering Systems with Two Time Delays

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Abstract

An elementary analysis is developed to determine the stability region of certain classes of ordinary differential equations with two delays. Our analysis is based on determining stability switches first where an eigenvalue is pure complex, and then checking the conditions for stability loss or stability gain. In the cases of both stability losses and stability gains Hopf bifurcation occurs giving the possibility of the birth of limit cycles.

1 Introduction

Delay differential equations have many applications in quantitative sciences including economics, biology, engineering among others. The single delay case is well established in the literature (Hayes, 1950, Matsumoto and Szidarovszky, 2013), however the presence of a second delay makes the models much more complicated. The works of Hale (1979) and Hale and Huang (1993) can be considered as major contributions. Matsumoto and Szidarovszky (2012) developed a simple analytic method, which is limited to examine only some special model variants. Gu et al. (2005) developed a geometric approach applicable for analyzing a more general class of models.

In this paper two particular models are examined and the two major approaches illustrated. A brief simulation study illustrates the theoretical findings.

2 Model 1

We will first examine the asymptotical stability of the delay differential equation

$$\dot{x}(t) + Ax(t-\delta) + Bx(t-\eta) = 0 \tag{1}$$

where A and B are positive constants. The characteristic equation can be obtained by looking for the solution in the exponential form $\alpha e^{\lambda t}$. By substitution,

$$\alpha\lambda e^{\lambda t} + A\alpha e^{\lambda(t-\delta)} + B\alpha e^{\lambda(t-\eta)} = 0$$

or

$$\lambda + Ae^{-\lambda\delta} + Be^{-\lambda\eta} = 0. \tag{2}$$

Introduce the new variables

$$\omega = \frac{A}{A+B}, \ 1-\omega = \frac{B}{A+B}, \ \bar{\lambda} = \frac{\lambda}{A+B}$$

$$\gamma_1 = \delta(A+B) \text{ and } \gamma_2 = \eta(A+B)$$

to reduce equation (2) to the following:

$$\bar{\lambda} + \omega e^{-\bar{\lambda}\gamma_1} + (1-\omega)e^{-\bar{\lambda}\gamma_2} = 0. \tag{3}$$

Because of symmetry we can assume that $\omega \geq 1/2$. In order to find the stability region in the (γ_1, γ_2) plane we will first characterize the cases when an eigenvalue is pure complex, that is, when $\bar{\lambda} = i\upsilon$. We can assume that $\upsilon > 0$, since if $\bar{\lambda}$ is an eigenvalue, its complex conjugate is also an eigenvalue. Substituting $\bar{\lambda} = i\upsilon$ into equation (3) we have

$$i\upsilon + \omega e^{-i\upsilon\gamma_1} + (1-\omega)e^{-i\upsilon\gamma_2} = 0.$$

If there is no delay, then $\gamma_1 = \gamma_2 = 0$ and equation (3) becomes

$$\bar{\lambda} + 1 = 0$$

with a negative eigenvalue $\bar{\lambda} = -1$, so the system is asymptotically stable.

In the special case of $\gamma_1 = 0$, the equation becomes

$$i\upsilon + \omega + (1-\omega)e^{-i\upsilon\gamma_2} = 0.$$

The real and imaginary parts imply that

$$\omega + (1-\omega)\cos(\upsilon\gamma_2) = 0$$

$$\upsilon - (1-\omega)\sin(\upsilon\gamma_2) = 0.$$

We can assume first $\omega > 1/2$, so from the first equation

$$\cos(\upsilon\gamma_2) = -\frac{\omega}{1-\omega} < -1$$

so no stability switch is possible. If $\omega = 1/2$, then

$$\cos(\upsilon\gamma_2) = -1$$

implying that $\sin(\upsilon\gamma_2) = 0$ and so $\upsilon = 0$ showing that there is no pure complex root. Hence for $\gamma_1 = 0$ the system is asymptotically stable with all $\gamma_2 \geq 0$.

