Delayed Dynamics in Heterogeneous Competition with Product Differentiation

Akio Matsumoto
Department of Economics
Chuo University
Hachioji, Tokyo, 192-0393, Japan
akiom@tamacc.chuo-u.ac.jp

Ferenc Szidarovszky
Systems and Industrial Engineering Department
University of Arizona
Tucson, Arizona, 85721-0020, USA
szidar@sie.arizona.edu

April 18, 2007

Abstract

Heterogeneous duopolies with product differentiation and isoelastic price functions are examined in which one firm is quantity setter and the other is price setter. The reaction functions and the Cournot-Bertrand (CB) equilibrium are first determined. It is shown that the best response dynamics with continuous time scales and without time delays is always locally asymptotically stable. This stability can be however lost in the presence of time delays. Both fixed and continuously distributed time delays are examined, stability conditions derived and the stability regions determined and illustrated. The results are compared to Cournot-Cournot (CC) and Bertrand-Bertrand (BB) dynamics. It turns out that continuously distributed lags have smaller instabilizing effect on the equilibria than fixed lags, and both homogeneous (CC and BB) competitions are more stable than the heterogeneous competitions.

1 Introduction

The effect of information delay in dynamic economic models has been introduced by Invernizzi and Medio(1991), and its application to dynamic oligopolies has been examined by Chiarella and Khomin(1996) and Chiarella and Szidarovszky(2001). In investigating concave quantity adjusting oligopolies, they have shown that the presence of information delay has an instabilizing effect on the equilibrium. Furthermore a complete stability analysis was performed in the case of continuously distributed time lags, and the occurrence of Hopf bifurcation was verified giving the possibility of the birth of limit cycles around the equilibrium. No such study was given for price adjusting oligopolies, for heterogeneous competitions and for markets with isoelastic demand functions.

In this paper, heterogeneous duopolies will be examined with product differentiation and isoelastic inverse demand functions. The paper develops as follows. In Section 2 the best responses of the firms will be determined and the Nash equilibrium computed. In Section 3, the stability of the equilibrium will be examined without time delays. Dynamics with time delays will be studied, the one with fixed time delays in Section 4 and then the one with continuously distributed delays in Section 5. In Section 6, we will also compare these two approaches. In Section 7 our results on heterogeneous duopolies will be compared to the homogeneous cases of quantity and price adjusting duopolies, and finally, concluding remarks will be given in Section 8.

2 Cournot-Bertrand Equilibrium

A duopoly is considered with product differentiation and isoelastic demand functions. The inverse demand functions of firms 1 and 2 are assumed to have the following forms:

$$p_1 = \frac{1}{x_1 + \theta_1 x_2} \tag{1}$$

and

$$p_2 = \frac{1}{\theta_2 x_1 + x_2}, \tag{2}$$

where θ_1 and θ_2 denote the degrees of production differentiation and fulfill Assumption 1 below.

Assumption 1. $0 < \theta_i < 1$.

In the following, we consider the Cournot-Bertrand (CB henceforth) competition in which firm 1 is a quantity-setter and firm 2 is a price-setter. By interchanging the firms we can have the identical Bertrand-Cournot (BC) competition. Let c_1 and c_2 denote the constant marginal costs. First we will find the best responses of the firms in terms of their decision variables x_1 and p_2. From (1) and (2),

$$x_2 = \frac{1}{p_2} - \theta_2 x_1$$

and

$$x_1 = \frac{1}{p_1} - \theta_1 x_2,$$

so the profit of firm 1 can be written as

$$\begin{aligned} \pi_1 &= (p_1 - c_1)x_1 \\ &= \left(\frac{1}{x_1 + \theta_1 \left(\frac{1}{p_2} - \theta_2 x_1 \right)} - c_1 \right) x_1 \\ &= \frac{p_2 x_1}{\theta_1 + x_1 p_2 (1 - \theta_1 \theta_2)} - c_1 x_1. \end{aligned}$$

Assuming interior optimum, the first order condition implies that

$$\frac{\partial \pi_1}{\partial x_1} = \frac{p_2 \theta_1}{\left(\theta_1 + x_1 p_2 (1 - \theta_1 \theta_2)\right)^2} - c_1 = 0.$$

Therefore the best response of firm 1 is the following:

$$x_1 = \frac{1}{1 - \theta_1 \theta_2} \left(\sqrt{\frac{\theta_1}{p_2 c_1}} - \frac{\theta_1}{p_2} \right). \tag{3}$$