Assume now that $\gamma_1 > 0$, $\gamma_2 \geq 0$. The real and imaginary parts give two equations:

$$\omega\cos(\upsilon\gamma_1) + (1-\omega)\cos(\upsilon\gamma_2) = 0 \tag{4}$$

and

$$\upsilon - \omega \sin(\upsilon\gamma_1) - (1-\omega)\sin(\upsilon\gamma_2) = 0. \tag{5}$$

We consider the case of $\omega > 1/2$ first and the symmetric case of $\omega = 1/2$ will be discussed later. Introduce the variables

$$x = \sin(\upsilon\gamma_1) \text{ and } y = \sin(\upsilon\gamma_2),$$

then (4) implies that

$$\omega^2(1-x^2) = (1-\omega)^2(1-y^2)$$

or

$$-\omega^2 x^2 + (1-\omega)^2 y^2 = 1 - 2\omega. \tag{6}$$

From (5),

$$\upsilon - \omega x - (1-\omega)y = 0$$

implying that

$$y = \frac{\upsilon - \omega x}{1-\omega} \tag{7}$$

Combining (6) and (7) yields

$$-\omega^2 x^2 + (1-\omega)^2 \left(\frac{\upsilon - \omega x}{1-\omega}\right)^2 = 1 - 2\omega$$

from which we can conclude that

$$x = \frac{\upsilon^2 + 2\omega - 1}{2\upsilon\omega} \tag{8}$$

and then from (7),

$$y = \frac{\upsilon^2 - 2\omega + 1}{2\upsilon(1-\omega)}. \tag{9}$$

Equations (8) and (9) provide a parameterized curve in the (γ_1, γ_2) plane:

$$\sin(\upsilon\gamma_1) = \frac{\upsilon^2 + 2\omega - 1}{2\upsilon\omega} \text{ and } \sin(\upsilon\gamma_2) = \frac{\upsilon^2 - 2\omega + 1}{2\upsilon(1-\omega)}. \tag{10}$$

In order to guarantee feasibility we have to satisfy

$$-1 \leq \frac{\upsilon^2 + 2\omega - 1}{2\upsilon\omega} \leq 1 \tag{11}$$

and

$$-1 \leq \frac{\upsilon^2 - 2\omega + 1}{2\upsilon(1-\omega)} \leq 1. \tag{12}$$

Simple calculation shows that with $\omega > 1/2$ these relations hold if and only if

$$2\omega - 1 \leq \upsilon \leq 1.$$

From (10) we have four cases for γ_1 and γ_2, since

$$\gamma_1 = \frac{1}{\upsilon}\left\{\sin^{-1}\left(\frac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right)+2k\pi\right\}$$

or

$$\gamma_1 = \frac{1}{\upsilon}\left\{\pi-\sin^{-1}\left(\frac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right)+2k\pi\right\} \quad (k=0,1,2,...)$$

and similarly

$$\gamma_2 = \frac{1}{\upsilon}\left\{\sin^{-1}\left(\frac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)}\right)+2n\pi\right\}$$

or

$$\gamma_2 = \frac{1}{\upsilon}\left\{\pi-\sin^{-1}\left(\frac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)}\right)+2n\pi\right\} \quad (n=0,1,2,...).$$

However from (4) we can see that $\cos(\upsilon\gamma_1)$ and $\cos(\upsilon\gamma_2)$ must have different signs, so we have only two possibilities:

$$L_1(k,n): \begin{cases} \gamma_1 = \dfrac{1}{\upsilon}\left\{\sin^{-1}\left(\dfrac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right)+2k\pi\right\} \\ \\ \gamma_2 = \dfrac{1}{\upsilon}\left\{\pi-\sin^{-1}\left(\dfrac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)}\right)+2n\pi\right\} \end{cases} \tag{13}$$

and

$$L_2(k,n): \begin{cases} \gamma_1 = \dfrac{1}{\upsilon}\left\{\pi-\sin^{-1}\left(\dfrac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right)+2k\pi\right\} \\ \\ \gamma_2 = \dfrac{1}{\upsilon}\left\{\sin^{-1}\left(\dfrac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)}\right)+2n\pi\right\} \end{cases} \tag{14}$$