The profit of firm 2 has the form

$$\begin{aligned} \pi_2 &= (p_2 - c_2)x_2 \\ &= (p_2 - c_2) \left(\frac{1}{p_1} - \theta_1 x_2 \right) \\ &= 1 - \frac{c_2}{p_2} - p_2 \theta_2 x_1 + c_2 \theta_2 x_1 \end{aligned}$$

with first order condition

$$\frac{\partial \pi_2}{\partial p_2} = \frac{c_2}{p_2^2} - \theta_2 x_1 = 0,$$

implying that the best response of firm 2 is

$$p_2 = \sqrt{\frac{c_2}{\theta_2 x_1}}. \tag{4}$$

To characterize the CB equilibrium it is convenient to re-define these reaction functions in the quantity space (x_1, x_2):

$$\theta_1(\theta_2 x_1 + x_2) = c_1(x_1 + \theta_1 x_2)^2$$

and

$$\theta_2 x_1 = c_2(\theta_2 x_1 + x_2)^2,$$

which are simple consequences of relations (2), (3) and (4). Dividing the first equation by the second and introducing new variables $z = \frac{x_2}{x_1}$ and $c = \frac{c_2}{c_1}$ give, after arrangements,

$$c\frac{\theta_1}{\theta_2}(z + \theta_2) = \left(\frac{1 + \theta_1 z}{\theta_2 + z}\right)^2. \tag{5}$$

The intersection of the right hand side and the left hand side functions determines the CB equilibrium ratio of outputs.

Let us denote the left hand side by $f(z)$,

$$f(z) = c\frac{\theta_1}{\theta_2}(z + \theta_2),$$

and the right hand side by $g(z)$,

$$g(z) = \left(\frac{1 + \theta_1 z}{\theta_2 + z}\right)^2.$$

Then it is easy to see that

$$\begin{aligned} g(0) &= \frac{1}{\theta_2^2} > 0, \\ g'(z) &= -\frac{2(1 - \theta_1\theta_2)(1 + \theta_1 z)}{(z + \theta_2)^3} < 0, \\ g''(z) &= \frac{2(1 - \theta_1\theta_2)(3 - \theta_1\theta_2 + 2\theta_1 z)}{(z + \theta_2)^4} > 0. \end{aligned}$$

Clearly, $f(z)$ strictly increases, linear and $f(0) = c\theta_1$. The graph of functions $f(z)$ and $g(z)$ are illustrated in Figure 1. A unique positive intersection $\alpha = \alpha(c, \theta_1, \theta_2)$ exists if and only if the following condition holds:

Assumption 2. $c < \dfrac{1}{\theta_1 {\theta_2}^2}$.

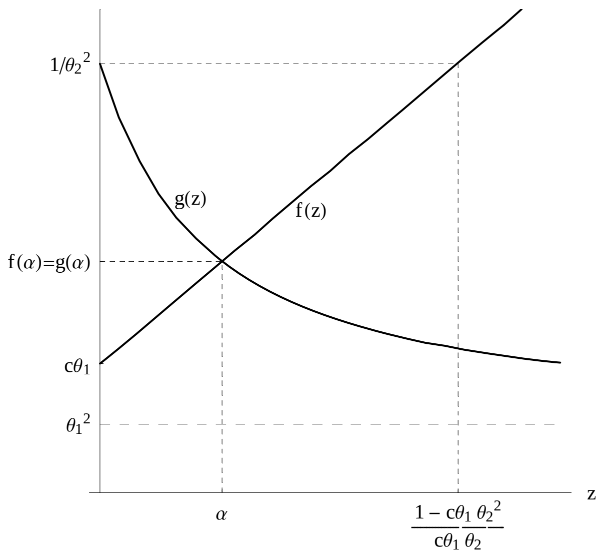

Figure 1. Determination of optimal output ratio.

It is also easy to see, as shown in Figure 1, that

$$\alpha < \frac{1 - c\theta_1 {\theta_2}^2}{c\theta_1\theta_2}.$$

The equilibrium of the CB duopoly is clearly given by

$$x_1^* = \frac{\theta_2}{c_2(\alpha + \theta_2)^2},$$

$$p_2^* = \frac{c_2(\alpha + \theta_2)}{\theta_2}.$$

3 Continuous Dynamics without Time Delays

The best response CB dynamic system has the form

$$\begin{aligned}\dot{x}_1 &= k_1\left(\frac{1}{1-\theta_1\theta_2}\left(\sqrt{\frac{\theta_1}{p_2c_1}}-\frac{\theta_1}{p_2}\right)-x_1\right),\\ \dot{p}_2 &= k_2\left(\sqrt{\frac{c_2}{\theta_2x_1}}-p_2\right).\end{aligned}$$