For each $\upsilon \in [2\omega-1,1]$ these equations determine the values of γ_1 and γ_2. At the initial point $\upsilon = 2\omega-1$, we have

$$\frac{\upsilon^2+2\omega-1}{2\upsilon\omega} = 1 \text{ and } \frac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)} = -1$$

and if $\upsilon = 1$, then

$$\frac{\upsilon^2+2\omega-1}{2\upsilon\omega} = 1 \text{ and } \frac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)} = 1.$$

Therefore the starting point and end point of $L_1(k,n)$ are given as

$$\gamma_1^s = \frac{1}{2\omega-1}\left(\frac{\pi}{2}+2k\pi\right),\ \gamma_2^s = \frac{1}{2\omega-1}\left(\frac{3\pi}{2}+2n\pi\right)$$

and

$$\gamma_1^e = \frac{\pi}{2}+2k\pi \text{ and } \gamma_2^e = \frac{\pi}{2}+2n\pi.$$

Similarly, the starting and end points of $L_2(k,n)$ are as follows:

$$\gamma_1^S = \frac{1}{2\omega-1}\left(\frac{\pi}{2}+2k\pi\right),\ \gamma_2^S = \frac{1}{2\omega-1}\left(-\frac{\pi}{2}+2n\pi\right)$$

and

$$\gamma_1^E = \frac{\pi}{2}+2k\pi \text{ and } \gamma_2^E = \frac{\pi}{2}+2n\pi.$$

With fixed value of k, $L_1(k,n)$ and $L_2(k,n)$ have the same end point, however the starting point of $L_1(k,n)$ is the same as that of $L_2(k,n+1)$. Therefore the segments $L_1(k,n)$ and $L_2(k,n)$ with fixed k form a continuous curve with $n = 0,1,2,....$ They are shown in Figure 1 for $k=0$. The curves $L_1(0,n)$ are shown in red color and curves $L_2(0,n)$ are given in blue.

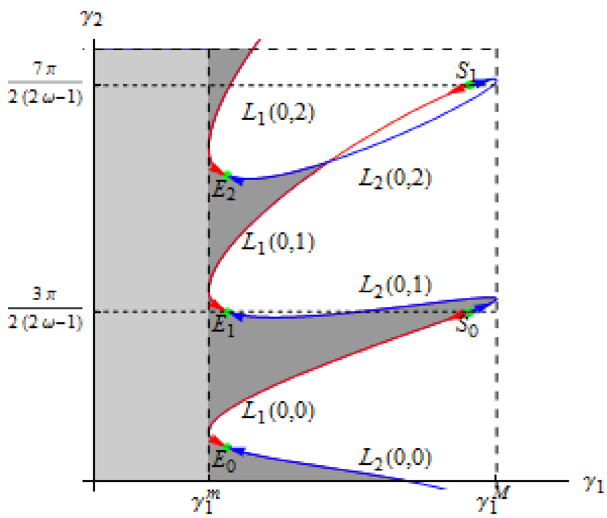


Figure 1. Partition curve in the (γ_1, γ_2) plane with fixing $k=0$.

Consider first the segment $L_1(k,n)$. Since $\left(\upsilon^2-2\omega+1\right)/(2\upsilon(1-\omega))$ is strictly increasing in υ, γ_2 is strictly decreasing in υ. By differentiation

$$\left.\frac{\partial\gamma_1}{\partial\upsilon}\right|_{L_1} = -\tfrac{1}{\upsilon^2}\left(\upsilon\gamma_1+\tan(\upsilon\gamma_2)\right). \tag{15}$$

Consider next segment $L_2(k,n)$, similarly to (15) we can shown that

$$\left.\frac{\partial\gamma_1}{\partial\upsilon}\right|_{L_2} = -\frac{1}{\upsilon^2}(\upsilon\gamma_1+\tan(\upsilon\gamma_2))$$

which is the same as in $L_1(k,n)$, since from (14), $\cos(\upsilon\gamma_1)<0$. Similarly

$$\left.\frac{\partial\gamma_2}{\partial\upsilon}\right|_{L_2} = -\frac{1}{\upsilon^2}\left(\upsilon\gamma_2+\tan(\upsilon\gamma_1)\right) \tag{16}$$

where we used again equation (4).

We will next examine the directions of the stability switches on the different segments of the curves $L_1(k,n)$ and $L_2(k,n)$. We fix the value of γ_2 and select γ_1 as the bifurcation parameter, so the eigenvalues are functions of $\gamma_1 : \bar{\lambda} = \lambda(\gamma_1)$. By differentiating the characteristic equation (3) implicitly with respect to γ_1 we have

$$\frac{d\bar{\lambda}}{d\gamma_1} + \omega e^{-\bar{\lambda}\gamma_1}(-\frac{d\bar{\lambda}}{d\gamma_1}\gamma_1 - \bar{\lambda}) + (1-\omega)e^{-\bar{\lambda}\gamma_2}\left(-\frac{d\bar{\lambda}}{d\gamma_1}\gamma_2\right) = 0$$

implying that

$$\frac{d\bar{\lambda}}{d\gamma_1} = \frac{\bar{\lambda}\omega e^{-\bar{\lambda}\gamma_1}}{1 - \omega\gamma_1 e^{-\bar{\lambda}\gamma_1} - (1-\omega)\gamma_2 e^{-\bar{\lambda}\gamma_2}} \tag{17}$$

From the characteristic equation we have

$$(1-\omega)e^{-\bar{\lambda}\gamma_2} = -\bar{\lambda} - \omega e^{-\bar{\lambda}\gamma_1},$$

so

$$\frac{d\bar{\lambda}}{d\gamma_1} = \frac{\bar{\lambda}\omega e^{-\bar{\lambda}\gamma_1}}{1 + \bar{\lambda}\gamma_2 + \omega(\gamma_2 - \gamma_1)e^{-\bar{\lambda}\gamma_1}}$$

If $\bar{\lambda} = i\upsilon$, then

$$\frac{d\bar{\lambda}}{d\gamma_1} = \frac{i\upsilon\omega(\cos(\upsilon\gamma_1) - i\sin(\upsilon\gamma_1))}{1 + i\upsilon\gamma_2 + \omega(\gamma_2 - \gamma_1)(\cos(\upsilon\gamma_1) - i\sin(\upsilon\gamma_1))}$$

and the real part of this expression has the same sign as

$$\begin{aligned} \upsilon\omega\sin(\upsilon\gamma_1)[1 + \omega(\gamma_2 - \gamma_1)\cos(\upsilon\gamma_1)] + \upsilon\omega\cos(\upsilon\gamma_1)[\upsilon\gamma_2 - \omega(\gamma_2 - \gamma_1)\sin(\upsilon\gamma_1)] \\ = \upsilon\omega\left[\sin(\upsilon\gamma_1) + \upsilon\gamma_2\cos(\upsilon\gamma_1)\right] \end{aligned}$$

Hence

$$\mathrm{Re}\left(\frac{d\bar{\lambda}}{d\gamma_1}\right) \gtreqless 0 \text{ if and only if } \sin(\upsilon\gamma_1) + \upsilon\gamma_2\cos(\upsilon\gamma_1) \gtreqless 0$$

Consider first the case of crossing any segment $L_1(k,n)$ from the left. Here $\upsilon\gamma_1 \in (0, \pi/2]$, so both $\sin(\upsilon\gamma_1)$ and $\cos(\upsilon\gamma_1)$ are positive. Hence stability is lost everywhere on any segment of $L_1(k,n)$. Consider the case when crossing the segments of $L_2(k,n)$ from the left. Stability is lost when γ_2 increases in υ and stability is gained when γ_2 decreases in υ. At all intersections with $L_1(k,n)$ and $L_2(k,n)$ Hopf bifurcation occurs giving the possibility of the birth of limit cycles.