Its Jacobian is

$$J=\begin{pmatrix}-k_1 & k_1\gamma_1\\ k_2\gamma_2 & -k_2\end{pmatrix},$$

where γ_i is the derivative of firm i's reaction function at the CB equilibrium. Simple calculation shows that

$$\gamma_1=\frac{\theta_1\theta_2^2}{(1-\theta_1\theta_2)c_2^2(\alpha+\theta_2)^2}\left(1-\frac{1}{2}\sqrt{\frac{c(\alpha+\theta_2)}{\theta_1\theta_2}}\right)$$

and

$$\gamma_2=-\frac{c_2^2(\alpha+\theta_2)^3}{2\theta_2^2},$$

furthermore by using (5), we have

$$\gamma_1\gamma_2=\frac{(1-\theta_1\theta_2)-\alpha\theta_1-\theta_1\theta_2}{4(1-\theta_1\theta_2)}.$$

The characteristic equation is

$$\lambda^2+(k_1+k_2)\lambda+k_1k_2(1-\gamma_1\gamma_2)=0.$$

Since

$$\gamma_1\gamma_2=\frac{(1-\theta_1\theta_2)-\alpha\theta_1-\theta_1\theta_2}{4(1-\theta_1\theta_2)}<\frac{1}{4}, \tag{6}$$

both coefficients are positive, therefore the real parts of the eigenvalues are confirmed to be negative. Therefore the CB dynamic system is always locally asymptotically stable under Assumptions 1 and 2.

4 Continuous Dynamics with Fixed Time Delays

Let us denote the reaction functions of the firms by $R_1(p_2)$ and $R_2(x_1)$ and assume that each firm i has a fixed time delay T_i on its competitor's variable. The delayed dynamic system is

$$\begin{aligned} \dot{x}_1(t) &= k_1 \left\{ R_1(p_2(t-T_1)) - x_1(t) \right\}, \\ \dot{p}_2(t) &= k_2 \left\{ R_2(x_1(t-T_2)) - p_2(t) \right\}. \end{aligned} \tag{7}$$

By linearizing these equations, we have a linear system:

$$\begin{aligned} \dot{x}_{1\delta}(t) &= k_1\gamma_1 p_{2\delta}(t-T_1) - k_1 x_{1\delta}(t), \\ \dot{p}_{2\delta}(t) &= k_2\gamma_2 x_{1\delta}(t-T_2) - k_2 p_{2\delta}(t), \end{aligned} \tag{8}$$

where $x_{1\delta}$ and $p_{2\delta}$ denote the deviations of x_1 and p_2 from their equilibrium levels. By looking for the solution in the usual exponential forms $x_{1\delta} = e^{\lambda t}u$ and $p_{2\delta} = e^{\lambda t}v$, and substituting these functions into equations (8), we have

$$\begin{pmatrix} -k_1 - \lambda & k_1\gamma_1 e^{-\lambda T_1} \\ k_2\gamma_2 e^{-\lambda T_2} & -k_2 - \lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Nontrivial solution exists only if the determinant of the coefficient matrix is zero. Therefore the characteristic equation is

$$\lambda^2 + (k_1 + k_2)\lambda + k_1 k_2 - k_1 k_2 \gamma e^{-\lambda\tau} = 0, \tag{9}$$

where we introduce two new variables, τ denoting the sum of the two time lags and γ denoting the product of the derivatives of the reaction functions,

$$\tau = T_1 + T_2 \text{ and } \gamma = \gamma_1\gamma_2.$$

First we show that $\lambda = 0$ cannot be an eigenvalue. Suppose $\lambda = 0$, then the characteristic function becomes

$$k_1 k_2 (1 - \gamma) > 0,$$

which implies that $\lambda = 0$ cannot be a solution. In order to understand the stability switches of the delayed dynamic system, it is crucial to determine the critical values of time lag at which the characteristic equation may have a pair of conjugate pure imaginary roots.

Suppose now that $\lambda = i\omega$, with $\omega > 0$, is a solution of the characteristic equation. Introducing a new variable $\theta = \omega\tau$, we have

$$-\omega^2 + k_1k_2 - k_1k_2\gamma\cos\theta = 0 \tag{10}$$

and

$$(k_1 + k_2)\omega + k_1k_2\gamma\sin\theta = 0. \tag{11}$$

Thus, by moving $-\omega^2 + k_1k_2$ and $(k_1 + k_2)\omega$ to the right hand sides, squaring and adding the resulted equations, we obtain

$$(k_1k_2\gamma)^2 = (-\omega^2 + k_1k_2)^2 + (k_1 + k_2)^2\omega^2.$$

Hence

$$\omega^4 + (k_1^2 + k_2^2)\omega^2 + (k_1k_2)^2(1 - \gamma^2) = 0. \tag{12}$$

Introducing $z = \omega^2$ makes the left hand side a quadratic polynomial in terms of z,

$$\varphi(z) = z^2 + (k_1^2 + k_2^2)z + (k_1k_2)^2(1 - \gamma^2). \tag{13}$$

Notice that

$$\varphi(0) = (k_1k_2)^2(1 - \gamma^2)$$

and

$$\varphi'(0) = (k_1^2 + k_2^2) > 0,$$

furthermore the discriminant of the quadratic polynomial (13) is always positive. For $\gamma \in [-1, \frac{1}{4})$, $\varphi(0) > 0$ with $\varphi'(0) > 0$ implying that $\varphi(z) = 0$ has real solutions, both are negative. Thus there are no imaginary solutions such as $\lambda = i\omega$ with $\omega > 0$.