We can also show that at any intersection with $L_1(k,n)$ or $L_2(k,n)$ the complex root is single. Otherwise $\lambda = i\upsilon$ would satisfy both equations

$$\lambda + \omega e^{-\lambda\gamma_1} + (1-\omega)e^{-\lambda\gamma_2} = 0$$

and

$$1 - \omega\gamma_1 e^{-\lambda\gamma_1} - (1-\omega)\gamma_2 e^{-\lambda\gamma_2} = 0,$$

from which we have

$$e^{-\lambda\gamma_1} = \frac{1+\lambda\gamma_2}{(\gamma_1-\gamma_2)\omega} \text{ and } e^{-\lambda\gamma_2} = \frac{-1-\lambda\gamma_1}{(\gamma_1-\gamma_2)(1-\omega)}.$$

By substituting $\lambda = i\upsilon$ and comparing the real and imaginary parts yield

$$\sin(\upsilon\gamma_1) + \upsilon\gamma_2\cos(\upsilon\gamma_1) = \sin(\upsilon\gamma_2) + \upsilon\gamma_1\cos(\upsilon\gamma_2) = 0.$$

Therefore this intersection is at an extremum in υ of a segment $L_1(k,n)$ and also at an extremum of a segment $L_2(\bar{k},\bar{n})$ which is impossible.

Assume next that $\omega = 1/2$. Then equations (4) and (5) imply that

$$\cos(\upsilon\gamma_1) + \cos(\upsilon\gamma_2) = 0$$

$$\upsilon - \tfrac{1}{2}\left(\sin(\upsilon\gamma_1) + \sin(\upsilon\gamma_2)\right) = 0$$

and the curves $L_1(k,n)$ and $L_2(k,n)$ are simplified as follows:

$$L_1(k,n) : \begin{cases} \gamma_1 = \dfrac{1}{\upsilon}\left(\sin^{-1}(\upsilon) + 2k\pi\right) \\ \\ \gamma_2 = \dfrac{1}{\upsilon}\left(\pi - \sin^{-1}(\upsilon) + 2n\pi\right) \end{cases} \tag{18}$$

and

$$L_2(k,n) : \begin{cases} \gamma_1 = \dfrac{1}{\upsilon}\left(\pi - \sin^{-1}(\upsilon) + 2k\pi\right) \\ \\ \gamma_2 = \dfrac{1}{\upsilon}\left(\sin^{-1}(\upsilon) + 2n\pi\right). \end{cases} \tag{19}$$

The stability switching curves are shown in Figure 2 in which the stability region is the gray area.

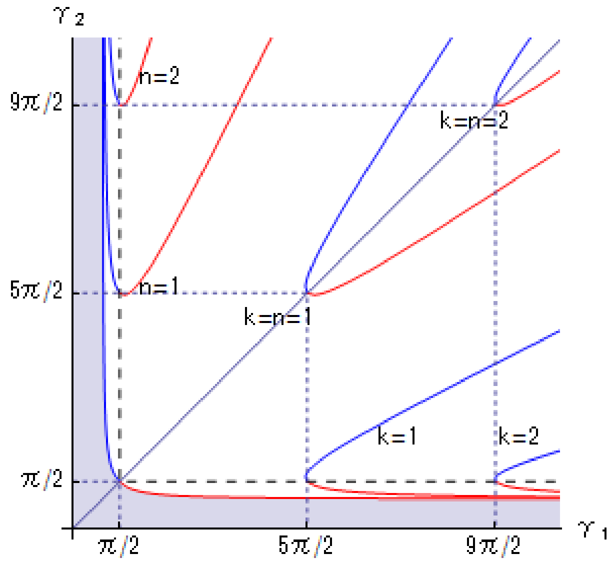


Figure 2. Partition curve in the (γ_1, γ_2) plane with $\omega = \frac{1}{2}$

3 Model 2

Next, we consider equation

$$\dot{x}(t) + ax(t) - b\dot{x}(t-\delta) - cx(t-\eta) = 0. \tag{20}$$

Here $a, b, c > 0$ and $b < 1,\ c < a$. Similarly to the previous model it is easy to prove that the system is stable without delays and also with a single delay, when either $\delta = 0$, $\eta = 0$ or $\delta = \eta$. The corresponding characteristic equation is obtained by substituting an exponential solution, $x(t) = e^{\lambda t}u$,