In order to prove that the equilibrium is always locally asymptotically stable, we will show that with sufficiently small $|\gamma|$, this is the case. Let $\lambda = a + ib$ be a solution of equation (9) with $a \geq 0$. Substituting this solution into the equation we have

$$a^2 - b^2 + 2abi + (k_1 + k_2)(a + bi) + k_1k_2\left[1 - \gamma e^{-a\tau}(\cos b\tau - i\sin b\tau)\right] = 0.$$

The imaginary part of this equation shows that

$$2ab + (k_1 + k_2)b + k_1k_2\gamma e^{-a\tau}\sin b\tau = 0,$$

which can be written as

$$2a = -(k_1 + k_2) - C$$

where

$$|C| = k_1 k_2 |\gamma| e^{-a\tau} \tau \left| \frac{\sin b\tau}{b\tau} \right| \leq k_1 k_2 \tau |\gamma| .$$

So, if $|\gamma| < \frac{k_1+k_2}{k_1 k_2 \tau}$, then $a < 0$, which completes the proof. Hence we have:

Theorem 1 *If* $-1 \leqq \gamma < \dfrac{1}{4}$, *no stability switching occurs and the delayed system is always locally asymptotically stable.*

Consider next the case where $\gamma < -1$. In this case, $\varphi(0) < 0$ and $\varphi'(0) > 0$, therefore, $\varphi(z) = 0$ has one positive and one negative root,

$$z_{\pm} = \omega_{\pm}^2 = \frac{-(k_1^2 + k_2^2) \pm \sqrt{(k_1^2 + k_2^2)^2 - 4(k_1 k_2)^2 (1 - \gamma^2)}}{2}. \tag{14}$$

In this case, only one imaginary root, $\lambda = i\omega_+$, exists with $\omega_+ > 0$. We select τ as the bifurcation parameter. The critical level of time lag for which stability switching may occur is determined by

$$\tau^* = \frac{\theta^*}{\omega_+},$$

where

$$\cos\theta^* = \frac{k_1 k_2 - \omega_+^2}{k_1 k_2 \gamma}$$

and

$$\sin\theta^* = -\frac{(k_1 + k_2)\omega_+}{k_1 k_2 \gamma} > 0$$

as consequences of the definition of θ and relations (10) and (11).

We have

$$k_1 k_2 - \omega_+^2 = \frac{(k_1 + k_2)^2 - \sqrt{(k_1^2 + k_2^2)^2 - 4(k_1 k_2)^2 (1 - \gamma^2)}}{2}.$$

By comparing the squares of the two terms of the numerator, it is easy to see that

$$sign\left\{k_1 k_2 - \omega_+^2\right\} = sign\left\{\left[(k_1 + k_2) - \sqrt{k_1 k_2}\gamma\right]\left[(k_1 + k_2) + \sqrt{k_1 k_2}\gamma\right]\right\}.$$

The first factor of the right hand side is always positive. Define next

$$\bar{\gamma} = -\frac{k_1 + k_2}{\sqrt{k_1 k_2}}. \qquad (15)$$

It is easy to see that $\bar{\gamma} < -1$. The second factor is negative if and only if $\gamma < \bar{\gamma}$. Hence

$$k_1 k_2 - \omega_+^2 \begin{cases} \geq 0 & \text{if } \bar{\gamma} \leq \gamma < -1, \\ < 0 & \text{otherwise.} \end{cases}$$

Notice in addition that by using equation (12), we have

$$\begin{aligned} (k_1 k_2 - \omega_+^2)^2 &= k_1^2 k_2^2 - 2k_1 k_2 \omega_+^2 + \omega_+^4 \\ &= k_1^2 k_2^2 \gamma^2 - (k_1 - k_2)^2 \omega_+^2 \\ &< k_1^2 k_2^2 \gamma^2, \end{aligned}$$

so $\left| k_1 k_2 - \omega_+^2 \right| < |k_1 k_2 \gamma|$, therefore θ^* can be uniquely defined for all $\gamma < -1$. If $k_1 k_2 - \omega_+^2 \geq 0$, then $\theta^* \in [\frac{\pi}{2}, \pi)$ and if $k_1 k_2 - \omega_+^2 < 0$, then $\theta^* \in (0, \frac{\pi}{2})$.

In order to observe stability switching, we need to determine the sign of the derivative of the real part of the purely imaginary root. We can think of the roots of the characteristic equation as continuous functions in terms of the delay τ. By implicit differentiation of equation (9) we have

$$\left\{ 2\lambda + (k_1 + k_2) + k_1 k_2 \gamma \tau e^{-\lambda\tau} \right\} \frac{d\lambda}{d\tau} = -k_1 k_2 \gamma \lambda e^{-\lambda\tau}.$$

For convenience, we study $(d\lambda/d\tau)^{-1}$ instead of $d\lambda/d\tau$. We have

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = -\frac{2\lambda + (k_1 + k_2) + k_1 k_2 \gamma \tau e^{-\lambda\tau}}{k_1 k_2 \gamma \lambda e^{-\lambda\tau}}$$

and from the characteristic equation we obtain

$$e^{-\lambda\tau} = \frac{\lambda^2 + (k_1 + k_2)\lambda + k_1 k_2}{k_1 k_2 \gamma}.$$

Thus

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = -\frac{2\lambda + (k_1 + k_2)}{\lambda(\lambda^2 + (k_1 + k_2)\lambda + k_1 k_2)} - \frac{\tau}{\lambda}.$$

Therefore,

$$\begin{aligned}
sign\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\} &= sign\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\} \\
&= sign\left\{\operatorname{Re}\left(-\frac{2\lambda+(k_1+k_2)}{\lambda(\lambda^2+(k_1+k_2)\lambda+k_1k_2)}\right)\right\} \\
&= sign\left\{\operatorname{Re}\left(\frac{(k_1+k_2)+i2\omega}{(k_1+k_2)\omega^2-i(k_1k_2-\omega^2)\omega}\right)\right\} \\
&= sign\left\{\frac{(k_1^2+k_2^2+2\omega^2)}{(k_1k_2\gamma)^2}\right\} > 0.
\end{aligned}$$

Consequently, the crossing of the imaginary axis is from left to right as τ increases and thus resulting in the loss of stability. In summary, we have

Theorem 2 *Given $\gamma < -1$, the delayed dynamic system with fixed time delays is locally asymptotically stable when $\tau < \tau^*$ and unstable when $\tau > \tau^*$ where $\tau^* = \theta^*/\omega_+$ with*

$$\omega_+^2 = \frac{-(k_1^2+k_2^2)+\sqrt{(k_1^2+k_2^2)^2-4(k_1k_2)^2(1-\gamma^2)}}{2},$$
$$\sin\theta^* = -\frac{(k_1+k_2)\omega_+}{k_1k_2\gamma} > 0,$$
$$\cos\theta^* = \frac{k_1k_2-\omega_+^2}{k_1k_2\gamma},$$

where $\theta^* \in [\frac{\pi}{2},\pi)$ if $k_1k_2-\omega_+^2 \geq 0$, and $\theta^* \in (0,\frac{\pi}{2})$ otherwise.

5 Dynamics with Continuously Distributed Lags

Matsumoto and Szidarovszky(2006) derived the stability conditions in the delayed Cournot competition with continuously distributed time lags. Instead of assuming fixed time delays, they assumed that the lags were continuously distributed with exponential kernel functions. Similarly, in equations (7), $p_2(t-T_1)$ and $x_1(t-T_2)$ are now replaced by their expectations:

$$p_2^e(t) = \int_0^t w(t-s,T_1)p_2(s)ds$$

and

$$x_1^e(t) = \int_0^t w(t-s,T_2)x_1(s)ds,$$

where

$$w(t-s,T) = \frac{1}{T}e^{-\frac{t-s}{T}}.$$

So a system of Volterra-type integro-differential equations is obtained. As it has been shown in Matsumoto and Szidarovszky(2006), the characteristic equation of the system has the special form,

$$(\lambda+k_1)(\lambda+k_2)(1+\lambda T_1)(1+\lambda T_2) - k_1k_2\gamma = 0, \tag{16}$$

where $\gamma = \gamma_1\gamma_2$ as before. This is a quartic equation for λ,

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

with coefficients

$$\begin{aligned}
a_0 &= T_1T_2, \\
a_1 &= T_1+T_2+T_1T_2(k_1+k_2), \\
a_2 &= 1+T_1T_2k_1k_2+(k_1+k_2)(T_1+T_2), \\
a_3 &= k_1+k_2+k_1k_2(T_1+T_2), \\
a_4 &= k_1k_2(1-\gamma).
\end{aligned}$$

Since all coefficients are positive, the Routh-Hurwitz criterion (see Szidarovszky and Bahill(1998)) implies that all eigenvalues have negative real parts if and only if

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0 \text{ and } \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} > 0.$$

Simple algebra shows that the second order determinant is always positive. The sign of the third order determinant depends on the value of γ. Solving the inequality for γ gives the following result:

Theorem 3 *For the dynamic system with continuously distributed time lags, the equilibrium is locally asymptotically stable if*

$$\gamma > -\frac{(k_1+k_2)(T_1+T_2)(1+k_1T_1)(1+k_2T_1)(1+k_1T_2)(1+k_2T_2)}{k_1k_2(T_1+T_2+T_1T_2(k_1+k_2))^2},$$

and is unstable if this relation is violated with strict opposite inequality.