$$\lambda + a - b\lambda e^{-\delta\lambda} - ce^{-\eta\lambda} = 0. \tag{21}$$

Dividing its both sides by $a + \lambda$ and introducing the new functions,

$$a_1(\lambda) = -\frac{b\lambda}{a+\lambda} \text{ and } a_2(\lambda) = -\frac{c}{a+\lambda}$$

simplify equation (21),

$$a(\lambda) = 1 + a_1(\lambda)e^{-\delta\lambda} + a_2(\lambda)e^{-\eta\lambda} = 0. \tag{22}$$

The terms of this function are shown in Figure 3.

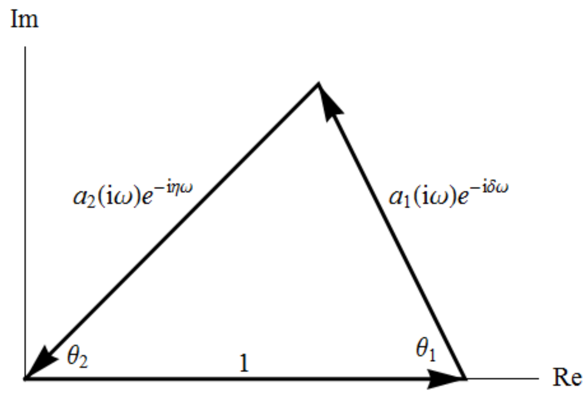


Figure 3. Triangle formed by 1, $|a_1(i\omega)|$ and $|a_2(i\omega)|$

Suppose that $\lambda = i\omega$ with $\omega > 0$, then

$$a_1(i\omega) = -\frac{b\omega^2}{a^2+\omega^2} - i\frac{ab\omega}{a^2+\omega^2} \tag{23}$$

and

$$a_2(i\omega) = -\frac{ac}{a^2+\omega^2} + i\frac{c\omega}{a^2+\omega^2}. \tag{24}$$

Their absolute values are

$$|a_1(i\omega)| = \frac{b\omega}{\sqrt{a^2+\omega^2}} \text{ and } |a_2(i\omega)| = \frac{c}{\sqrt{a^2+\omega^2}}$$

and their arguments are

$$\arg(a_1(i\omega)) = \tan^{-1}\left(\frac{a}{\omega}\right) + \pi \text{ and } \arg(a_2(i\omega)) = \pi - \tan^{-1}\left(\frac{\omega}{a}\right).$$

The triangle can be above the real line and also under the real line. In the two cases the following relations hold for angles θ_1 and θ_2:

$$\arg\left(a_1(i\omega)\right) - \delta\omega \pm \theta_1 = \pi + 2n\pi \tag{25}$$

and

$$\arg\left(a_2(i\omega)\right) - \eta\omega \mp \theta_2 = \pi + 2m\pi \tag{26}$$

In a triangle consisting of three line segments, the length of the sum of any two adjacent line segments is not shorter than the length of the remaining line segment,

$$1 \leq |a_1(i\omega)| + |a_2(i\omega)|,$$

$$|a_1(i\omega)| \leq 1 + |a_2(i\omega)|,$$

and

$$|a_2(i\omega)| \leq 1 + |a_1(i\omega)|\,.$$

Substituting the absolute values renders these three conditions to the following two conditions,

$$f(\omega) = (1-b^2)\omega^2 - 2bc\omega + a^2 - c^2 \leq 0$$

and

$$g(\omega) = (1-b^2)\omega^2 + 2bc\omega + a^2 - c^2 \geq 0.$$

Both $f(\omega)$ and $g(\omega)$ have the same discriminant,

$$D = 4[c^2 - a^2(1-b^2)].$$

In the following we draw attention to the case of $D > 0$, otherwise $f(\omega) > 0$ for all ω implying no stability switch. Solving $g(\omega) = 0$ gives the solutions

$$\omega_1 = \frac{-bc - \sqrt{c^2 - a^2(1-b^2)}}{1-b^2} \text{ and } \omega_2 = \frac{-bc + \sqrt{c^2 - a^2(1-b^2)}}{1-b^2}$$

and so does solving $f(\omega) = 0$,

$$\omega_3 = \frac{bc - \sqrt{c^2 - a^2(1-b^2)}}{1-b^2} \text{ and } \omega_4 = \frac{bc + \sqrt{c^2 - a^2(1-b^2)}}{1-b^2}.$$