6 Comparison of The Two Approaches

We compared the effects caused by fixed and continuously distributed lags on stability by determining and illustrating the stability regions in the two cases. Figure 2 shows the division of the (T_1, c) space in which the upper bold curve and the lower bold curve are the boundaries between the stable regions and the unstable regions in the cases of fixed lags and continuously distributed lags, respectively. In particular, from Theorems 2, 3 and relation (6), these boundaries are given by

$$(T_1+T_2)\omega_+ = \cos^{-1}\left(\frac{k_1k_2-\omega_+^2}{k_1k_2\gamma^{CB}(c,\theta_1,\theta_2)}\right),$$

where ω_+ is obtained from equation (14), and

$$\gamma^{CB}(c,\theta_1,\theta_2) = -\frac{(k_1+k_2)(T_1+T_2)(1+k_1T_1)(1+k_2T_1)(1+k_1T_2)(1+k_2T_2)}{k_1k_2(T_1+T_2+T_1T_2(k_1+k_2))^2}.$$

In both cases $\gamma^{CB}(c,\theta_1,\theta_2)$ endogenously determines the product of derivatives of the reaction functions and is given as

$$\gamma^{CB}(c,\theta_1,\theta_2) = \frac{(1-\theta_1\theta_2)-\alpha(c,\theta_1,\theta_2)\theta_1-\theta_1\theta_2}{4(1-\theta_1\theta_2)}.$$

We set $k_1 = k_2 = 0.8$, $\theta_1 = \theta_2 = 0.8$ and $T_2 = 2.5$ in Figure 2 in which D means distributed time lags and F means fixed time lags. It is numerically confirmed that there are only small differences between the symmetric ($k_1 = k_2$ and $\theta_1 = \theta_2$) and nonsymmetric ($k_1 \neq k_2$ and/or $\theta_1 \neq \theta_2$) cases. It is also proved that the dynamics with fixed time lags is unstable below the upper bold curve and stable above it. Similarly, the dynamics with continuously distributed time lags is unstable below the lower bold curve and stable above the curve. As we have proved earlier, the CB dynamics without time delay is always locally asymptotically stable. By introducing time lags, this stability might be lost and an instability region develops. Notice that in the middle shaded region surrounded by the bold curves, the CB equilibrium is stable under continuously distributed time lags and unstable under fixed time lags. In other words, the instability region is larger when fixed time lags are assumed. Therefore it is demonstrated that the destabilizing effect of fixed time lags is larger than that in the case of continuously distributed time lags.

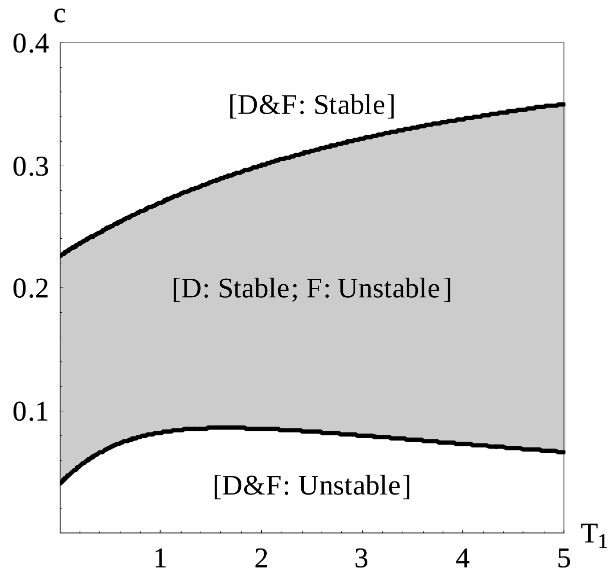

Figure 2. Stability regions in CC competition.