Since both ω_1 and ω_2 are negative and both ω_3 and ω_4 are positive, the two conditions, $f(\omega) \leq 0$ and $g(\omega) \geq 0$, are satisfied when ω is in interval $[\omega_3, \omega_4]$.

The internal angles, θ_1 and θ_2, of the triangle in Figure 3 can be calculated by the law of cosine as

$$\theta_1(\omega) \quad = \cos^{-1}\left(\frac{a^2 + (1+b^2)\omega^2 - c^2}{2b\omega\sqrt{a^2+\omega^2}}\right) \tag{27}$$

and

$$\theta_2(\omega) \quad = \cos^{-1}\left(\frac{a^2 + (1-b^2)\omega^2 + c^2}{2c\sqrt{a^2+\omega^2}}\right). \tag{28}$$

Solving equations (25) and (26) for δ and η yields

$$\delta^{\pm}(\omega, k) = \frac{1}{\omega}\left[\tan^{-1}\left(\frac{a}{\omega}\right) + \pi + (2k-1)\pi \pm \theta_1(\omega)\right].$$

and

$$\eta^{\mp}(\omega, k) = \frac{1}{\omega}\left[-\tan^{-1}\left(\frac{\omega}{a}\right) + \pi + (2n-1)\pi \mp \theta_2(\omega)\right],$$

so we have again two stability switching curves with fixed values of k and n,

$$L_1(k, n) = \{\delta^{+}(\omega, k), \eta^{-}(\omega, n)\}$$

and

$$L_2(k, n) = \{\delta^{-}(\omega, k), \eta^{+}(\omega, n)\}$$

They are shown in Figure 4 for the case of $k = n = 1$. They have the same initial point S and arrive at the same end point E as ω increases from ω_3 to ω_4. With fixed $\eta = \eta^0$ by increasing the value of δ, stability is lost at point A and regained at point B. These curves are shifted to the right by increasing the value of k and up by increasing the value of n.

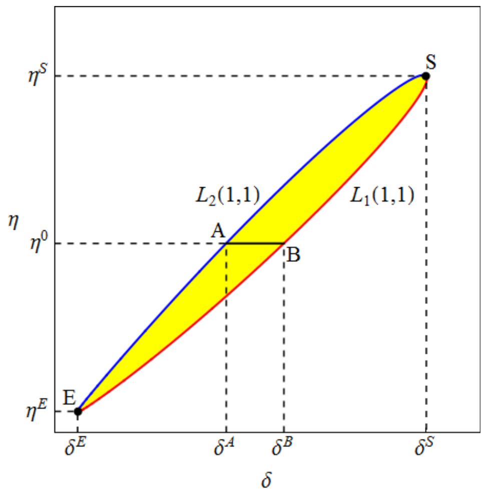
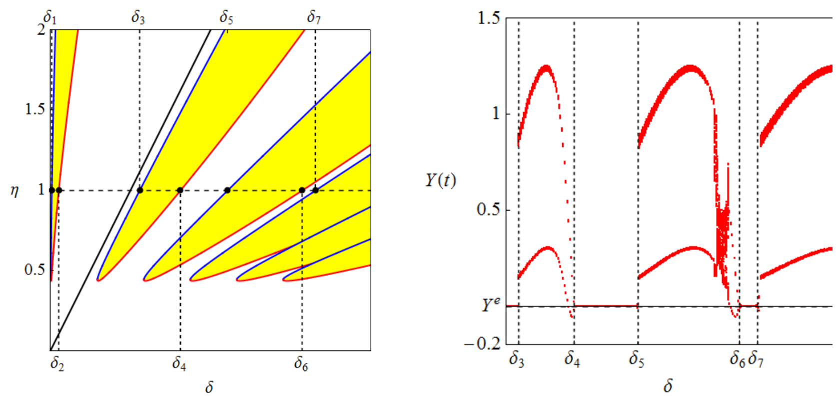