7 Heterogeneous vs Homogeneous Competition

In Matsumoto and Szidarovszky(2006), the delayed continuous dynamic system is considered under the traditional Cournot competition (abbreviated as Cournot-Cournot or CC competition) in which both firms are quantity-setters. In this case, the firms have the best response functions

$$R_1(x_2) = \sqrt{\frac{\theta_1 x_2}{c_1}} - \theta_1 x_2$$

and

$$R_2(x_1) = \sqrt{\frac{\theta_2 x_1}{c_2}} - \theta_2 x_1.$$

Equation (5) has the form

$$c\frac{\theta_1}{\theta_2} z = \left(\frac{1+\theta_1 z}{\theta_2 + z}\right)^2, \tag{17}$$

which has always a unique positive solution $\alpha^C = \alpha^C(c, \theta_1, \theta_2)$ if Assumption 1 is fulfilled. The CC equilibrium is given as

$$x_1^C = \frac{\theta_2}{c_2(\theta_2 + \alpha^C)^2}$$

and

$$x_2^C = \frac{\alpha^C \theta_2}{c_2(\theta_2 + \alpha^C)^2}.$$

The Jacobian of the corresponding best response dynamics has the same form as in the CB duopoly case with the only difference that in the CC case, $\alpha = \alpha^C$ is the solution of (17), and

$$\gamma^C := \gamma_1^C \gamma_2^C = \frac{1}{4}\left\{1 + \theta_1\theta_2 - \left(\alpha^C \theta_1 + \frac{1}{\alpha^C}\theta_2\right)\right\}.$$

It can be proved that with appropriate values of α^C, $\gamma_1^C \gamma_2^C$ can have any real value between $-\infty$ and $\frac{1}{4}$ similarly to the CB case. Figure 3 illustrates the stability regions in the CC competition case. The dotted upper and lower curves have the same meanings as the upper and lower bold curves in Figure 2. Thus it can be said that under CC competition, fixed time lags have the larger destabilizing effect in comparison with continuously distributed time lags.

The numerical results of Figures 2 and 3 are repeated in a different way in Figure 4, where it can be seen that both dotted boundary curves of CC competition are located below the corresponding boundary curves of CB competition. In the middle shaded region between these boundary curves, the equilibrium is stable under CC competition and unstable under CB competition. Consequently, the stability regions are much larger for CC dynamics than for CB dynamics showing that CC competition is more stable than CB competition.

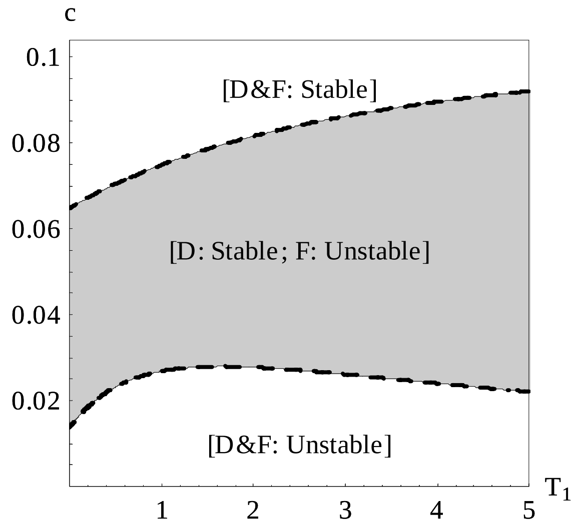

Figure 3. Stability regions in CB competition.

In Matsumoto and Szidarovszky(2006), the case of Bertrand-Bertrand (BB) competition has been also discussed when both firms are price-setters. In this case the reaction functions of the two firms are

$$p_1 = \sqrt{\frac{c_1 p_2}{\theta_1}}$$

and

$$p_2 = \sqrt{\frac{c_2 p_1}{\theta_2}}.$$

The equilibrium prices and outputs are

$$p_1^B = \sqrt[3]{\frac{c_1^2 c_2}{\theta_1^2 \theta_2}}, \; p_2^B = \sqrt[3]{\frac{c_2^2 c_1}{\theta_2^2 \theta_1}},$$

$$x_1^B = \frac{1}{1-\theta_1\theta_2} \sqrt[3]{\frac{\theta_1^2 \theta_2}{c_1 c_2}} \left(\sqrt[3]{\frac{1}{c_1}} - \sqrt[3]{\frac{\theta_1^2 \theta_2}{c_2}} \right)$$

and

$$x_2^B = \frac{1}{1-\theta_1\theta_2} \sqrt[3]{\frac{\theta_2^2 \theta_1}{c_1 c_2}} \left(\sqrt[3]{\frac{1}{c_2}} - \sqrt[3]{\frac{\theta_2^2 \theta_1}{c_1}} \right).$$

Simple calculation shows that the form of the Jacobian of the corresponding dynamic system is the same as in the CB case with the only difference that in the BB case

$$\gamma^B = \gamma_1^B \gamma_2^B = \frac{1}{4}.$$

Because of this special value of γ^B, the BB dynamics is always locally asymptotically stable without and also with time delays with both fixed and continuously distributed time lags. Notice that the results of Section 3, Theorem 1 (with $\gamma = \frac{1}{4}$) and Theorem 3 can be applied in the BB case without any limitation.

If we call both BB and CC competitions *homogeneous*, in which both firms are either price-setters or quantity setters and CB competition *heterogeneous*, in which behavioral specification of one firm is different from that of the other firm, then we can summarize our results of this section as follows: both homogeneous competitions are much more stable than the heterogeneous competition regardless of the forms of the time delays.

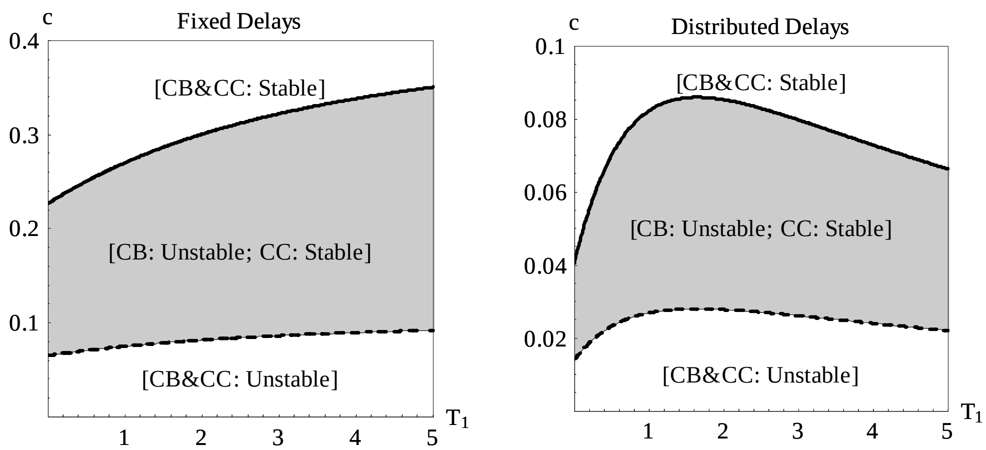

Figure 4. Stability regions in CC and CB competitions.

8 Concluding Remarks

In this paper, heterogeneous duopolies were examined, where one firm is quantity-setter and the other is price-setter. After the reaction functions of the firms were determined, the unique CB-equilibrium was computed. Continuous dynamics were investigated without time delays, and with fixed and continuously distributed time lags. The CB dynamics without time delay is always locally asymptotically stable. This stability can be lost in the presence of time delays. Complete stability analysis was presented for two types of time lags, and the stability regions were illustrated. We have seen that fixed time lags have larger destabilizing effect on the dynamics than continuously distributed time lags. Similar conclusion could be reached in the case of CC duopolies, however the instability regions were much smaller in the case of CC dynamics than in CB dynamics. Therefore we can conclude that if any one of the firms of a CC competition changes from quantity setting behavior to price setting policy, then it has a destabilizing effect on the equilibrium. However, if the other firm also switches to price setting policy, then the corresponding dynamic system becomes always locally asymptotically stable with and without time delays. Hence, homogeneous competitions in which both firms are either quantity setters or price setters are more stable than the heterogeneous competitions in which one firm is quantity setter and the other is price setter.

In our study we considered only exponential weighting functions for continuously distributed time delays, more complex weighting functions will be examined in a future paper.

Acknowledgement The authors highly appreciate financial supports from the Japan Ministry of Education, Culture, Sports, Science and Technology (Grant-in-Aid for Scientific Research (A) 16203019), the US Air Force Office of Scientific Research (MURI grant N00014-03-1-0510) and Chuo University Grant for Special Research(0382). The usual disclaimer applies.

References

[1] Chiarella, C. and A. Khomin, "An Analysis of the Complex Dynamic Behavior of Nonlinear Oligopoly Models with Time Lags," *Chaos, Solitons and Fractals*, vol.7, no.12, pp.2049-2065, 1996.

[2] Chiarella, C. and F. Szidarovszky, "The Birth of Limit Cycles in Nonlinear Oligopolies with Continuously Distributed Information Lags," in Dror, M., P. L'Ecuyer and F. Szidarovszky (eds) *Modeling Uncertainty*, Kluwer Academic Publishers, Boston, 249-268, 2001.

[3] Invernizzi, S., and A. Medio, "On Lags and Chaos in Economic Dynamic Models," *Journal of Mathematical Economics*, vol.20, pp. 521-550, 1991.

[4] Kuang, Y., *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, 1993.

[5] Matsumoto, A. and F. Szidarovszky, "Delayed Nonlinear Cournot and Bertrand Dynamics with Product Differentiation," *Discussion Paper No.89 of the Institute of Economic Research*, Chuo University, Tokyo, Japan, 2006, forthcoming in *Journal of Nonlinear Dynamics, Psychology, and Life Science.*

[6] Szidarovszky, F. and T. A. Bahill, *Linear Systems Theory*, CRC Press, Boca Raton/London, 1998.