Figure 4. Partition curve with $k = 1$ and $n = 1$

4 Simulations

In the first case, Figure 5(A) shows the six cigar-shaped domains obtained for $k = 0, 1, 2, 3, 4, 5$ and $n = 1$ and their lower parts are colored in yellow. We fix $\eta = 1$ and increase δ from 1 to 4 along the dotted horizontal line. The system is stable until $\delta = \delta_1$, when stability is lost. It is regained at $\delta = \delta_2$ and system remains stable until $\delta = \delta_3$ where stability is lost, and regained again at $\delta = \delta_4$, and so on. So stability is lost at points $\delta_1,\ \delta_3,\ \delta_5$ and δ_7 and stability is regained at points δ_2, δ_4 and δ_6. The bifurcation diagram shown in Figure 5(B) well demonstrates these observations.

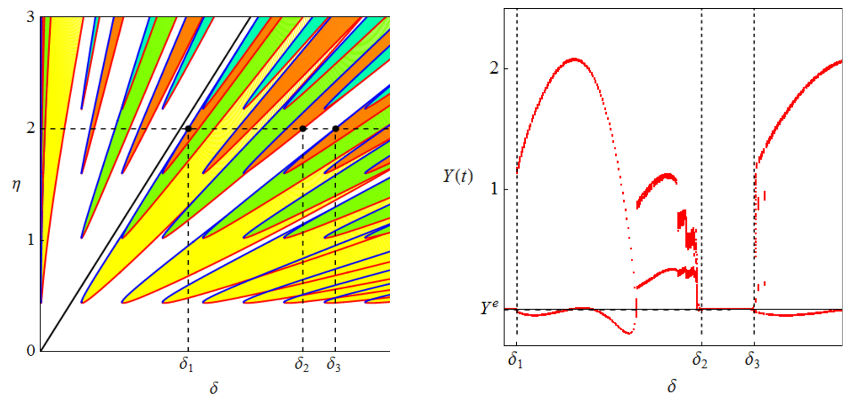


(A) Partition curves (B) Bifurcation diagram

Figure 5. Stability switches with $\eta = 1$

In the second simulation we illustrate the curves $L_1(k,n)$ and $L_2(k,n)$ for $k = 0, 1, ..., 8$ and $n = 1, 2, 3, 4$ in Figure 6(A). The yellow domains are surrounded by $L_1(k,1)$ and $L_2(k,1)$, which are the same as in Figure 5(A). The green regions are surrounded by $L_1(k,2)$ and $L_2(k,2)$, and the orange and blue regions by $L_1(k,3)$ and $L_2(k,3)$ and by $L_1(k,4)$ and $L_2(k,4)$, respectively. The value of $\eta = 2$ is now selected. The dotted horizontal line crosses the stability switching curves many times, but not all intersections are stability switches. For example, between δ_1 and δ_2 the system is unstable regardless of several intersections between them. At $\delta = \delta_2$ stability is regained, and lost again at $\delta = \delta_3$. The bifurcation diagram shown in Figure 6(B) well illustrates these findings.

Let (δ, η) be any point in the positive quadrant and not on the stability switching curves and consider the line segment connecting points $(0, \eta)$ and (δ, η). Let L be the number of intersections of this segment with the stability switching curves with stability loss and G the number of intersections with stability gain. The system is stable for (δ, η) if $G \geq L$, otherwise unstable.



(A) Partion curves (B) Bifurcation diagram

Figure 6. Stability switches with $\eta = 2$

5 Conclusions

Two particular engineering models were examined. Both are first order ordinary differential equations with two delays. The stability switching curves were first determined where an eigenvalue is pure complex, and then the stability and instability regions were demonstrated. In the first case an elementary analytic approach was used, and in the second case a geometric approach was shown. This approach could be also used for solving the first model as well, however the more simple analytic approach cannot be used for the second model without major changes.

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